CHAPTER 2

(1,1), (1,0) and (0,1) Edge-Magic / 
Vertex-Magic Graphs
2.1 A Brief Introductory Note

We investigate the existence of certain labeling for graphs which have some magic property. Aided by the use of certain number-theoretic properties, we try to classify the ordinary graphs that are finite, simple and undirected.

Kotzig and Rosa [78, 79] proved:

1. $K_{m,n}$ has a magic labeling (of type (1,1) edge-magic) for all $m$ and $n$.

2. $C_n$ has a (1,1) edge-magic labeling for all $n \geq 3$.

3. odd-disjoint union of $P_2$ has a (1,1) edge-magic labeling.

4. $K_n$ has a (1,1) edge-magic labeling iff $n = 1$ to 6.

Balakrishnan and Kumar [22] proved that $2K_2 \lor K_n^c$ is (1,1) edge-magic iff $n = 3$.

Ringel and Llado [116] proved that a $(p, q)$ graph is not (1,1) edge-magic if $q$ is even and $p + q \equiv 2 \pmod{4}$ and each vertex has odd degree. Lee, Pigg and Cox [85] conjectured that a cubic graph with $q$ edges is (1,1) edge-magic iff $q = 2 \pmod{4}$.

Let $G(p, q)$ be any connected graph. Suppose that for any bijection $f : V(G) \rightarrow \{1, \ldots, p\}$, the corresponding induced edge labeling $f_E : E(G) \rightarrow \mathbb{Z}^+$ forms the set of consecutive integers $\{r, r+1, \ldots, q+r-1\}$ for some $r(= f_E(e) = f(u)+f(v)) \in \mathbb{Z}^+$, where $e = (u, v) \in E(G)$, then $G(p, q)$ is (1,1) edge-magic. This can be easily seen once we define a bijection $f_1 : V(G) \cup E(G) \rightarrow \{1, \ldots, p+q\}$ by $f_1(V(G)) = f(V(G))$ and $f_1(E(G)) = \{p+q+r, p+q+r-1, \ldots, p+r+1\}$. It is easy to observe that (i) $K_1$ and $K_{3,3}$ are both (1,1) edge-magic and (1,1) vertex-magic, whereas $K_2$ is (1,1) edge-magic (ii) In any (1,1) edge-magic graph $G(p, q)$ if there exists an edge
$e = (u, v)$ with $f(u) = 1$, $f(v) = 2$, then $f(e) = p + q$.

2.2 Some Results on (1,1) Edge-Magic Graphs

**Definition 2.2.1** Two magic labeling $g, g_0$ of a graph $G(p,q)$ are equal, $g = g_0$, if there exists an automorphism $\alpha$ of $G$ such that $g(\alpha u) = g_0(u)$ for all elements $u \in G$. (By elements of $G$, we mean both the vertices and the edges).

**Definition 2.2.2** Given a magic labeling $g$ of a graph $G(p,q)$ the labeling $g_0$ such that $g_0(x) = p + q + 1 - g(x)$ for all elements $x \in G$ is said to be complementary to $g$.

**Definition 2.2.3** Two magic labeling $g_1$ and $g_0$ of $G$ are said to be equivalent if $g_1 = g_0$ or $g_1 = g_0^c$.

**Theorem 2.2.4** The $n-1$ stars, $K_{1,n-1}$ for $n \geq 2$ are (1,1) edge-magic.

*Proof.* Let $x$ be the central vertex of $K_{1,n-1}$ of degree $(n-1)$ and let $x_1, \ldots, x_{n-1}$ be its pendant vertices, then $xx_i$ for $1 \leq i \leq n-1$ are the $(n-1)$ edges of $K_{1,n}$. Define a bijection $g_1 : V(K_{1,n}) \cup E(K_{1,n}) \rightarrow \{1, \ldots, 2n-1\}$ as follows $g_1(x) = 1; g_1(x_i) = i + 1$ for $1 \leq i \leq n-1; g_1(xx_i) = 2n - i$ for $1 \leq i \leq n-1$. Then one can check that $g_1$ is a (1,1) edge-magic labeling with common edge count $k_0 = 2n + 2$. \hfill $\Box$

**Theorem 2.2.5** Any odd disjoint union of $P_3$ is (1,1) edge-magic where $P_3$ is the path on 3 vertices.

*Proof.* Let $G = (2t + 1)P_3$. Denote the two degree vertices in $G$ by $u_1, \ldots, u_{2t+1}$.
Denote the vertices of degree one adjacent to the left of $u_i$ by $v_{2t+2-i}$ and those adjacent to the right of $u_i$ by $w_{2t+2-i}$ for $1 \leq i \leq 2t + 1$. Denote the edge joining $u_i$ and $v_{2t+2-i}$ by $e_i$ and the edge joining $u_i$ and $w_{2t+2-i}$ by $e^*_i$ for $1 \leq i \leq 2t + 1$.

Define a bijection $g_2 : V(G) \cup E(G) \to \{1, \ldots, 10t + 5\}$ as follows:

$$g_2(u_i) = \begin{cases} 2t + 2, & \text{if } i = 1 \\ 2t + 3, & \text{if } i = 2 \\ 3t + 3, & \text{if } i = 3, \text{ etc} \\ 3t + 4, & \text{if } i = 4, \text{ etc}; \end{cases}$$

$$g_2(v_i) = \begin{cases} \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 3t + 2, & \text{if } i = 2t + 1 \\ 4t + 2, & \text{if } i = 2t \end{cases}$$

$$g_2(w_i) = \begin{cases} 4t + 3, & \text{if } i = 1 \\ 4t + 4, & \text{if } i = 3, \text{ etc} \\ 5t + 4, & \text{if } i = 4, \text{ etc}; \end{cases}$$

$$g_2(w_i) = \begin{cases} 5t + 3, & \text{if } i = 2t + 1 \\ 6t + 3, & \text{if } i = 2t \end{cases}$$
Then one can check that $g_2$ is a (1,1) edge-magic labeling with common edge count $k_0 = 13t + 8$.

**Conjecture 2.2.6** The disjoint union of $2t$ copies of $P_3, t \geq 1$, has a (1,1) edge-magic labeling.

The above conjecture was raised by Yegnanarayanan in [147]. The conjecture has been settled positively by Manickam and Marudai in [99].

**Problem 2.2.7** Determine the values of $m$ and $n$ such that $mP_n$ is (1,1) edge-magic.

**Theorem 2.2.8** The graph $G = P_n \lor K_1$ is a (1,1) edge-magic, where $\lor$ indicates join.
Proof. Let \( x \) denote the graph \( K_1 \) and \( P_n = v_1v_2\ldots v_n \). Define a bijection \( g_3 : V(G) \cup E(G) \to \{1, \ldots, 3n\} \) as follows: \( g_3(x) = 1; \ g_3(v_1) = 2; \ g_3(v_2) = 3; \ g_3(v_3) = 5; \ g_3(v_{i+1}) = g_3(v_i) + g_3(v_{i-1}) - g_3(v_{i-2}) \) for \( 3 \leq i \leq n - 1; \ g_3(v_{n+1-i}v_{n-i}) = 3i + 1 \) for \( 1 \leq i \leq n - 1; \ g_3(xv_i) = 3n + 2 - g_3(v_i) \) for \( 1 \leq i \leq n \). Then one can check that \( g_3 \) is a \((1,1)\) edge-magic labeling with common edge count \( k_0 = 3n + 3 \).

**Definition 2.2.9** Consider the graph \( G_i = P_i \times K_3 \), where \( \times \) stands for the cartesian product of a path on \( i \) vertices \( (i \geq 2) \) with a complete graph on three vertices. Obtain a new graph \( G_i \odot K_{1,n} \) called the generalized \( n \)-crown, by introducing \( n \) new pendant edges at each vertex of the outermost \( K_3 \) in \( G_i \).

**Theorem 2.2.10** For \( i \geq 2 \), the graph \( G_i = P_i \times K_3 \) is \((1,1)\) edge-magic.

Proof. The following algorithm indicates a labeling pattern:

**Step 1.** Orient the edges of \( G_i \) in the anti-clockwise direction starting away from any fixed vertex of the inner most \( K_3 \) and continue the orientation to the next \( K_3 \) surrounding it via an edge from the final vertex of the previous orientation.

**Step 2.** If there are \( tK'_3 \)'s at any \( G_i \), then assign the label 1 to that vertex from which the orientation has started in the inner most \( K_3 \) as indicated in Step 1 and assign the labels 2, 3, \ldots, 3t to those respective vertices by strictly following the orientation.

**Step 3.** Assign the label \( 9t - 3 \) to the edge joining the vertices with labels 1 and 2, so that its edge count is \( 9t \). Assign appropriate edge labels to the rest of the edges in the graph so that \( 9t \) is the common edge count.

\( \square \)
Theorem 2.2.11  For $i \geq 2$, the graph $G_i \odot K_{1,n}$ is (1,1) edge-magic.

Proof. The following algorithm indicates a labeling pattern.

Step 1. Orient the edges of $G_i$ in $G_i \odot K_{1,n}$ exactly in Step 1 of the Theorem 2.2.10.

Step 2. If there are $tK_3$'s at any $G_i$ in $G_i \odot K_{1,n}$ then label the vertices of $G_i$ exactly as in the Step 2 of Theorem 2.2.10.

Step 3. We are now at the vertex of the outer-most $K_3$ in $G_i \odot K_{1,n}$ bearing the label $3t$. Assign to the $n$—new pendant vertices adjacent to it the labels $3t+1, 3t+4, \ldots, 3t+3n-2$ respectively in order and go to the vertex in the outer-most $K_3$ bearing the label $3t-2$. Assign to the $n$—new pendant vertices adjacent to it the labels $3t+2, 3t+5, \ldots, 3t+3n-1$ respectively in order and go to the vertex bearing the label $3t-1$ in the same outer-most $K_3$. Assign to the $n$—new pendant vertices adjacent to it the labels $3t+3, 3t+6, \ldots, 3t+3n$ respectively in order.

Step 4. Assign the label $6n + 9t - 3$ to the edge joining the vertices with labels 1
and 2, so that its edge count is $6n + 9t$.

**Step 5.** Assign appropriate edge labels to the rest of the edges in $G_i \odot K_{1,n}$ (including the pendant edges) so that $6n + 9t$ is the common edge count.

**Figure 2.2.** Illustrates the algorithm for $G_2 \odot K_{1,n}$ of Theorem 2.2.11

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**Theorem 2.2.12** The 1-crown $C_n \odot K_{1,1}$ is (1,1) edge-magic, where $C_n = v_1 v_2 \ldots v_n$ is the cycle of length $n$.

**Proof.** We split the proof into three cases. We give an explicit algorithm for case (i) and leave it to the readers to construct similar algorithm for the other two cases. However we do give the labels of the vertices of the cycle in all the cases. Let $v_i^1$ denote the pendant vertices and $v_i v_i^1$ the pendant edges of $C_n \odot K_{1,1}$. 


Case (i): $n = 2k + 1$

**Step 1.** Label the vertices of $C_n$ as follows: $g_4(v_i) = i$ if $i \equiv 1 \pmod{2}$ and $g_4(v_i) = i + n$ if $i \equiv 0 \pmod{2}$.

**Step 2.** First observe that $g_4(v_i) + g_4(v_{i+1})$ and $g_4(v_{2k+1}) + g_4(v_1)$ for $1 \leq i \leq 2k$ forms the set \{2k+4, 2k+6, 2k+8, ..., 6k+2, 2k+2\}. Assign appropriate labels to $v^1_i$ in such a way that $g_4(v_i) + g_4(v^1_i)$ results in the set \{2k+3, 2k+5, 2k+7, ..., 6k+3\}.

**Step 3.** Assign appropriate labels to the rest of the edges including the pendant edges so that the common edge count is $10k + 6$.

Case (ii): $n = 4k$

$$g_5(v_i) = \begin{cases} 
 i, & \text{for } i = 1, 3, 5, ..., 2k - 1; \\
 i + 1, & \text{for } i = 2k, 2k + 2, 2k + 4, ..., 4k - 2; \\
 n + i, & \text{for } i = 2k + 1, 2k + 3, 2k + 5, ..., 4k - 3, k \geq 2; \\
 n + i + 1, & \text{for } i = 2, 4, 6, ..., 2k - 2, k \geq 2; \\
 2, & \text{for } i = 4k - 1; \\
 2n - 2, & \text{for } i = 4k.
\end{cases}$$

Case (iii): $n = 4k + 2$
Theorem 2.2.13  The $n$-crown $C_{2t+1} \odot K_{1,n}$ is (1,1) edge-magic for all $n$.

Proof. Let $V(C_{2t+1} \odot K_{1,n}) = \{v_1, v_2, \ldots, v_{2t+1}; v_{11}, v_{12}, \ldots, v_{1,n}; v_{2,1}, v_{2,2}, \ldots v_{2,n}; \ldots; v_{2t+1,1}, v_{2t+1,2}, \ldots, v_{2t+1,n}\}$. Here the cycle vertices are denoted by single subscripts of $v$ and $n$ pendent vertices corresponding to each of the cycle vertex are denoted by the double subscripts of $v$. Then clearly $E(C_{2t+1} \odot K_{1,n}) = \{v_i v_{i+1} \text{ for } 1 \leq i \leq 2t; v_j v_{jk} \text{ for } 1 \leq 2t+1, 1 \leq k \leq n\}$. Now define a function $f : V(C_{2t+1} \odot K_{1,n}) \to \{1, 2, \ldots, (4t+2)(n+1)\}$ as follows: $f(v_i) = i$ if $i \equiv 1 \pmod{2}$ and $f(v_i) = i + 2t + 1$ if $i \equiv 0 \pmod{2}$ so that $f(v_i) + f(v_{i+1})$ and $f(v_{2i+1}) + f(v_i)$ for $1 \leq i \leq 2t$ forms the set $\{2t+4, 2t+6, \ldots, 6t+2, 2t+2\}$. Assign appropriate labels to $v_{jk}, 1 \leq j \leq 2t+1, 1 \leq k \leq n$ in such a way that $f(v_j) + f(v_{j+1})$ for all $1 \leq j \leq 2t+1$ results in the set $\{2t+3, 2t+5, \ldots, 6t+3\}$ and $f(v_j) + f(v_{jk}) = \{(2t+1)(k+1) + 1, (2t+1)(k+1) + t + 2, (2t+1)(k+1) + 2, (2t+1)(k+1) + t + 3, \ldots, (2t+1)(k+1) + t, (2t+1)(k+2), (2t+1)(k+1) + t + 1\}$ for all $2 \leq k \leq n$. Assign appropriate labels to the rest of the edges including the pendant edges so

$$g_6(v_i) = \begin{cases} 
\frac{1}{2}(i+1), & \text{for } i = 1, 3, 5, \ldots, 2k + 1; \\
\frac{1}{2}(i + 3), & \text{for } i = 2k + 3, 2k + 5, 2k + 7, \ldots, 4k + 1; \\
6k + 3, & \text{for } i = 2; \\
k + 2, & \text{for } i = 2k + 2; \\
2k + 3, & \text{for } i = 4k + 2; \\
\frac{i}{2} + 2k + 2, & \text{for } i = 4, 6, \ldots, 2k, k \geq 2; \\
\frac{i}{2} + 2k + 1, & \text{for } i = 2k + 4, 2k + 6, \ldots, 4k, k \geq 2.
\end{cases}$$

$\square$
that the common edge count is \((2t + 1)(2n + 3) + 1\).

\[\textbf{Conjecture 2.2.14} \text{ The generalized } n\text{-crown } C_{2t} \odot K_{1,n} \text{ is (1,1) edge-magic for all } n \text{ and } t.\]

\[\textbf{Remark 2.2.15} \text{ Ringel [115] in his paper exhibited a (1,1) edge-magic labeling for } P_6 \text{ and } P_7, \text{ the path on Six and Seven vertices respectively. In fact, his labeling generalizes to } P_n \text{ for any } n.\]

\[\textbf{Theorem 2.2.16} \text{ Paths } P_n \text{ are (1,1) edge-magic.}\]

**Proof.** We divide the proof into two cases.

**Case:** 1 When \(n = 2m\)

Let \(V(P_{2m}) = \{u_1, \ldots, u_{2m-1}, u_{2m}\}\) and \(E(P_{2m}) = \{e_1 = (u_1, u_2), e_2 = (u_2, u_3), \ldots, e_{2m-2} = (u_{2m-2}, u_{2m-1}), e_{2m-1} = (u_{2m-1}, u_{2m})\}\). Define a bijection \(f : V(P_{2m}) \cup E(P_{2m}) \rightarrow \{1, 2, \ldots, 4m - 2, 4m - 1\}\) as follows: \(f(u_{2i-1}) = 2i - 1\) for \(1 \leq i \leq m\); \(f(u_{2i}) = 2m + (2i - 1)\) for \(1 \leq i \leq m\); \(f(e_{2m-i}) = 4m - 2j\) for \(1 \leq i \leq m\); \(1 \leq j \leq 2m - 1\). Then one can check that \(f\) is (1,1) edge-magic labeling of \(P_{2m}\) with common edge count 6m. \(f(u_2) = 2m + 1, f(u_4) = 2m + 3, f(u_6) = 2m + 5, \ldots, f(u_{2m-2}) = 4m - 3, f(u_{2m}) = 4m - 1; f(e_1) = 4m - 2, f(e_2) = 4m - 4, f(e_3) = 4m - 6, \ldots, f(e_{2m-2}) = 4, f(e_{2m-1}) = 2.\)

**Case:** 2 when \(n = 2m - 1\)

Let \(V(P_{2m-1}) = \{u_1, \ldots, u_{2m-2}, u_{2m-1}\}\) and \(E(P_{2m-1}) = \{e_1 = (u_1, u_2), e_2 = (u_2, u_3), \ldots, e_{2m-3} = (u_{2m-3}, u_{2m-2}), e_{2m-2} = (u_{2m-2}, u_{2m-1})\}\). Define a bijection \(f : V(P_{2m-1}) \cup E(P_{2m-1}) \rightarrow \{1, 2\ldots, 4m - 4, 4m - 3\}\) as follows:
\( f(u_1) = 2m - 1, f(u_3) = 2m + 1, f(u_5)2m + 3, \ldots, f(u_{2m-3}) = 4m - 5, f(u_{2m-1}) = 4m - 3; f(u_2) = 1, f(u_4) = 3, f(u_6) = 5), \ldots, f(u_{2m-4}) = 2m - 5, f(u_{2m-2}) = 2m - 3; f(u_1) = 4m - 4; f(e_2) = 4m - 6, f(e_3) = 4m - 8, \ldots, f(e_{2m-3}) = 4, f(e_{2m-2}) = 2; \)

then one can check that \( f \) is a \((1,1)\) edge-magic labeling of \( P_{2m-1} \) with common edge count \( 6m \).

\( \square \)

**Note 2.2.17** One can also use computers to search a \((1,1)\) edge-magic labeling of \( P_n \). We give below an algorithm which also computes a \((1,1)\) edge-magic labeling.

**Algorithm:**

1. Begin

2. Include all necessary headers.

3. Initialize a count \( n = 1 \) for number of rows.

4. Assign count \( c \) for odd number of rows as 6 and that for even number of rows as

   \[ c_1 = c + 2. \]

5. Get the number of rows in a variable \( n \).

6. Calculate edges=number of rows and vertices=edges+1.

7. Calculate edges+vertices and store it in a variable say \( b \).

8. Start from the count \( k = 3 \) till the number of rows/edges, increment \( k \) by 2 and

    perform the following:

    8.1 initialize array count to 1\((i = 1)\) and assign first position of the array to \( k \) value.

    8.2 start from count \( j = 2 \) till \( k \) value, increment \( j \) value by 2 and perform the

        following.
8.2.1 assign next position of array to $j$

8.2.2 assign next position of array to count ($c$ or $c_1$) - (array of previous edge $(i-2)+j$)

8.3 increment row count $n$ by 1.

8.4 check if $n$ is odd or even,

if odd go to step 8.5,

otherwise go to step 8.6.

8.5 increment $c$ by 6 and assign $c$ to count.

8.6 increment $c_1$ by $c + 2$ and assign $c_1$ to count.

8.7 display all rows and their contents using array.

9. Stop.

**Theorem 2.2.18** If $n \geq 4$, there is no labeling which is both $(1,1)$ edge-magic and $(1,1)$ vertex-magic on the complete graph $K_n$.

*Proof.* Denote the vertices of $K_n$ by $v_i$ for $1 \leq i \leq n$. Suppose that a bijection $g_{14}: V(K_n) \cup E(K_n) \to \{1, \ldots, n + n(n - 1)/2\}$ is both $(1,1)$ edge-magic and $(1,1)$ vertex magic. Then we have, $(n - 1)f(v_1) + \sum_{i=2}^{n} f(v_i) + \sum_{j=2}^{n} f(v_1v_j) = (n - 1)k_0$ and $f(v_1) + \sum_{j=2}^{n} f(v_1v_j) = k_1$. That is, $(n - 2)f(v_1) + \sum_{i=2}^{n} f(v_i) = (n - 1)k_0 - k_1$. Repeating the argument for the vertex $v_2$, we get for $i \neq 2(n - 2)f(v_2) + \sum_{i=1}^{n} f(v_i) = (n - 1)k_0 - k_1$. So $(n - 3)[f(v_1) - f(v_2)] = 0$. That is, $f(v_1) = f(v_2)$, a contradiction. 

**Observation 2.2.19** $K_1$ and $K_3$ are both $(1,1)$ edge-magic and $(1,1)$ vertex-magic, whereas $K_2$ is $(1,1)$ edge-magic and $(1,1)$ vertex-antimagic.
Observation 2.2.20 If a graph $G(p, q)$ is $(1,1)$ edge-magic, then $\sum_{v \in V(G)} \deg_G(v)f(v) + \sum_{e \in E(G)} f(e) = qk_0$. where $k_0$ is the common edge count. To see this, let $f$ be a $(1,1)$ edge-magic labeling of $G(p, q)$. Since $f$ is a bijection, each vertex label $f(v), v \in V(G)$ occurs exactly $\deg(v)$ times, once each in the calculation of the edge count of $\deg(v)$ distinct edges in $G$ and each edge label $f(e), e \in E(G)$ occurs exactly once and that too in the calculation of its own count.

Observation 2.2.21 Kotzig and Rosa [78] proved that the disjoint union of $n$ copies of $P_2$ has a $(1,1)$ edge-magic labeling iff $n$ is odd. We consider the problem of determining the existence or non-existence of a $(1,1)$ edge-magic labeling of the graph $G = 2tP_3$. Let $P^t_3 = (v^t_1e^t_1v^t_2e^t_2v^t_3)$. Let $A = \{v_1, ..., v_{2t}\}$ denote the set of all two degree vertices in $G$. Suppose that there exists a $(1,1)$ edge-magic labeling $f : V(G) \cup E(G) \rightarrow \{1, ..., 10t\}$. Then it follows that the common edge count $c(e)$ is such that $c(e) \in B$, where $B = \{k_0 : 4tk_0 = \sum_{v \in A} f(v) + 5t(10t + 1)\}$. Set $N = \sum_{v \in A} f(v)$. Since $N$ can be at most $18t^2 + t$, it follows that $B$ is finite.

We now give an algorithm for finding one such $f$ in case if it exists.

**Step 1.** Determine the all possible values of $c(e)$ such that $N + 5t(10t + 1) \equiv 0 \pmod{4t}$.

**Step 2.** If $c(e) = kr$ for some $r$, then collect the distinct $2t$-partitions of $N$ such that $kr$ is the quotient of the congruence in Step 1.

**Step 3.** Pick a partition of $N$ from Step 2 and set each of its $2t$ individual entries as the labels of the two degree vertices in $A$.

**Step 4.** Label the rest of the $8t$ elements (vertices and edges) of $G$ such that $kr$ is
the common edge count.

**Step 5.** If Step 4 is implemented successfully then stop. Otherwise pick another 2t
partition from Step 3 and go to Step 4, else go to Step 6.

**Step 6.** If Step 3 is exhausted then stop and conclude that $G$ has no such labeling.

We have tested the above algorithm successfully for $t = 1, \ldots, 4$. Let us discuss the
case $t = 1$ a little and leave the rest for the readers. For $t = 1$, $B = \{15, 16, 17, 18\}$.
If $c(e) = 15, 16, 17$ or 18, then $N$ is respectively 5,9,13 and 17. The all possible
2-partitions of 5,9,13 and 17 are respectively:

5:(1,4),(2,3);9:(1,8),(2,7),(3,6),(4,5);13:(3,10),(4,9),(5,8),(6,7);17:(7,10),(8,9).

Out of these partitions only (2,7) and (4,9) of 9 and 13 respectively produces a (1,1)
edge-magic labeling for $G = 2P_3$. That is,

$P_3^1 : (10, 4, 2, 5, 9); P_3^2 : (6, 3, 7, 8, 1); c(e) = 16$ or

$P_3^1 : (8, 5, 4, 3, 10); P_3^2 : (2, 6, 9, 1, 7); c(e) = 17$.

For $t = 2$, $B = \{14, 22, 30, 38\}$. A 4-partition $(1,2,3,8)$ of 14 produces a (1,1) edge-
magic labeling for $G = 4P_3$. That is,

$P_3^1 : (17, 10, 1, 9, 18); P_3^2 : (15, 11, 2, 14, 12);$

$P_3^3 : (20, 5, 3, 6, 19); P_3^4 : (16, 4, 8, 7, 13); c(e) = 28$.

For $t = 3$, $B = \{15, 27, 39, 51\}$. A 6-partition $(1,2,3,4,5,12)$ of 27 produces a (1,1) edge-
magic labeling for $G = 6P_3$. That is,

$P_3^1 : (24, 16, 1, 14, 26); P_3^2 : (20, 19, 2, 17, 22);$

$P_3^3 : (23, 15, 3, 13, 25); P_3^4 : (27, 10, 4, 9, 28);$

$P_3^5 : (30, 6, 5, 7, 29); P_3^6 : (18, 11, 12, 8, 21); c(e) = 41$. 

41
For $t = 4$, $B = \{44, 60, 76\}$. A 8-partition $(1, 2, 3, 4, 5, 6, 10, 13)$ of $44$ produces a $(1,1)$ edge-magic labeling for $G = 8P_3$. That is,

$P_3^1 : (15, 38, 1, 18, 35); P_3^2 : (28, 24, 2, 12, 40);$

$P_3^3 : (22, 29, 3, 25, 26); P_3^4 : (27, 23, 4, 19, 31);$

$P_3^5 : (32, 17, 5, 16, 33); P_3^6 : (34, 14, 6, 9, 39);$

$P_3^7 : (36, 8, 10, 7, 37); P_3^8 : (21, 20, 13, 11, 30); c(e) = 54.$

**Remark 2.2.22** The above mentioned algorithm can be generalized to any number of copies of $P_n$.

**Theorem 2.2.23** Every graph on $p \leq 8$ vertices can be embedded as a subgraph of some $(1,1)$ edge-magic graph.

**Proof.** As $K_p$ is $(1,1)$ edge-magic for $p \leq 6$, nothing to prove.

So consider $K_7$ and let $V(K_7) = \{v_i : 1 \leq i \leq 7\}$. Form a new graph $G = K_7(v_5, v_6, v_7) \odot (K_{1,2}, K_{1,m}, K_{1,n})$, where $G$ is the graph obtained from $K_7$ by identifying the central vertex of $K_{1,2}, K_{1,m}, K_{1,n}$ with $v_5, v_6$ and $v_7$ respectively.

Denote the pendant vertices of $G$ by $v_5^i, v_6^i, v_7^i$ and $v_i^j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Suppose that $f : V(G) \to \{1, \ldots, m + n + 9\}$ is a bijection defined as follows:

$f(v_1) = 1; f(v_2) = 2; f(v_i) = f(v_{i-1}) + f(v_{i-2}); f(v_5^1) = 4; f(v_5^2) = 9; f(v_6^1) = 6; f(v_6^2) = 7; f(v_6^3) = 12; f(v_6^4) = 23; f(v_6^m) = 8t + B, B = 9, 12, \ldots, 14$ and 15 if $m = 5t + D, D = 0, 1, \ldots, 3$ and 4 and $t \geq 2$ respectively in order; $f(v_7^1) = 8t + B_0, B_0 = 2, 3$ and 8 if $n = 3t + D_0, D_0 = -2, -1$ and 0 and $t \geq 1$ respectively in order.

Then the induced labeling $f_E : E(G) \to \mathbb{Z}^+$ defined for some $r = f_E(e) = f(u) + f(v)$
where \( e = (u, v) \in E(G) \), forms the set \( A \) of consecutive and distinct integers:

\[
A = \bigcup_{i=1}^{4} A_i \quad \text{where} \quad A_1 = \{3, \ldots, 11, 13, \ldots, 16, 18, 21, \ldots, 24, 26, 29, 34\} \quad \text{from the label of the vertices of } K_7; \quad A_2 = \{12, 17\} \quad \text{from the label of pendant vertices adjacent to } v_5; \quad A_3 = \{19, 20, 25, 27, 28, 30, 33, 35, 36, 8t+22, 8t+25, 8t+26, 8t+27, 8t+28\} \quad \text{from the label of pendant vertices adjacent to the vertex } v_6; \quad A_4 = \{8t+23, 8t+24, 8t+29\} \quad \text{from the label of the pendant vertices adjacent to the vertex } v_7. \quad \text{Now we define a bijection } f_1 : V(G) \cup E(G) \to \{1, \ldots, 2(m + n) + 32\} \quad \text{by } f_1(V(G)) = f(V(G)) \quad \text{and} \quad f_1(E(G)) = \{2(m + n) + 32, 2(m + n) + 31, \ldots, m + n + 10\}. \quad \text{Then one can check that } f_1 \quad \text{is a required labeling with common edge count } 2(m + n) + 35. \quad \text{Finally as } K_7 \subseteq G, \quad \text{the result follows.}

Now consider \( K_8 \) and let \( V(K_8) = \{v_i : 1 \leq i \leq 8\} \). Form a new graph \( G \) from \( K_8 \) by (i) pasting a star of size \( K_{1,1} \) at each of \( \{v_i : 1 \leq i \leq 3\} \); (ii) pasting a star of size \( K_{1,3} \) at each of \( \{v_i : 4 \leq i \leq 5\} \); (iii) pasting a star of size \( K_{1,17} \) at the vertex \( v_6 \); (iv) pasting a star of size \( K_{1,m+30} \) at the vertex \( v_7 \); (v) pasting a star of size \( K_{1,n+18} \) at the vertex \( v_8 \). Denote the pendant vertices of \( G \) by \( \{v^1_i : 1 \leq i \leq 8\}; \{v^2_i : 4 \leq i \leq 8\}; \{v^3_i : 4 \leq i \leq 8\}; \{v^4_j : 4 \leq j \leq 17\}; \{v^r : 4 \leq r \leq m + 30\}; \{v^s : 4 \leq s \leq n + 18\}. \quad \text{Suppose that } f : V(G) \to \{1, \ldots, m + n + 82\} \quad \text{is a bijection defined as follows: } f(v_1) = 1; f(v_2) = 2; f(v_3) = f(v_{i-1}) + f(v_{i-2}); f(v_1) = 11; f(v_2) = 15; f(v_3) = 16; f(v_4) = 27; f(v_5) = 28; f(v_6) = 35; f(v_7) = 12; f(v_8) = 30; f(v_9) = 33; f(v_{10}) = 36; f(v_{11}) = 37; f(v_{12}) = 43; f(v_{13}) = 44; f(v_{14}) = 45; f(v_{15}) = 51; f(v_{16}) = 59; f(v_{17}) = 61; f(v_{18}) = 62; f(v_{19}) = 63; f(v_{20}) = 67; f(v_{21}) = 69; f(v_{22}) = 70; f(v_{23}) = 77; f(v_{24}) = 78; f(v_{25}) = 85; f(v_{26}) = 93; f(v_{27}) = 101; f(v_{28}) = 4; f(v_{29}) = 6; f(v_{30}) = 7; f(v_{31}) = 9; f(v_{32}) = 10; f(v_{33}) = 43 \right\}
\[ i + 16, 6 \leq i \leq 9; f(v^0_i) = 38; f(v^1_i) = 40; f(v^2_i) = 41; f(v^3_i) = i + 333, 13 \leq i \leq 17; f(v^4_i) = i + 38, 18 \leq i \leq 20; f(v^5_i) = 60; f(v^6_i) = 63; f(v^7_i) = 64; f(v^8_i) = i + 47, 24 \leq i \leq 29; f(v^9_i) = 80; f(v^{10}_i) = 13t + B, B = 56, 57, 58, 60, \ldots, 64 \text{ and } 65 \text{ for } m = 13t + D, D = 5, 6, 7, 9, \ldots, 13 \text{ and } 14 \text{ and } t \geq 2 \text{ respectively in order; } f(v^{11}_i) = 13t + B_0, B_0 = 59, 67 \text{ for } m = 13t + D_0, D_0 = 8, 16 \text{ and } t \geq 3 \text{ respectively in order } f(v^1_i) = 14; f(v^1_i) = i + 15, 2 \leq i \leq 5; f(v^5_i) = 26; f(v^6_i) = 29; f(v^7_i) = 31; f(v^8_i) = 32; f(v^9_i) = 39; f(v^{10}_i) = 42; f(v^{11}_i) = i + 40, 12 \leq i \leq 15; f(v^{12}_i) = 65; f(v^{13}_i) = 66; f(v^{14}_i) = 68; f(v^{15}_i) = 13t + 53 \text{ for } n = 13t - 7 \text{ and } t \geq 2; f(v^{16}_i) = 13t + 55 \text{ for } n = 13t - 6 \text{ and } t \geq 2 \]. Then the induced edge labeling \( f_E : E(G) \to \mathbb{Z}^+ \) as above, forms the set \( A \) of consecutive and distinct integers. \( A = \bigcup_{i=1}^{9} A_i \), where \( A_1 = \{3, \ldots, 11, 13, \ldots, 16, 18, 21, \ldots, 24, 26, 29, 34, \ldots, 37, 39, 42, 47, 55\} \) from the vertex labels of \( K_8; A_2 = \{12\} \) from the label of the pendant vertex adjacent to \( v_1; A_3 = \{17\} \) from the label of the pendant vertex adjacent to \( v_2; A_4 = \{19\} \) from the label of the pendant vertex adjacent to \( v_3; A_5 = \{32, 33, 40\} \) from the label of the pendant vertex adjacent to the vertex \( v_4; A_6 = \{20, 38, 41\} \) from the label of the pendant vertex adjacent to the vertex \( v_5; A_7 = \{49, 50, 56, \ldots, 58, 64, 72, 74, 75, 80, 82, 83, 90, 91, 98, 106, 114\} \) from the label of the pendant vertex adjacent to the vertex \( v_6; A_8 = \{26, 27, 28, 30, 31, 43, \ldots, 46, 59, 61, 62, 67, \ldots, 71, 77, \ldots, 79, 81, 84, 85, 92, \ldots, 97, 101, 13t + B^*, B^* = 77, \ldots, 86\} \) and \( 88 \) from the label of the pendant vertex adjacent to the vertex \( v_7; A_9 = \{48, 51, \ldots, 54, 60, 63, 65, 66, 73, 76, 86, \ldots, 89, 99, 100, 102, 13t + 87, 13t + 89\} \). Now we define a bijection \( f_1 : V(G) \cup E(G) \to \{1, \ldots, 2(m + n) \} \) by \( f_1(V(G)) = f(V(G)) \) and \( f_1(E(G)) = \{2(m + n) + 184, 2(m + n) + \).
183, \ldots, m+n+83\}$. Then one can check that $f_1$ is a required labeling with common edge count $2(m + n) + 187$. Finally as $K_8 \subseteq G$, the result follows and the proof is complete. 

**Conjecture 2.2.24** Every graph on $p \geq 9$ vertices can be embedded as a subgraph of some $(1,1)$ edge-magic graph.

**Theorem 2.2.25** Every graph on $p$ vertices with $9 \leq p \leq 12$ can be embedded as a subgraph of some $(1,1)$ edge-magic graph.

**Comments.** Consider $K_p$ for $9 \leq p \leq 12$. Form a new graph $G$ from $K_p$ by adding a number of vertices and joining each of them to appropriate vertices of $K_p$. We do this using computers to match the following requirement: How many vertices are needed to add to $K_p$ and how their adjacency with the vertices of $K_p$ can be defined so that the resulting graph $G$ is connected and there exists a bijection $f : V(G) \to \{1, \ldots, p(G)\}$ with the induced edge labeling $f_E : E(G) \to \mathbb{Z}^+$ forming the set of consecutive integers $\{r, r+1, \ldots, q(G) + r - 1\}$ for some $r(= f_E(e) = f(u) + f(v)) \in \mathbb{Z}^+$ where $e = (u, v) \in E(G)$. We give a detailed proof for $p = 9$. It is routine to organize the details for $p = 10$ to $12$ in a similar manner. But we give the complete vertex labels of the corresponding $G'$s extending $K_p$ for $p = 10$ to $12$ in Tables 1 to 3. In these figures the pendant vertex/vertices with their respective labels adjacent to a vertex of $K_p$ with its label are indicated by an arrow.

**Proof.** Let $(K_9) = \{v_i : 1 \leq i \leq 9\}$. Form a new graph $G = K_9(v_1, v_2, \ldots, v_8) \bullet (K_{1,1}, K_{1,2}, K_{1,3}, K_{1,2}, K_{1,2}, K_{1,19}, K_{1,21})$, where $G$ is the graph obtained from $K_9$
by identifying the central vertex of $K_{1,1}, K_{1,2}, K_{1,3}, K_{1,2}, K_{1,2}, K_{1,19}, K_{1,21}$, with $v_1, \ldots, v_8$ respectively. Denote the pendant vertices of $G$ by $v_1^1, v_2^1, v_3^1, v_4^1, v_5^1, v_6^1, v_7^1, 1 \leq i \leq 19, v_8^1, 1 \leq i \leq 21$. Suppose that $f : V(G) \to \{1, \ldots, 60\}$ is a bijection defined as follows: $f(v_1) = 1; f(v_2) = 2; f(v_i) = f(v_{i-1})+f(v_{i-2}); f(v_1^1) = 16; f(v_2^1) = 10; f(v_3^1) = 18; f(v_4^1) = 29; f(v_1^2) = 23; f(v_2^2) = 26; f(v_3^2) = 47; f(v_1^3) = 11; f(v_2^3) = 42; f(v_3^3) = 20; f(v_4^3) = 31; f(v_1^4) = 4; f(v_2^4) = 6; f(v_3^4) = 9; f(v_1^5) = 17; f(v_2^5) = 19; f(v_3^5) = 22; f(v_4^5) = 24; f(v_5^5) = 30; f(v_6^5) = 31 + i, 1 \leq i \leq 2; f(v_7^{10+i}) = 42 + i, \text{ for } 1 \leq i \leq 4; f(v_7^{14+i}) = 55 + i, \text{ for } 1 \leq i \leq 5; f(v_8^1) = 7; f(v_8^2) = 12; f(v_8^{2+i}) = 13 + i, \text{ for } 1 \leq i \leq 2; f(v_8^3) = 25; f(v_8^{3+i}) = 26 + i, \text{ for } 1 \leq i \leq 2; f(v_8^{7+i}) = 34 + i, \text{ for } 1 \leq i \leq 7; f(v_8^{14+i}) = 47 + i, \text{ for } 1 \leq i \leq 7$. Then the induced edge-labeling $f_E : E(G) \to Z^+$ forms the set $A$ of consecutive and distinct integers $A = \bigcup A_i$ with $i = 1$ to $9$, where $A_1 = \{3, 4, \ldots, 11, 13, \ldots, 16, 18, 21, \ldots, 24, 26, 29, 34, \ldots, 37, 39, 42, 47, 55, \ldots, 58, 60, 63, 68, 76, 89\}$ from the vertex labels of $K_9; A_2 = \{17\}$ from the label of the vertex adjacent to $v_1; A_3 = \{12, 26\}$ from the label of the vertices adjacent to $v_2; A_4 = \{32\}$ from the label of the vertex adjacent to $v_3 ; A_5 = \{28, 31, 52\}$ from the label of the vertices adjacent to $v_4 ; A_6 = \{19, 50\}$ from the label of the vertices adjacent to $v_5 ; A_7 = \{33, 44\}$ from the label of the vertices adjacent to $v_6; A_8 = \{25, 27, 30, 38, 40, 43, 45, 51, 53, 54, 64$ to $67, 77$ to $81\}$ from the label of the vertices adjacent to $v_7; A_9 = \{41, 46, 48, 49, 59, 61, 62, 69$ to $75, 82$ to $88\}$ from the label of the vertices adjacent to $v_8$. Now define a bijection $f_1 : V(G) \cup E(G) \to \{1, \ldots, 147\}$ by $f_1(V(G)) = f(V(G))$ and $f_1(E(G)) = \{147, 146, \ldots, 61\}$. Then one can check that $f_1$ is a required labeling with the common edge count 150.
Table 2.1. Complete vertex labels of $G'$ extending $K_p$ with $p = 10$

<table>
<thead>
<tr>
<th>$K_{10}$</th>
<th>→</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>→ 16, 24</td>
</tr>
<tr>
<td>2</td>
<td>→ 10, 18, 31</td>
</tr>
<tr>
<td>3</td>
<td>→ 29, 50</td>
</tr>
<tr>
<td>5</td>
<td>→ 23, 26, 39, 47</td>
</tr>
<tr>
<td>8</td>
<td>→ 11, 37, 42, 63, 76</td>
</tr>
<tr>
<td>13</td>
<td>→ 60, 68</td>
</tr>
<tr>
<td>21</td>
<td>→ 6, 9, 17, 19, 22, 30, 43, 44, 56, 57, 58</td>
</tr>
<tr>
<td>34</td>
<td>→ 7, 12, 14, 15, 20, 27, 28, 32, 33, 35, 36, 38, 40, 41, 48, 49, 51</td>
</tr>
<tr>
<td>55</td>
<td>→ 4, 25, 45, 46, 66, 67, 87, 88</td>
</tr>
<tr>
<td>89</td>
<td>–</td>
</tr>
</tbody>
</table>
Table 2.2. Complete vertex labels of $G'$ extending $K_p$ with $p = 11$

<table>
<thead>
<tr>
<th></th>
<th>(1\rightarrow)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>11, 16, 29, 37</td>
</tr>
<tr>
<td>2</td>
<td>18, 23, 26, 31, 52</td>
</tr>
<tr>
<td>3</td>
<td>50, 84</td>
</tr>
<tr>
<td>5</td>
<td>39, 47, 60, 68, 81</td>
</tr>
<tr>
<td>8</td>
<td>24, 42, 58, 63, 71, 73, 76, 97, 110, 131</td>
</tr>
<tr>
<td>13</td>
<td>6, 14, 32, 94, 102, 115, 118, 123</td>
</tr>
<tr>
<td>21</td>
<td>10, 19, 22, 27, 40, 53, 56, 57, 165</td>
</tr>
<tr>
<td>(K_{11})</td>
<td>34 (\rightarrow) 7, 12, 15, 17, 28, 33, 35, 36, 38, 48, 49, 54, 59, 61, 69, 70, 74, 75, 77 to 80, 82, 90 to 92, 95, 96, 98 to 101, 103, 116, 117, 119 to 122, 124, 136 to 143, 145, 157 to 164, 166, 178 to 187</td>
</tr>
<tr>
<td></td>
<td>55 (\rightarrow) 4, 9, 20, 25, 30, 41, 43 to 46, 51, 62, 64 to 67, 72, 83, 85 to 88, 93, 104 to 109, 111 to 114, 125 to 130, 132 to 135, 146 to 156, 167 to 177</td>
</tr>
<tr>
<td></td>
<td>89 –</td>
</tr>
<tr>
<td></td>
<td>144 –</td>
</tr>
</tbody>
</table>
Table 2.3. Complete vertex labels of \( G' \) extending \( K_p \) with \( p = 12 \)

<table>
<thead>
<tr>
<th>( K_{12} )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<tr>
<td>5</td>
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<td>8</td>
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<td>13</td>
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<td>21</td>
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<td>34</td>
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<td>55</td>
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<tr>
<td>89</td>
</tr>
<tr>
<td>144</td>
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<tr>
<td>233</td>
</tr>
</tbody>
</table>
Theorem 2.2.26  The graph $G$ has (1,1) edge-magic labeling iff it has both (1,1) edge-odd magic labeling and (1,1) edge-even magic labeling.

Proof. Necessary Part: Suppose that $G$ is a graph of $p$ vertices and $q$ edges and $h : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ is an (1,1) edge-magic labeling with common edge count $k_0$. Now define $f : V(G) \cup E(G) \rightarrow \{1, 3, \ldots, 2(p + q) - 1\}$; and $g : V(G) \cup E(G) \rightarrow \{0, 2, \ldots, 2(p + q) - 2\}$ such that $f(v_i) = 2h(v_i) - 1$ and $f(e_i) = 2h(e_i) - 1; g(v_i) = 2h(v_i) - 2$ and $g(e_i) = 2h(e_i) - 2$ for all $i$. Then one can check that $f$ and $g$ form respectively (1,1) edge-odd magic labeling with common edge count $2k_0 - 3$, and (1,1) edge-even magic labeling with common edge count $2k_0 - 6$.

Sufficiency Part: Suppose that $G$ is a graph on $p$ vertices and $q$ edges and $f : V(G) \cup E(G) \rightarrow \{1, 3, \ldots, 2(p + q) - 1\}$ is an (1,1) edge-odd magic with common edge count $k_1$ and $g : V(G) \cup E(G) \rightarrow \{0, 2, \ldots, 2(p + q) - 2\}$ is an edge-even magic with common edge count $k_2$. Then $h : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ defined by $h(v_i) = \frac{1}{4}[f(v_i) + g(v_i) + 3]$ for all $1 \leq i \leq n; h(v_{n+1-i}v_{n-i}) = \frac{1}{4}[f(v_{n+1-i}v_{n-i}) + g(v_{n+1-i}v_{n-i}) + 3]$ for all $1 \leq i \leq n$ is an (1,1) edge-magic labeling with common edge count $\frac{1}{4}[k_1 + k_2 + 9] = k_0$.

Theorem 2.2.27  Every graph on $p \leq 8$ vertices can be embedded as a subgraph of some (1,1) edge-odd magic graph and (1,1) edge-even magic graph.

Proof. In [147] it was proved that every graph on $p \leq 8$ vertices can be embedded as a subgraph of some (1,1) edge-magic graph. By Theorem 2.2.26, it follows that such a graph also admits both (1,1) edge-odd magic labeling and (1,1) edge-even magic labeling. Hence the result follows.
Theorem 2.2.28 If a graph $G(p, q)$ is $(1, 1)$ edge-magic, then $\sum_{v \in V(G)} deg_G(v)f(v) + \sum_{e \in E(G)} f(e) = qk_0$, where $k_0$ is the common edge count.

Proof. Let $f$ be a $(1, 1)$ edge-magic labeling of $G(p, q)$. Since $f$ is a bijection, each vertex label $f(v), v \in V(G)$ occurs exactly $deg(v)$ times, once each in the calculation of the edge count of $deg(v)$ distinct edges in $G$ and each edge label $f(e), e \in E(G)$ occurs exactly once and that too in the calculation of its own count. \[\square\]

Observation 2.2.29 If a graph $G(p, q)$ is $(1, 1)$ edge-magic, then $\sum_{v \in V(G)} (deg_G(v) - 1)f(v) = qk_0 - (p + q + 1)(p + q)/2$.

Theorem 2.2.30 If $A = \{a_i : 1 \leq i \leq R$ and $a_1 < \ldots < a_n\}$ is a subset of integers in $[1,N]$ such that $B = \{a_r - a_s : a_r, a_s \in A$ and $s < r\}$ are distinct, then $R \leq N^{1/2} + N^{1/4} + 1$.

Remark 2.2.31 The above result was due to Erdos in [48]. An improvement of this was obtained later by Bose and Chowla in [38]. One can also see [66] for an exhaustive information. We present here a slightly different proof, enabling to compute the 0-constant in the error term better.

Proof. Let us define $A_j = a_{r+j} - a_r : 1 \leq r \leq R - j$. Then $|A_j| = R - j$ and $\sum_{b \in A_j} b = \sum_{1 \leq r \leq R-j} (a_{r+j} - a_r) = (a_R - a_1) + (a_{R-1} - a_2) + \ldots + (a_{R-j+1} - a_j) \leq jN$. Let $D = \cup_{1 \leq j \leq k} A_j$. Then $|D| = kR - k(k + 1)/2$ and $\sum_{d \in D} d \leq \sum_{1 \leq j \leq k} jN = (k(k + 1)/2)N$. Now the elements of $D$ are distinct by definition, and $\sum_{d \in D}$ is at least the sum of first $|D|$ natural numbers. Hence $\sum_{d \in D} d \geq ((kR - k(k + 1)/2(kR - k(k + 1)/2 + 1))/2$. Comparing the upper and lower bound of $\sum_{d \in D} d$, we get that
$$(k(k+1)N)^{\frac{1}{2}} \geq kR - k(k+1)/2. \text{ That is } (N(k+1)/k)^{\frac{1}{2}} \geq R - (k+1)/2 \text{ and } R \leq N^{\frac{1}{2}}(1 + (1/2k)) + (k+1)/2. \text{ Now set } k\lfloor N^{\frac{1}{2}} \rfloor + 1 \text{ to get the result.}$$

**Remark 2.2.32** Now following Erdos, we shall call any set $A$ as in the Theorem 2.2.30 as a Sidon set.

**Corollary 2.2.33** The complete graph $K_p$ is not $(1, 1)$ edge-magic if $p \geq 17$.

*Proof.* Assume that $K_p$ is $(1,1)$ edge-magic. Then there exists a $(1,1)$ edge-magic labeling $f : V(K_p) \cup E(K_p) \to \{1, \ldots, (p + p(\frac{p-1}{2}) = N\}$. If $e_i = (u_i, v_i)$ for $1 \leq i \leq 2$ are any two edges, then $c(e_1) = c(e_2)$ implies $f(u_1) - f(u_2) = f(v_1) - f(v_2)$.

Now as $\{f(u) : u \in V(K_p)\}$ is a Sidon set in $[1, N]$, we get by the Theorem 2.2.30, that $p = |V(K_p)| \leq N^{\frac{1}{2}} + N^{\frac{1}{4}} + 1$. This yields a contradiction if $p \geq 17$ is clear.

For, put $N = x^4$, so that $p(p + 1) = 2x^4$ and $2x^4 \leq (p + (1/2))^2$. Now $p + (1/2) \leq N^{\frac{1}{2}} + N^{\frac{1}{4}} + 3/2$ implies that $\sqrt{2x^2} \leq x^2 + x + 3/2$ and $x \leq 3.45$. \hfill \square

### 2.3 Some Results on $(1, 0)$ and $(0, 1)$ Edge-Magic Graphs

**Observation 2.3.1** If a graph $G(p,q)$ is $(1,0)$ edge-magic, then

$$\sum_{v \in V(G)} \deg(v)f(v) = qk_2,$$

where $k_2$ is the common edge count.

We note that the graph $G = nK_2$ ($n$-disjoint copies of $K_2$) is $(1,0)$ edge-magic. For, let $E(G) = \{(u_i,v_i) : u_i,v_i \in V(G) \text{ and } 1 \leq i \leq n\}$. Define a bijection $f : V(G) \to \{1, \ldots, 2n\}$ by $f(u_i) = i$ and $f(v_{n+1-i}) = n + i$ for $1 \leq i \leq n$. Then one can check that $f$ is a $(1,0)$ edge-magic labeling with common edge-count $c(v) = 2n + 1$. Further we observe that no graph (without isolated vertices) is $(1,0)$ edge-magic.
This is because, as $G(p, q)$ is connected we have $q \geq p - 1$ and moreover irrespective of the parity of $p$, any $(1,0)$ edge-magic labeling of $G$ can have at most $\frac{p}{2}$ distinct pairs of integers $(a, b)$ such that $a + b = p + 1$. Further note that because of the above, no simple connected graph is a subgraph of some $(1,0)$ edge-magic graph.

**Theorem 2.3.2** Let $G$ be an $r$-regular $(p, q)(0,1)$ edge-magic graph. Then

(i) if $r \equiv 1 \pmod{2}$, then $p \equiv 2 \pmod{4}$

(ii) if $r \equiv 2 \pmod{4}$ and $p \equiv 0 \pmod{2}$, then $G$ contains no component of an odd order

(iii) If $p > 2$, then $r > 2$.

**Proof.** First observe that for an $r$-regular $(p, q)$ graph $G$ that $k_p = 2f(e) = q(q + 1), e \in E(G)$ where $k$ is the common edge count and $q = rp/2$ implies $r = r(1 + rp/2)/2$. Now to see that (i) is true, suppose that $p \equiv 0 \pmod{2}$. Then $k = r(1 + (rp/2))/2$ implies that $k$ is not an integer. But this is a contradiction to the fact that the common edge count $k$ is a sum of integer labels of the edges of $G$. Further as the order of an odd regular graph is even it follows that $p \equiv 2 \pmod{4}$.

Now to see that (ii) is true, let us assume that $G$ contains a component $C_0$ of an odd order. Then from $k = r(1 + (rp/2))/2$ it follows that $k$ is odd. Hence $k|V(C_0)|$ is odd. But then this implies that $k|V(C_0)| = 2\sum f(e), e \in E(C_0)$, a contradiction.

Now it is easy to see (iii) as a regular graph of degree one is magic if and only if it is connected (ie., $p$ is equal to 2) and a 2-regular graph is never magic. □

**Theorem 2.3.3** If a graph $G(p, q)$ is $(1,0)$ edge-magic, then $\sum_{v \in V(G)} deg(v)f(v) = qk_2$, where $k_2$ is the common edge count.
Proof. We omit the proof as they are easy to see.

**Theorem 2.3.4** Let $G$ be an $r$–regular $(p,q)(0,1)$ edge-magic graph. Then

(i) if $r \equiv 1 \pmod{2}$, then $p \equiv 2 \pmod{4}$

(ii) if $r \equiv 2 \pmod{4}$ and $p \equiv 0 \pmod{2}$, then $G$ contains no component of an odd order

(iii) If $p > 2$, then $r > 2$.

Proof. First observe that for an $r$-regular $(p,q)$ graph $G$ that $k_p = 2f(e) = q(q + 1), e \in E(G)$ where $k$ is the common edge count and $q = rp/2$ implies $r = r(1 + rp/2)/2$. Now to see that (i) is true, suppose that $p \equiv 0 \pmod{2}$. Then $k = r(1 + (rp/2))/2$ implies that $k$ is not an integer. But this is a contradiction to the fact that the common edge count $k$ is a sum of integer labels of the edges of $G$.

Further as the order of an odd regular graph is even it follows that $p \equiv 2 \pmod{4}$. Now to see that (ii) is true, let us assume that $G$ contains a component $C_0$ of an odd order. Then from $k = r(1 + (rp/2))/2$ it follows that $k$ is odd. Hence $k|V(C_0)|$ is odd. But then this implies that $k|V(C_0)| = 2 \sum f(e), e \in E(C_0)$, a contradiction. Now it is easy to see (iii) as a regular graph of degree one is magic if and only if it is connected (ie., $p$ is equal to 2) and a 2-regular graph is never magic.

**Theorem 2.3.5** If $G$ is a nice $(1, 1)$ edge-magic 2-regular graph, then $G \odot K_n^c$ is a nice $(1, 1)$ edge-magic for every positive integer $n$.

Proof. Let $f$ be a nice $(1, 1)$ edge-magic labeling of $G$ with common edge count $k$. Assume that $H$ is a component of $G \odot K_n^c$. Then $H \cong C_r \odot K_n^c$ for some integer
Let $V(H) = \{v_i : i \in z_r \} \cup \{u_{ij} : i \in z_r \text{ and } 1 \leq j \leq n \}$ and $E(H) = \{v_iv_{i+1} : i \in z_r \} \cup \{v_iu_{ij} : i \in z_r \text{ and } 1 \leq j \leq n \}$, where $z_r$ denotes the set of integers modulo $r$.

Then $f|_H$ extends to a labeling $g$ of $H$ as follows:

$g(v_i) = (n+1)f(v_i) - n$, $g(v_{i-1}v_i) = nf(v_{i-1}v_i)$,

$g(u_{ij}) = (n+1)f(v_{i-1}) - n + j$, $g(v_{i-1}u_{ij}) = nf(v_{i-1}v_i) - j$, where $i \in z_r$ and $1 \leq j \leq n$. Therefore $f$ extends likewise in every component of $G \odot K_n^c$, and a nice $(1,1)$ edge-magic labeling of $G \odot K_n^c$ is obtained with common edge count $n(k - 2) + 2$.

**Theorem 2.3.6** Let $G$ be a graph of even order $p \geq 4$ having the nice $(1,1)$ edge-magic labeling $f$ with the property that $\max \{f(u) + f(v) : uv \in E(G)\} = \frac{3p}{2}$. Then the graph $H$ obtained by attaching $n$ pendant edges to each vertex of $G$ except the vertex $v$ with $f(v) = p$ is nice $(1,1)$ edge-magic for every positive integer $n$.

**Proof.** Let $V(G) = \{v_1, \ldots, v_p\}$. Then take a nice $(1,1)$ edge-magic labeling $f$ of $G$ with common edge count $k$ satisfying the property that $f(v) = i$ for $i = 1, \ldots, p$.

Now, define the graph $H$ as follows: $V(H) = V(G) \cup \{w_i^j : 1 \leq i \leq p - 1 \text{ and } 1 \leq j \leq n \}$ and $E(H) = E(G) \cup \{v_iw_i^j : 1 \leq i \leq p - 1 \text{ and } 1 \leq j \leq n \}$.

By producing an argument similar to the one in Theorem 2.2.4 the vertex labeling $g : V(H) \rightarrow \{1, \ldots, p(n+1) - n\}$ such that $g(v) = f(v)$ for every vertex $v$ of $G$ and $g(w_i^j) = i + \left(\frac{p}{2}\right) + (p-1)j + 1$ if $1 \leq i \leq \left(\frac{p}{2}\right) - 1 \text{ and } 1 \leq j \leq n; i + \left(\frac{p}{2}\right) + (p-1)(j-1) + 1$, if $\frac{p}{2} \leq i \leq p - 1 \text{ and } 1 \leq j \leq n$ extends to a nice $(1,1)$ edge-magic labeling of $H$ with common edge count $k + 2n(p - 1)$.

**Theorem 2.3.7** $L_n$ is nice $(1,1)$ edge-magic when $n$ is odd.
Observation 2.3.8 A graph $G = (V, E)$ is said to be $(0,1)$ edge-magic (modulo $p = |V|$) if there exists a bijection $f : E(G) \to \{1, \ldots, q = |E|\}$ such that the induced mapping $g : V(G) \to \mathbb{N}$ defined by $g(u) = \sum_{(u,v) \in E(G), v \in V(G)} f(u, v) \pmod{p}$ is a constant map. By careful analysis we infer that a procedure can be drawn in general to label graphs of this type. Suppose that $G = C_n \times K_2$ is a prism with $V(G) = p = 2n, p \equiv 2 \pmod{4}$ and $|E(G)| = q = \frac{3p}{2}$. We proceed as follows to produce a modulo $p$ $(0,1)$ edge-magic labeling: Start at any edge of one of the cycles. Assign successive integers from $1, 2, \ldots, \frac{p}{2}$ to every alternative edge on the cycle in clockwise direction. When we arrive at a previously labeled edge then, continue the labeling in the same direction and same way beginning with the edge of the other cycle that is parallel to the first labeled edge on the first cycle. Now to label the linking edges between two cycles in $G$, start with the label $p + 1$. [Observe that we have used the labels $1, 2, \ldots, p$ successively to label the edges of the outer and inner cycles of $G$]. Assign the label $(p + 1)$ to that linking edge which joins the vertex common to the edges of the inner cycle with labels $p - \lfloor n/2 \rfloor$ and $p$. Then assign the rest of the labels $p + 2, p + 3, \ldots, q$ to other linking edges in the anti-clockwise direction. The above method took the advantage of the property of a cubic graph viz., A cubic graph on $p$ vertices will have $3p/2$ edges and the set of integers from $1$ to $3p/2$ can be partitioned into the sets $\{1, 2, \ldots, p/2\}, \{(p/2) + 1, (p/2) + 2, \ldots, p\}$ and $\{p + 1, p + 2, \ldots, 3p/2\}$. 
2.4 (1,1), (1,0) and (0,1) Vertex-Magic Labeling

**Theorem 2.4.1** If a graph $G(p, q)$ is (1, 1) vertex-magic, then $q(q + 1) = 2pk - [(p + q)^2 + (p + q)]$.

*Proof.* It follows, since in any (1,1) vertex-magic labeling $f$, each vertex label occurs only in its own count and each edge gives its count to only two vertices of $G$. \qed

**Theorem 2.4.2** If a graph $G(p, q)$ is (0, 1) vertex-magic then $q(q + 1) = pk$, where $k$ is the common vertex count.

*Proof.* Assume that $G(p, q)$ is (0, 1) vertex-magic. Then as each edge contributes its label to the vertex-count of exactly two distinct vertices we are through. \qed

**Theorem 2.4.3** The cycles are not (0,1) vertex-magic.

*Proof.* Suppose that $f : E(C_n) \rightarrow \{1, \ldots, n\}$ is a (0,1) vertex-magic labeling. Then $c(v) = n + 1$. Clearly $C_3$ is not (0,1) vertex-magic is observed earlier as $C_3 = K_3$.

If $n = 2t$ or $2t + 1$ with $t \geq 2$, then there are only $t$ pairs $(a, b)$ possible such that $a + b = n + 1$. But then an edge joining the final vertex of one pair with the initial vertex of the immediate successor pair would possess a edge label which is either less than $n + 1$ or more than $n + 1$, a contradiction. \qed

**Corollary 2.4.4** The paths are not (0, 1) vertex-magic.

*Proof.* It follows on the basis of a similar reasoning as in the Theorem 2.4.3. \qed

**Theorem 2.4.5** Every graph is a subgraph of some (1,0) vertex-magic graph.
Proof. We claim that $K_p$ is (1,0) vertex-magic for all $p$ so that the result follows. Let $V(K_p) = \{v_i : 1 \leq i \leq p\}$. Define a bijection $f : V(K_p) \to \{1, \ldots, p\}$ such that $f(v_i) = i$ for $1 \leq i \leq p$. Then one can check that $f$ is a $(1,0)$ vertex-magic labeling of $K_p$ with common vertex count $c(v) = p(p + 1)/2$. \qed

Observation 2.4.6 If a graph $G(p,q)$ is (0, 1) vertex-magic then $q(q + 1) = pk$, where $k$ is the common vertex count. To see this, assume that $G(p,q)$ is (0,1) vertex-magic. Then as each edge contributes its label to the vertex-count of exactly two distinct vertices we are through.

Observation 2.4.7 $K_n$ is not (0,1) vertex-magic if $n \equiv 0 \pmod{4}$. To see this, suppose $K_{4t}$ is (0,1) vertex-magic, then $4t^2(4t - 1)^2 + 2t(4t - 1) = 4tk$ implies $(4t - 1)(8t^2 - 2t + 1) = 2k$, a contradiction.

Observation 2.4.8 It is obvious that $K_1$ and $K_2$ are (0,1) vertex-magic and $K_3$ is not so. Stewart in [127, 128] proved that $K_5$ is not (0, 1) vertex-magic and $K_n$ for the rest of $n$’s are (0, 1) vertex-magic.

Observation 2.4.9 Of all the five regular polyhedra viz, the cube, the tetrahedron, the dodecahedron, the octahedron and the icosahedron, the octahedron is the only graph that is (0,1) vertex-magic. A (0,1) vertex-magic labeling is shown in Figure 2.3.
Figure 2.3. An example for a (0, 1) vertex-magic labeling

Figure 2.4 exhibits a (0,1) vertex-magic labeling for $K_{3,3}$

Observation 2.4.10 If a graph $G(p,q)$ is (1, 1) vertex-magic, then $q(q + 1) = 2pk - [(p + q)^2 + (p + q)]$. It follows, since in any (1,1) vertex-magic labeling $f$, each vertex label occurs only in its own count and each edge gives its count to only two vertices of $G$.

2.5 Concluding Remarks

In [152] we have obtained a number of results on nice (1,1) edge-magic graphs. It is interesting to mention that if $G$ is a $r$-regular nice (1,1) edge-magic with the common edge count where $r$ belongs to $Z^+$ is odd then its common edge count is $(4p + q + 3)/2$. Hence the $n$-dimensional cube $Q_n$ which is $n$-regular nice (1,1) edge-
magic if and only if \( n = 1 \). As every nice \((1,1)\) edge-magic graph contains at least two vertices of degree less than 4, it follows that the minimum degree is at most 3.

So the Whitney’s famous inequality, viz., \( k(G) \leq k_1(G) \leq 3 \) holds for every nice \((1,1)\) edge-magic graph \( G \), where \( k(G) \) and \( k_1(G) \) denote the vertex connectivity and edge-connectivity of \( G \), respectively. Furthermore the toroidal mesh \( C_m \times C_n \) is also not nice \((1,1)\) edge-magic for every pair of integers \( m, n \geq 3 \).

Rosa [118] has showed that all graphs that admit \( \alpha \)-valuation are bipartite. Therefore, if \( G \) is a \((p, p-1)\) graph with an \( \alpha \)-valuation \( f \), then there exists partite sets \( V_1 \) and \( V_2 \) where \( p_1 = |V_1| \) and \( p_2 = |V_2| \). If \( f(V_1) = \{0, 1, K, p_1 - 1\} \) then \( G \) is a nice \((1,1)\) edge-magic. Hence, if \( T \) is a tree having an \( \alpha \)-valuation, then \( T \) is a nice \((1,1)\) edge-magic. A well known result by Gilbert [56] states that almost all graphs are connected, which implies that for almost all \((p, q)\) graphs we have \( q \geq p \). This combined with Graham and Sloane [60] result that almost all graphs are not harmonious and for result in [152] that if a \((p, q)\) graph \( G \) it follows that ‘Almost all graphs are not nice \((1,1)\) edge-magic.’

**Some Open Problems for Further Work**

1. Every graph on \( p \geq 9 \) vertices can be embedded as a subgraph of some \((1,1)\) edge-magic graph.

2. The generalized \( n \)-crown \( C_n \odot K_{1,n} \) is \((1,1)\) edge-magic.

3. The generalized \( n \)-crown \( C_{2t} \odot K_{1,n} \) is \((1,1)\) edge-magic for all \( n \) and \( t \).

4. Every graph can be embedded as a subgraph of some \((1,1)\) vertex-magic graph.
5. For what values of $m$ and $n$, is the graph $K_{m,n}$ (0,1) vertex-magic?

6. Every graph can be embedded as a subgraph of some (0,1) vertex-magic and some (0,1) edge-magic graph.

To conclude, in this chapter we have not only proved a number of classes of graphs (1,1) edge-magic, but also succeeded in devising an algorithm for the problem of determining the existence or non-existence of a (1,1) edge-magic labeling of the graph $G = 2tP_3$ and pointed out in the procedure, a way to find one such labeling if it exists. We also considered a difficult problem of showing the complete graph on $p$ vertices are edge-magic and succeeded with the help of computers positively for the cases $p = 1$ to 12. The concept of (1,1) odd edge-magic and (1,1) even edge-magic labeling were introduced and we succeeded in obtaining a necessary and sufficient condition for a graph to be a (1,1) edge-magic in terms of (1,1) odd edge-magic and (1,1) even edge-magic labeling.