CHAPTER I
FUZZY SUBSETS AND FUZZY TOPOLOGY

1.1 INTRODUCTION

The concept of a fuzzy subset was introduced and studied by L.A.Zadeh [61] in the year 1965. The subsequent research activities in this area and the related areas have found applications in many branches of science and engineering. C.L.Chang [11] introduced and studied fuzzy topological spaces in 1968 as a generalisation of topological spaces. Many researchers like R.H.Warren [55-57], C.K.Wong [58-60], S.E.Rodabaugh [47], R.Lowen [28], K.K.Azad [2-3], A.S.Mashhour [33-36], S.R.Malghan and S.S. Benchalli [31,32], M.N.Mukherjee and B.Ghosh [37], P.Sundaram [49] and many others have contributed to the development of fuzzy topological spaces.

The work carried out by M.E.Abd El-Monsef, I.M.Hanafy and S.N.Eldeeb[1], A.S.Bin Shahna [10], M.H.Ghanim, E.E.Kerre and A.S.Mashhour [18], Pu Pao-Ming and Liu Ying-Ming [46], M.K.Singal and Niti Prakash [48], A.Di Concilio and G.Gerla [12], Y.Gnanambal [19], G.Balasubramanian [7], P.Sundaram and N.Nagaveni [50-52] has been refered to, during the present work.

In this chapter, the concept of fuzzy subset is illustrated. Various operations on fuzzy sets such as union, intersection and complementation of fuzzy sets are included and a list of related properties is included. The concept of image and the inverse image of a fuzzy set under a function are included and the properties proved by C.L.Chang [11] and R.H.Warren [55] are given. Further the basic concepts and results on fuzzy topological spaces, from the work of C.L.Chang [11], R.H.Warren [55-57], C.K.Wong [59], S.R.Malghan and S.S. Benchalli [31,32] are presented, which are required in the subsequent chapters. Finally th
basics and the results on boundary of fuzzy sets from the work of R.H. Warren [57] are presented. Other preliminary ideas on fuzzy set theory can be found in [24,25].

1.2 THE CONCEPT OF A FUZZY SUBSET

Let X be a set and A be a subset of X. Let \( \chi_A : X \rightarrow \{0, 1\} \) be the characteristic function of A, defined as follows:

\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]

Thus an element \( x \in X \) is in A if \( \chi_A(x) = 1 \) and is not in A if \( \chi_A(x) = 0 \).

Hence A is characterised by its characteristic function \( \chi_A : X \rightarrow \{0, 1\} \).

Further note that A has the following representation:

\[
A = \{(x, \chi_A(x)) : x \in X \}
\]

Here \( \chi_A(x) \) may be regarded as the degree of belongingness of x to A, which is either 0 or 1. Hence A is a class of objects with degree of belongingness either 0 or 1.

L.A. Zadeh [61] introduced the class of objects with continuum grades of belongingness ranging between 0 and 1. He called such a class a fuzzy subset.

Let X be a set and \( \mu_A : X \rightarrow [0,1] \) be a function from X into the closed unit interval \([0,1]\), which may take any value between 0 and 1 for an element of X. Such a function is called a membership function or membership characteristic function. A fuzzy subset A in X is characterised by a membership function \( \mu_A : X \rightarrow [0,1] \) which associates with each point x in X, a real number \( \mu_A(x) \) between 0 and 1 which represents the degree or grade of membership or belongingness of x to A. If A is an ordinary subset of X, then \( \mu_A \) can take either 1 or 0 according as
Thus a fuzzy subset $A$ of a set $X$ has the following representation:

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A : X \to [0, 1]$ is the membership function. Equivalently a fuzzy subset $A$ in $X$ is defined as a function from $X$ into closed unit interval $[0, 1]$, since $A$ is characterised by its membership characteristic function.

1.2.1 Definition[11]: A fuzzy set $A$ in a set $X$ is defined to be a function $A : X \to [0, 1]$.

1.2.2 Example: Let $X = \{a, b, c, d, e\}$ and $\mu_A : X \to [0, 1]$ be a function defined by $\mu_A(a) = 0.4$, $\mu_A(b) = 0.7$, $\mu_A(c) = 0$, $\mu_A(d) = 0.8$, $\mu_A(e) = 1$. Then $A = \{(a, .4), (b, .7), (c, 0), (d, .8), (e, 1)\}$ is a fuzzy subset of $X$.

A fuzzy subset in $X$ is empty iff its membership function is identically zero on $X$ and it is denoted by $\emptyset$ or $\mu_\emptyset$. The set $X$ can be considered as a fuzzy subset of $X$ whose membership function is identically 1 on $X$ and is usually denoted by 1 or $\mu_X$ or $I_X$.

In fact, every subset of $X$ is a fuzzy subset of $X$ but not conversely. Hence the concept of a fuzzy subset is a generalisation of the concept of a subset.

1.3 OPERATIONS ON FUZZY SUBSETS

In this section, the extension of the notions of inclusion, union, intersection and complementation of ordinary subsets to fuzzy subsets and some of their properties are given. The definitions and properties contained in this section are from L.A.Zadeh [61] and A.Kaufmann [24]. Throughout this thesis the phrases "fuzzy subset" and "fuzzy set" are interchangeably used.

1.3.1 Definition: If $A$ and $B$ are any two fuzzy subsets of a set $X$, then "$A$ is said to be included in $B$" or "$A$ is contained in $B$" or "$A$ is less than
or equal to B” iff \( A(x) \leq B(x) \) for all \( x \) in \( X \) and is denoted by \( A \leq B \).

Equivalently, \( A \leq B \) iff \( \mu_A(x) \leq \mu_B(x) \) for all \( x \) in \( X \).

Note that every fuzzy subset is included itself and empty fuzzy subset is included in every fuzzy subset.

**1.3.2 Definition** : Two fuzzy subsets \( A \) and \( B \) of a set \( X \) are said to be equal, written \( A = B \), if \( A(x) = B(x) \) for every \( x \) in \( X \). Equivalently, \( A = B \) iff \( \mu_A(x) = \mu_B(x) \) for every \( x \) in \( X \).

Note that \( A \neq B \) if there exists atleast one \( x \) in \( X \) for which \( \mu_A(x) \neq \mu_B(x) \).

**1.3.3 Definition** : The complement of a fuzzy subset \( A \) in a set \( X \), denoted by \( 1 - A \), is the fuzzy subset of \( X \) defined by \( 1 - A(x) \) for all \( x \) in \( X \). That is \( 1 - \mu_A(x) \) for all \( x \) in \( X \).

Note that \( 1 - (1 - A) = A \).

**1.3.4 Definition** : The union of two fuzzy subsets \( A \) and \( B \) in a set \( X \), denoted by \( A \vee B \), is a fuzzy subset in \( X \) defined by

\[
(A \vee B)(x) = \max\{A(x), B(x)\}, \text{ for all } x \in X.
\]

Equivalently, \( \mu_{A \vee B}(x) = \max\{\mu_A(x), \mu_B(x)\} \) for all \( x \) in \( X \).

It can be shown that \( (A \vee B) \vee C = A \vee (B \vee C) \) for any three fuzzy subsets \( A, B, C \) of \( X \).

In general, the union of a family of fuzzy subsets \( \{A_\lambda : \lambda \in \Lambda\} \) is a fuzzy subset denoted by \( \bigvee_{\lambda \in \Lambda} A_\lambda \) and defined by

\[
(\bigvee_{\lambda \in \Lambda} A_\lambda)(x) = \sup\{A_\lambda(x) : \lambda \in \Lambda\}, \text{ for all } x \in X.
\]

**1.3.5 Definition** : The intersection of two fuzzy subsets \( A \) and \( B \) in a set \( X \), denoted by \( A \wedge B \), is a fuzzy subset in \( X \) defined by

\[
(A \wedge B)(x) = \min\{A(x), B(x)\}, \text{ for all } x \in X.
\]

Equivalently, \( \mu_{A \wedge B}(x) = \min\{\mu_A(x), \mu_B(x)\} \) for all \( x \) in \( X \).
It can be shown that \((A \wedge B) \wedge C = A \wedge (B \wedge C)\) for any three fuzzy subsets \(A, B, C\), of \(X\).

In general, the intersection of a family of fuzzy subsets \(\{A_\lambda : \lambda \in \Lambda\}\) is a fuzzy subset denoted by \(\bigwedge_{\lambda \in \Lambda} A_\lambda\) and defined by

\[
(\bigwedge_{\lambda \in \Lambda} A_\lambda)(x) = \inf \{A_\lambda(x) : \lambda \in \Lambda\}, \text{ for all } x \text{ in } X.
\]

**1.3.6 Theorem [24, 25 and 61]**: Let \(X\) be any set and \(A, B, C\) be fuzzy subsets of \(X\). The following results hold good.

1. \(A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)\)
2. \(A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)\)
3. \(A \wedge 0 = 0\), where 0 is the empty fuzzy set.
4. \(A \vee 0 = A\), where 0 is the empty fuzzy set.
5. \(A \wedge X = A\)
6. \(A \vee X = X\)
7. \(1 - (A \vee B) = (1 - A) \wedge (1 - B)\)
8. \(1 - (A \wedge B) = (1 - A) \vee (1 - B)\)
9. \(A - B = A \wedge (1 - B)\)

Thus the above properties are clear extensions of the basic set theoretic properties to fuzzy subsets.

Note that, the following properties which are true in the case of set theory are no longer true in case of fuzzy set theory.

1. \(A \wedge (1 - A) = 0\) except for \(A = 0\) or \(A = X\)
2. \(A \vee (1 - A) = X\) except for \(A = 0\) or \(A = X\)

**1.4 FUZZY SUBSETS INDUCED BY MAPPINGS**

In this section we mention the definitions of image and inverse image defined by L.A. Zadeh [61] and related properties proved by C.L. Chang [11] and R.H. Warren [55].
1.4.1 Definition[61]: Let $f: X \rightarrow Y$ be a function from a set $X$ into a set $Y$. Let $A$ be a fuzzy set in $X$ and $B$ be a fuzzy set in $Y$.

1) The inverse image of $B$ under $f$, written $f^{-1}(B)$, is a fuzzy set in $X$, defined by $[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x)$, for each $x$ in $X$.

2) The image of $A$ under $f$, written $f(A)$, is a fuzzy set in $Y$, defined by $[f(A)](y) = \text{Sup}\{A(z): z \in f^{-1}(y)\}$, for each $y \in Y$ where $f^{-1}(y) = \{x \in X: f(x) = y\}$.

1.4.2 Theorem[11]: Let $f$ be a function from a set $X$ into a set $Y$.

The following results hold good.

1) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set $B$ in $Y$.

2) $f(1 - A) \geq 1 - f(A)$, for any fuzzy set $A$ in $X$.

3) $A \leq B$ implies $f(A) \leq f(B)$, for any two fuzzy sets $A$, $B$ in $X$.

4) $C \leq D$ implies $f^{-1}(C) \leq f^{-1}(D)$, for any two fuzzy sets $C$, $D$ in $Y$.

5) $A \leq f^{-1}[f(A)]$, for any fuzzy set $A$ in $X$.

6) $B \geq f[f^{-1}(B)]$, for any fuzzy set $B$ in $Y$.

7) Let $g$ be a function from $Y$ to $Z$. Then $(g \circ f)^{-1}(C) = f^{-1}[g^{-1}(C)]$, for any fuzzy set $C$ in $Z$.

In addition to above properties R.H. Warren [55] proved the following.

1.4.3 Theorem[55]: Let $f$ be a function from a set $X$ into a set $Y$. If $A_i, i \in I$ are fuzzy sets in $X$ and $B_j, j \in J$ are fuzzy sets in $Y$, where $I$ and $J$ denote the indexed sets, then the following results are true.

1) $f[f^{-1}(B)] = B$, if $f$ is onto.

2) $f\left( \bigwedge_{i \in I} A_i \right) \leq \bigwedge_{i \in I} f(A_i)$

3) $f^{-1}\left( \bigwedge_{j \in J} B_j \right) = \bigwedge_{j \in J} f^{-1}(B_j)$

4) $f\left( \bigvee_{i \in I} A_i \right) = \bigvee_{i \in I} f(A_i)$
5) \( f^{-1} \left( \bigvee_{j \in J} B_j \right) = \bigvee_{j \in J} f^{-1}(B_j) \)

6) \( f \left[ f^{-1} (B) \wedge A \right] = B \wedge f(A) \).

In this thesis, the word "crisp" has been used frequently. Crispness is the qualitative opposite of fuzziness, although technically, it is a special case.

**1.4.4 Example**: Let \( X = \{x, y, z\} \) be a set and \( A \) be a fuzzy subset of \( X \) defined by \( \mu_A(x) = 1, \mu_A(y) = 0, \mu_A(z) = 1 \), then \( A = \{(x, 1), (y, 0), (z, 1)\} = \{x, z\} \) is a crisp subset of \( X \).

### 1.5 FUZZY TOPOLOGICAL SPACES

C.L. Chang [11] in the year 1968, introduced the notion of fuzzy topological spaces as an application of fuzzy sets to general topological spaces. Since then several researchers have contributed to the development of fuzzy topological spaces. In this section, some basic concepts on fuzzy topological spaces, which may be used in the sequel, are included.

**1.5.1 Definition [11]**: Let \( X \) be a set and \( T \) be a family of fuzzy subsets of \( X \). The family \( T \) is called a fuzzy topology on \( X \) iff \( T \) satisfies the following axioms:

1) \( \mu_\emptyset, \mu_x \in T \), that is 0, 1 \( \in T \).

2) If \( \{A_\lambda : \lambda \in \Lambda\} \subset T \) then \( \bigvee_{\lambda \in \Lambda} A_\lambda \in T \).

3) If \( G, H \in T \) then \( G \wedge H \in T \).

The pair \((X, T)\) is called a fuzzy topological space (abbreviated as fts). The members of \( T \) are called open fuzzy sets in \( X \). A fuzzy set \( A \) in \( X \) is said to be closed in \( X \) iff \( 1 - A \) is an open fuzzy set in \( X \).

**1.5.2 Remark**: Every topological space is a fuzzy topological space but not conversely.
For example, let \( X = \{a,b,c\} \) be a set and \( A = \{(a,.5), (b,.7), (c,1)\} \) be a fuzzy subset of \( X \). Let \( T = \{0,1,A\} \), then \((X, T)\) is a fts which is not a topological space.

The concept of closure of a fuzzy set was introduced in [41].

**1.5.3 Definition [41]**: Let \( X \) be a fts and \( A \) be a fuzzy set in \( X \). Then \( \bigwedge \{B : B \text{ is closed fuzzy set in } X \text{ and } B \geq A\} \) is called the closure of \( A \) and is denoted by \( \overline{A} \) or \( \text{cl}(A) \).

The properties of closure of a fuzzy set are mentioned in the following.

**1.5.4 Theorem [55]**: Let \( A \) and \( B \) be two fuzzy sets in a fts \((X, T)\). Then the following results are true.
1) \( \overline{A} \) is a closed fuzzy set in \( X \)
2) \( \overline{A} \) is the least closed fuzzy set in \( X \) which is greater than or equal to \( A \).
3) \( A \) is closed iff \( A = \overline{A} \)
4) \( \overline{0} = 0 \), where \( 0 \) is the empty fuzzy set.
5) \( A = \overline{\overline{A}} \)
6) \( \overline{A} \lor B = \overline{A \lor B} \)
7) \( \overline{A} \land B \geq \overline{A} \land B \)
8) \( A \leq B \) implies \( \overline{A} \leq \overline{B} \)

The interior of a fuzzy set was defined by C.L.Chang [11] as follows.

**1.5.5 Definition [11]**: Let \( A \) and \( B \) be two fuzzy sets in a fuzzy topological space \((X, T)\) and let \( A \geq B \). Then \( B \) is called an interior fuzzy set of \( A \) if there exists \( H \in T \) such that \( A \geq H \geq B \). The least upper bound of all interior fuzzy sets of \( A \) is called the interior of \( A \) and is denoted by \( A^* \).
Some basic properties of interior which are extensions of the corresponding results in general topology, are given below.

**1.5.6 Theorem[11,55]** : Let \( X \) be a fts and \( A \) and \( B \) be two fuzzy sets in \( X \). The following results hold good.

1) \( A^* \) is an open fuzzy set in \( X \).
2) \( A^* \) is the largest open fuzzy set in \( X \) which is less than or equal to \( A \).
3) \( A \) is open iff \( A = A^* \)
4) \( A \leq B \) implies \( A^* \leq B^* \)
5) \((A^*)^* = A^* \)
6) \( A^* \land B^* = (A \land B)^* \)
7) \( A^* \lor B^* \leq (A \lor B)^* \)
8) \((1 - A)^* = 1 - \bar{A} \)
9) \( \bar{1 - A} = 1 - A^* \)

The concept of a boundary of a fuzzy set was introduced by R.H. Warren [57].

**1.5.7 Definition[57]**: Let \( A \) be a fuzzy set in a fts \((X,T)\). The fuzzy boundary of \( A \) is defined as the infimum of all closed fuzzy sets \( d \) in \( X \) with the property :

\[
d(x) \geq \bar{A}(x) \text{ for all } x \in X \text{ for which } (\bar{A} \land (\bar{1 - A}))(x) > 0.
\]

\( \text{Bd} (A) = A^b = \inf \{ d : d \text{ is closed fuzzy set and } d(x) \geq \bar{A}(x) \text{ for all } x \in X \text{ for which } (\bar{A} \land (\bar{1 - A}))(x) > 0 \} . \)

Note that \( \text{bd} (A) \) is a closed fuzzy set and \( \text{bd} (A) \leq \bar{A} \).

The properties of boundary of fuzzy set are mentioned in the following, which are proved by R.H. Warren [57].

**1.5.8 Theorem[57]** : Let \( A \) and \( B \) be fuzzy sets in a fts \((X,T)\). The following results hold good.

1) If \((\bar{A} \land (\bar{1 - A}))(x) > 0\), then \((\text{bd} (A))(x) = A^b(x) = \bar{A}(x) \)
2) If $\bar{A} \land (1-\bar{A}) = 0$, then $\text{bd}(A) = \land \{ \text{all closed fuzzy sets in } X \} = 0$
3) $\bar{A} = A^* \lor \text{bd}(A)$
4) $\text{bd}(A) \geq \bar{A} \land (1-A^*) > A^*$
5) $\bar{A} = A \lor \text{bd}(A)$
6) $\text{bd}(A^*) \lor \text{bd}(\bar{A}) \leq \text{bd}(A)$
7) $\text{bd}(A \lor B) \leq \text{bd}(A) \lor \text{bd}(B)$
8) $\text{bd}(A \land B) \leq \text{bd}(A) \lor \text{bd}(B)$
9) $A \lor B \lor \text{bd}(A \lor B) = A \lor B \lor \text{bd}(A) \lor \text{bd}(B)$
10) $\text{bd}(0) = 0$
11) $\text{bd}(\text{bd}(A)) \leq \text{bd}(A)$

The following are the additional properties of boundary of a fuzzy set proved by R.H. Warren [57].

1.5.9 Theorem[57] : Let $A$ and $B$ be fuzzy sets in a fts $(X,T)$. The following results hold good.
1) If $\bar{A} \land (1-A^*) = 0$, then $\text{bd}(A) = \text{bd}(1-A^*)$.
2) If $(\bar{A} \land (1-A^*)) (x) > 0$, then $(\text{bd}(A)) (x) \geq A(x)$.
3) If $\bar{A} \land (1-A^*) \neq 0$ and given $y \in X$ such that $1-A(y) = 0$ then
   $$(\text{bd}(A)) (y) = \land \{ \bar{d}(y) : \bar{d}(x) \geq \bar{A}(x) \text{ when } (\bar{A} \land 1-A^*) (x) > 0 \}.$$  
4) $A$ is closed iff $\text{bd}(A) \leq A$.
5) $\text{bd}(A) = 0$ iff $\bar{A} \land (1-A^*) = 0$.
6) $\text{bd}(A) \land A = 0$ iff $A$ is open and crisp.
7) $\text{bd}(A) = 0$ iff $A$ is open, closed and crisp.
8) $\bar{A} = (A^0)$ iff $\text{bd}(A) \leq (A^0)$.
9) If $A \leq B$ then, $\text{bd}(A) \leq B \lor \text{bd}(B)$.

1.5.10 Definition[20] : Let $(X,T)$ be a fts. A subfamily $B$ of $T$ is called a base for $T$ iff for each $A$ in $T$, there exists a subfamily $B_A$ of $B$ such that
A = \bigvee B_A. A subfamily S of T is called a subbase for T iff the family 
C = \{ \wedge \tau : \tau \text{ is a finite subfamily of } S \} \text{ is a base for } T.

The following characterisation of a base is due to R.H. Warren [55].

**1.5.11 Theorem**[55] : Let T be a fuzzy topology on a set X and \( \beta \) be a 
subfamily of T. Then the following two properties are equivalent.

1) \( \beta \) is a base for T

2) For each \( G \in T \), for each \( x \in X \), such that \( G(x) > 0 \) and for each real 
number \( \varepsilon > 0 \), there is a \( B \) in \( \beta \) such that \( \beta \preceq G \) and \( G(x) - B(x) < \varepsilon \).

The concept of relative fuzzy topology is due to R.H. Warren [55].

**1.5.12 Definition**[55] : Let \((X, T)\) be a fts and let \( A \) be a crisp subset of \( X \). 
Then the family \( T_A = \{ G/A : G \in T \} \) is a fuzzy topology on \( A \), where 
\( G/A \) is the restriction of \( G \) to \( A \). The fuzzy topology \( T_A \) is called the 
relative fuzzy topology on \( A \) or the fuzzy topology on \( A \) induced by the 
fuzzy topology \( T \) on \( X \). Also \((A, T_A)\) is called the subspace of \((X, T)\).

The concept of continuity of maps for fuzzy topological spaces was 

**1.5.13 Definition**[11] : Let \( f : X \rightarrow Y \) be a function from a fts \((X, T)\) into 
a fts \((Y, S)\). Then \( f \) is said to be fuzzy continuous (f-continuous) function 
iff for each \( B \in S \), \( f^{-1}(B) \in T \).

Some basic characterisations of f-continuous maps established 
by R.H. Warren are mentioned in the following.

**1.5.14 Theorem**[55] : Let \( f \) be a function from a fts \((X, T)\) into a fts 
\((Y, S)\). Then the following statements are equivalent.

1) \( f \) is a fuzzy - continuous (f-continuous) function

2) The inverse image of every closed fuzzy set in \( Y \) is a closed fuzzy set 
in \( X \).

3) The inverse image of every element of a subbase for \( S \) is in \( T \).

4) For every fuzzy set \( A \) in \( X \), \( f(A) \leq f(A) \)
5) For every fuzzy set $B$ in $Y$, $f^{-1}(B) \subseteq f^{-1}(B)$

6) If the set $G = \{(x, f(x) : x \in X\}$ has the fuzzy topology inherited as a subspace of $(X \times Y, T \times S)$, then the function $H : X \rightarrow G$ given by $H(x) = (x, f(x))$ is $f$-continuous.

The concepts of fuzzy open map and fuzzy closed map were defined by C.K. Wong [59].

**1.5.15 Definition [59]**: A function $f : X \rightarrow Y$ from a fts $X$ into a fts $Y$ is said to be $f$-open (or $f$-closed) iff for each open (resp. closed) fuzzy set $A$ in $X$, $f(A)$ is an open (resp. closed) fuzzy set in $Y$. 