CHAPTER 1

Introduction
1.1 Brief History of Graph Theory

Christian Huygens who lived during 1629-1695 was a great philosopher. He once expressed to his colleague Gottfried W. Leibniz a noted mathematician (belonging to the period 1646-1716) the need for an appropriate tool to deal with the geometry of position. Leonhard Euler who lived between 1707-1783 once had an opportunity to know this. Like several other problems lingering in him this issue also has occupied a place in his mind. He never realized at that point of time about the almighty’s intention to provide that tool to the whole world through him.

Later when Euler was relaxing once during 1735 to refurbish his energy for more serious mathematics research casually attempted a puzzle about seven bridges. As we expect he was able to find a solution also for it. The seven bridges belong to the East Prussian city of Königsberg (now Kaliningrad) on the banks of the River Pregel and Kneiphof an island. These bridges actually were present at a point where it branches into two parts in the pregel river. People living there are very familiar with the problem concerning identifying a path using which one could be able to cross all the seven bridges exactly once and retrace back to the starting point. After several attempts by many of them it was dumped as something impossible. Euler explained his ideas to the members of the Petersburg Academy on August 26, 1735. He later recorded his expositions in the form of an article that has appeared in the proceedings of the Petersburg Academy bearing the title Solutio Problematis ad Geometriam Situs Pertinentis (The solution to a problem relating to the geometry of position). This has marked the beginning of the wonderful subject called Graph
Theory and Euler was widely acknowledged as the Father of the same.

The evolution of graph theory is really amazing and simple day to day problems have led to tremendous outburst and brought unexpected innovations. For instance, consider the “Four Color Problem”. In 1852, De Morgan asked Hamilton regarding the sufficiency of four colors to color the map of any country so that no two of them with a common boundary are assigned the same color. This outwardly looking simple problem has defied all attempts at solution for a very long time and finally in 1976, Kenneth Appel and Wolfgang Haken managed a computer assisted proof and this has triggered the birth of a new proof technique through computers.

Graph theory suffers due to large number of definitions that are used inconsistently. For instance, A graph is a collection of nodes and lines that we call vertices and edges, respectively. Some prefers call a graph, others call a simple graph. What some call a multigraph, other just call a graph. Some call a graph labeled if the vertices are labeled, while others mean that the edges are labeled. Why are the definitions so confusing in graph theory? May be it is for mere convenience. Those who write about simple graphs make their lives easier by just calling them graphs and those who prefer to write about multigraphs call them graphs. So the message in all of this is that when we start to read a new result we should read the definitions carefully to conform with the ground rules. Graphs are much sought after way to model several situations in the creation (connections of wires/leads, logistics/transportation problems, pipelines between points with known capacities, family trees, organizational charts, among many more). This explains more than enough why graph theory has witnessed such monstrous growth in the past century.
1.2 Basic Definitions

The graphs considered in this thesis are finite, simple and undirected.

**Definition 1.2.1** In a graph $G$ we call two vertices $u, v \in V$ adjacent if $uv \in E(G)$. Vertices $u, v$ are also called as neighbors of each other. It is customary to use the notation $N(u)$ to refer the set of all neighbors of the vertex $u$.

**Definition 1.2.2** By the degree of a vertex $u$ we mean the number of edges incident upon it. We also denote it as $deg(u)$. We call a vertex an isolated vertex if the degree of it is 0 and a pendant vertex (or end vertex or a leaf) if the degree of it is 1. We use the symbol $\delta(G)$ to denote the minimum degree of a graph $G$ and is defined as $\delta(G) = \min_{u \in V} deg(u)$ and the symbol $\Delta(G)$ to denote the maximum degree of a graph $G$ and is defined as $\Delta(G) = \max_{u \in V} deg(u)$.

**Definition 1.2.3** We call a graph $G$ an $r$-regular graph if $\delta(G) = \Delta(G) = r$.

**Definition 1.2.4** We call the graph $H$ a subgraph of the graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also call $G$ a supergraph of $H$.

**Definition 1.2.5** By a spanning subgraph $H$ of a graph $G$ we mean a subgraph with $V(H) = V(G)$.

**Definition 1.2.6** By the complement of a graph $G$ we mean the graph $G^c$ with $V(G^c) = V(G)$ and we say that $uv \in E(G^c)$ whenever $uv \notin E(G)$. By $P_n$ and $C_n$ we mean a path and cycle of length $n - 1$ and $n$ respectively.

**Definition 1.2.7** A complete graph $K_n$ of order $n$ is a graph in which every two
Definition 1.2.8 A graph $G$ is called bipartite if $V$ can be partitioned into two nonempty subsets $V_1$ and $V_2$ in such a way that every edge in $E$ joins a vertex of $V_1$ with a vertex of $V_2$. If each vertex in $V_1$ is adjacent to all vertices in $V_2$, then $G$ is said to be complete bipartite, denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$.

Definition 1.2.9 We call the graphs $G_1$ and $G_2$ as vertex disjoint if $V(G_1) \cap V(G_2) = \phi$.

Definition 1.2.10 By the union of two vertex disjoint graphs $G_1$ and $G_2$ we mean the graph $G = G_1 \cup G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Definition 1.2.11 By a join of the graphs $G_1$ and $G_2$ we mean the graph $G = G_1 \vee G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$.

Definition 1.2.12 By the cartesian product of graphs $G_1$ and $G_2$ we mean the graph $G = G_1 \times G_2$ with $V(G) = V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G)$ if and only if $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, $x_2 = y_2$ and $x_1y_1 \in E(G_1)$.

Definition 1.2.13 A wheel $W_n$, with $n$ spokes, is a graph that has a center $x$ connected to all the $n$ vertices in cycle $C_n$.

Definition 1.2.14 A fan graph $F_n$ can be constructed from a wheel by deleting one edge on the $n$-cycle.

Definition 1.2.15 A friendship graph is a graph that can be represented as
\[ f_n = K_1 \lor nK_2. \]

**Definition 1.2.16** A book graph is a graph \( B_i = K_2 \lor K_i^c \), it has \( i + 2 \) vertices.

**Definition 1.2.17** A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path.

**Definition 1.2.18** The caterpillar can be seen as a sequence of stars \( S_1 \cup S_2 \cup \ldots \cup S_r \), where each \( S_i \) is a star with center \( c_i \) and \( n_i \) leaves, \( i = 1, 2, ..., r \), where the leaves of \( S_i \) include \( c_{i-1} \) and \( c_{i+1}, i = 2, ..., r - 1 \). We denote the caterpillar as \( S_{n_1, n_2, ..., n_r} \), where the vertex set is

\[
V(S_{n_1, n_2, ..., n_r}) = \{c_i : 1 \leq i \leq r\} \cup \bigcup_{i=2}^{r-1} \{x_i^j : 2 \leq j \leq n_i - 1\} \\
\cup \{x_i^1 : 1 \leq j \leq n_i - 1\} \cup \{x_i^2 : 2 \leq j \leq n_r\}
\]

and the edge set is

\[
E(S_{n_1, n_2, ..., n_r}) = \{c_i c_{i+1} : 1 \leq i \leq r - 1\} \cup \bigcup_{i=2}^{r-1} \{c_i x_i^j : 2 \leq j \leq n_i - 1\} \\
\cup \{c_1 x_1^j : 1 \leq j \leq n_1 - 1\} \cup \{c_r x_r^j : 2 \leq j \leq n_r\}.
\]

If \( r = 2 \) then the graph is called a double star.

**Definition 1.2.19** By a crown graph of graphs \( G_1 \) and \( G_2 \) we mean the graph \( G = G_1 \odot G_2 \) formed from a copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) in such a way that all vertices in the same copy of \( G_2 \) are connected with exactly one vertex of \( G_1 \) and each vertex of \( G_1 \) is connected to exactly one copy of \( G_2 \). Let \( n \) and \( m \) be positive integers, \( n \geq 3 \) and \( 1 \leq m < \frac{n}{2} \).
Definition 1.2.20 The generalized Petersen graph $P(n, m)$ is a graph that consists of an outer-cycle $y_0, y_1, y_2, ..., y_{n-1}$, a set of $n$ spokes $y_ix_i, 0 \leq i \leq n - 1$, and $n$ edges $x_ix_{i+m}, 0 \leq i \leq n - 1$, where all subscripts are taken modulo $n$.

Note that, for $m = 1$, the generalized Petersen graph $P(n, 1)$ is also known as a prism. The common notation for a prism with $2n$ vertices is $D_n$.

Definition 1.2.21 The generalized prism can be defined as the cartesian product $C_m \times P_n$ of a cycle on $m$ vertices with a path on $n$ vertices. Let $V(C_m \times P_n) = \{v_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ be the vertex set and $E(C_m \times P_n) = \{v_{i,j}v_{i+1,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ be the edge set, where $i$ is taken modulo $m$.

Definition 1.2.22 A generalized antiprism $A^n_m$ can be obtained by completing the generalized prism $C_m \times P_n$, by adding the edges $v_{i,j+1}v_{i+1,j}$, for $1 \leq i \leq m - 1, 1 \leq j \leq n - 1$, and the edges $v_{m,j+1}v_{1,j}$, for $1 \leq j \leq n - 1$. Let $V(A^n_m) = V(C_m \times P_n) = \{v_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ be the vertex set of $A^n_m$, and let $E(A^n_m) = E(C_m \times P_n) = \{v_{i,j+1}v_{i+1,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1\}$ be the edge set of $A^n_m$, where $i$ is taken modulo $m$.

1.3 Brief Introductory Note On Graph Labeling

Graph labeling are more vividly looked after area of research among graph theorists. A graph labeling is presumed as a function that assigns to the elements of a graph some numbers which can be either integers or real numbers so that some rules are adhered to. (By elements of a graph we mean either vertices or edges or both).
The assigned numbers are referred as labels. The usage of terms like valuations or numberings are also found in the literature. From what has been said so far one can carefully reason that a variety of graph concepts can be expressed in terms of graph labeling.

A useful reference for research in graph labeling can be attributed to [118] or [60]. The author in [118] called a function $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ with the label $|f(x) - f(y)|$ so that all such labels are distinct for all $xy \in E(G)$ a $\beta$-valuation. Further the authors in [58] analyzed this labeling and coined a new name graceful labeling. Thereafter serious attention was given regarding scope for applications besides a vivid academically inclined theoretical interest. The outcome of such a probe can be visualized in fields like x-ray crystallography, coding theory, radar, circuit design, astronomy, communication design etc., For more on graph labeling one can also see [3, 4, 12–20, 23, 24, 29, 30, 32–37, 40, 41, 43–47, 49, 51–54, 57, 61–63, 69–73, 75, 80–82, 84, 86, 87, 90, 94, 95, 97, 98, 100, 101, 103, 104, 107–110, 113–115, 117, 122, 123, 125, 126, 130–133, 135–137, 141, 142, 145].

Graph labeling is an injective mapping from elements of a graph (it can be vertices, edges, faces, or a combination of any of these) to a set of numbers (usually positive integers). In this thesis, we consider only a mapping from a set of vertices, set of edges, or a combination of both. If the domain of the mapping is the set of vertices (edges) then the labeling is called vertex labeling (edge labeling). There are three common possibilities when making an evaluation of the labeling of the graph: vertex-evaluation, edge-evaluation and face evaluation.
**Definition 1.3.1 Graceful Labeling** (Rosa [118])

A graph that is connected and comprising of \( p \) vertices and \( q \) edges is said to be graceful if one succeeds in the process of assigning distinct integers \( f(u) \) to the vertices from the set \( \{0, 1, ..., q\} \) with the property that if every edge \( uv \) is assigned the label \( |f(u) - f(v)| \) then all the edge labels are distinct and together form the set \( \{1, 2, ..., q\} \).

**Definition 1.3.2 \( \alpha \)-Valuation** (Rosa [118])

A graceful labeling \( f \) is referred as an \( \alpha \)-valuation if \( \exists x \in \mathbb{Z} \) such that \( f(u) \leq x < f(v) \) or \( f(v) \leq x < f(u) \) \( \forall uv \in E(G) \). Observe that the occurrence of an instance where some edge \( uv \in E(G) \) with \( |f(u) - f(v)| = 1 \) is common with every graceful labeling \( f \), we deduct that the integer \( x \) is uniquely determined by the \( \alpha \)-valuation.

Informally, by a graph labeling we mean an assignment of integers to the elements of a graph such as vertices, or edges or both subject to some specified conditions. These conditions are usually expressed on the basis of the values (called weights) of some evaluating function. The evaluating function will be simply to produce partial sums of the labeled elements of the graph. The partial sums will be either a (multi) set of vertex weights, obtained for each vertex by adding all the labels of a vertex and its adjacent edges, or a (multi) set of edge weights, obtained for each edge by adding the labels of an edge and its end points.
1.4 Why Additive Labeling is Interesting?

The vertices of a graph can be labeled in many different ways. One way to label vertices is with numbers. Just as the edges are assigned the integers that result out of absolute difference of the labels of their respective vertices, the edges can also be assigned the integers that result out of the sum labels of their respective vertices. There are a number of such interesting variations. These variations can also be termed as additive variations. These additive variations include harmonious labeling, felicitous labeling, sequential labeling, elegant labeling, magic labeling etc. We will now see them one by one. Although a vast amount of literature is available concerning these difference and additive labelings, a real motivating factor for this thesis is derived from my Supervisor’s interesting results concerning some of these labelings. Moreover it would be more appropriate to point out them as my real inspirations as it is with him, I have learnt the art of graph labeling and hence has responded to the urge within me to frame the title of this thesis as “Graph labeling and Its additive variations”.

1.5 Different Types of Additive Labeling

Definition 1.5.1 Sequential Labeling (Grace [59])

A graph $G$ with $p$ vertices and $q$ edges is said to have a sequential labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q - 1\}$ is such that when an edge $uv \in E(G)$ is assigned with the label $f(u) + f(v)$ then the resulting edge labels form a finite sequence of distinct successive integers.
Definition 1.5.2 Harmonious Labeling (Graham and Sloane [60])
A graph $G$ with $p$ vertices and $q$ edges is said to have a harmonious labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q - 1\}$ is such that when an edge $uv \in E(G)$ is assigned with the label $f(u) + f(v)$ (mod $q$) then the resulting edge labels form a sequence of distinct successive integers.

Definition 1.5.3 Felicitous Labeling (Lee et. al [88])
A graph $G$ with $p$ vertices and $q$ edges is said to have a felicitous labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q\}$ is such that when an edge $uv \in E(G)$ is assigned the label $f(u) + f(v)$ (mod $q$) then the resulting edge labels form a sequence of distinct successive integers.

Definition 1.5.4 Odd Edge Labeling (Yegnanarayanan et. al [151])
A graph $G$ with $p$ vertices and $q$ edges is said to have an odd edge labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q\}$ is such that when an edge $uv \in E(G)$ is assigned the label $f(u) + f(v)$ then the resulting edge labels can be expressed as the set $\{1, 3, 5, ..., 2q - 1\}$.

Definition 1.5.5 The wreath product $G \star H$ has vertex set $V(G) \times V(H)$ in which $(u_1, v_1)$ is adjacent to $(u_2, v_2)$ whenever $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

Definition 1.5.6 By $H_{n,n}$ we mean the graph with vertex set $V(H_{n,n}) = \{u_1, ..., u_n; v_1, ..., v_n\}$ and Edge set $E(H_{n,n}) = \{u_iv_j : 1 \leq i \leq j \leq n\}$.

Definition 1.5.7 By $mH$, we mean the graph consisting of $m$ disjoint copies of $H$.

Definition 1.5.8 By $P^k_n$, we mean the graph obtained from $P_n$ (the path on $n$
vertices) by joining each pair of vertices at distance \( k \) in \( P_n \).

**Definition 1.5.9** Consider the graph \( C_n \times P_m \). Let \( C_i^i, 1 \leq i \leq m \) denote the \( m \) cycles in the graph \( C_n \times P_m \), corresponding to each vertex \( v_i \) of \( P_m \). Add a new vertex \( v \) and join it to all the vertices of \( C_n^1 \) and \( C_n^m \). Call the resulting graph as \( C_{n,m} \).

**Definition 1.5.10** \( C_{n,m}^+ \) stand for the graph obtained from \( C_n \times P_m \) by taking two new distinct vertices, say, \( u, v \) and joining \( u \) to all the vertices of \( C_n^1 \) and \( v \) to all the vertices of \( C_n^m \).

**Definition 1.5.11** Given a graph \( G \), consider \( 2G \). Let \( u \) be any vertex of one copy of \( G \). Let \( u' \) be the corresponding vertex in the other copy of \( G \). Define \( G_{u,u'}^* = 2G + (uu') \). Let \( G^* \) denote any such graph \( G_{u,u'}^* \) for some pair \( (u, u') \) of corresponding vertices. Define, \((G^*)^0 = G, (G^*)^1 = G^* \) and \((G^*)^m = ((G^*)^{m-1})^*\).

**Definition 1.5.12** For any two disjoint graphs \( G_1 \) and \( G_2 \) with \( u \in V(G_1) \) and \( v \in V(G_2) \), the \( uv \) join of \( G_1 \) and \( G_2 \) denoted \( G_{uv} \) is obtained by joining \( u \) and \( v \) by means of a new edge.

**Definition 1.5.13** The middle graph of \( G \), \( M(G) \), is the graph with \( V(M(G)) = V(G) \cup E(G) \) and where two vertices in \( M(G) \) are adjacent if and only if they are adjacent edges of \( G \), or one is a vertex and another is an edge of \( G \) incident with it.

**Definition 1.5.14** The total graph \( T(G) \) is the graph with vertex set \( V(T(G)) = V(G) \cup E(G) \) and two vertices of \( T(G) \) are adjacent whenever they are neighbors in \( G \).
**Definition 1.5.15** For any two graphs $G$ and $H$, by $(G \circ H)(v)$, we mean the graph obtained by fusing a copy of $H$ at the vertex $v$ of $G$.

**Definition 1.5.16** A subgraph $H$ of a graph $G$ is said to be an even subgraph of $G$, if the degree of every vertex of $H$ is even in $H$.

**Definition 1.5.17** The generalized crown graph $C_m \odot K_{1,n}$ is obtained from $C_m, m \geq 3$ by adding at a vertex of $C_m$, $n$ new edges with $n$ new end vertices.

**Definition 1.5.18** By Helm graph $H(m)$, we mean the graph obtained from $C_m \odot K_{1,1}$, by adding a new vertex and joining it to every vertex of the unique cycle of the crown.

**Definition 1.5.19** The generalized odd $m$-web $W(m, 2k + 1)$ graph consists of $m$ cycles $C_{2k+1}$ together with a set of $2k + 2$ new vertices, whose vertex and edge sets are defined as follows: Set $V(C_i) = \{u^i_0, u^i_1, ..., u^i_{2k}\}, 1 \leq i \leq m$. Then $V(W(m, 2k + 1)) = \{u^i_0, u^i_1, ..., u^i_{2k}\} \cup \{u_0, u^1_{m+1}, u^1_{m+1}, ..., u^m_{2k+1}\}$, and $E(W(m, 2k + 1)) = \cup E(C_i) \cup \{u^i_ju^{i+1}_j : 1 \leq i \leq m, 0 \leq j \leq 2k\} \cup \{uu'_j : 0 \leq j \leq 2k\}$.

### 1.6 Some Motivational Results on Graph Labeling

The following results of Yegnanarayanan on harmonious labeling, felicitous labeling and odd edge labeling which appeared in [146, 150, 151] are the inspirational and motivational factors for me to work on graph labeling.

**Result 1.6.1** The graph $H_{n,n}$ is felicitous.
Result 1.6.2 The graph $P_n \star K_2^c$ is felicitous.

Result 1.6.3 The graph $(K_{1,m} : m)$ is felicitous.

(i) when a) $m \not\equiv 0 \pmod{3}$, or b) $m \equiv 3 \pmod{6}$ or c) $m \equiv 6 \pmod{12}$.

(ii) a) $(K_{1,2n} : m)$ for all $m \geq 1$ and $n \geq 2$ and b) $(K_{1,2l+1} : 2n + 1)$ when $n \geq l$
are felicitous.

Result 1.6.4 The graph $P_n^k$ is felicitous, if

(i) $k = n - 1$ and $n \not\equiv 2 \pmod{4}$ or

(ii) $k = 2t$, $t \geq 1$, $n \geq 3$ and $k < n - 1$.

Result 1.6.5 If $G$ is a graph on $p$ vertices with $1 \leq f(u) \leq p$ for each $u \in V(G)$ and

(i) if for each edge $e = uv \in E(G)$, $k + 1 \leq f^*(e) = f(u) + f(v) \leq q + k$, for some $k \geq 1$,

(ii) $q + k \geq p$ to ensure that the labeling is one-one and

(iii) $f$ is a felicitous labeling for $G$, then the graph $G \lor K_n^c$ is felicitous.

Result 1.6.6 The graph $K_{1,m} \lor K_n^c$ is felicitous.

Result 1.6.7 The graph $(C_3 \times P_m) \lor K_n^c$ is felicitous.

Result 1.6.8 The total graph $T(P_n)$ is harmonious.

Result 1.6.9 The graph $K_{1,n}^*$ is felicitous.

Result 1.6.10 If a graph $G$ admits an odd edge labeling, then $G^*$ is felicitous.
**Result 1.6.11** Suppose $G$ admits an odd edge labeling, then $(G^*)^m$ is felicitous for all $m \geq 1$.

**Result 1.6.12** Each family in the following collection $G = \{G_j\}, \infty \leq j \leq \forall$, of graphs is felicitous.

(i) $G_1 = (C_{4k}^*)^m$ for all $k \geq 2, m \geq 1$.

(ii) $G_2 = ((\text{Amal}(C_{4k}, u_{2k+3}, 2))^*)^m$ for all $m \geq 1, k \geq 2$, where the Amalgamation Amal $(G, u, n)$ for a graph $G$, a vertex $u$ of $G$ and a positive integer $n$ is defined as the graph obtained by concatenating $n$ disjoint copies of $G$ at $u$.

(iii) $G_3 = (K_{2n}^*)^m$ for all $m, n \geq 1$.

(iv) $G_4 = (P_n^*)^m$ for all $m, n \geq 1$.

(v) $G_5 = ((P_{2r} \circ P_{2k+1}(v_0))^*)^m$ for all $r \geq 1$ and $m \geq 1$.

(vi) $G_6 = ((P_{4r+1} \circ P_{2k+2}(v_0))^*)^m$ for all $r \geq 1$ and $m \geq 1$.

(vii) $G_7 = ((P_{4r+3} \circ P_{2k+3}(v_0))^*)^m$ for all $r \geq 1$ and $m \geq 1$.

(viii) $G_8 = ((P_t \times P_n)^*)^m$ for all $m \geq 1, t \geq 1$ and $n \geq 1$.

**Result 1.6.13** The graphs $G_{u,u'} = (K_{1,n} \cup K_{1,m}) + uu'$ where $u$ and $u'$ are either both central vertices or both of them end vertices is felicitous.

**Result 1.6.14** Let $G$ be a harmonious (or felicitous) graph with an even number of edges. Then every even subgraph $G'$ of $G$ contains an even number of odd edges.

**Result 1.6.15** No even graph with $4n + 2$ edges is felicitous.

**Result 1.6.16** If $G$ is an $r$-regular graph on $p$ vertices then $M(G)$ is not felicitous in the following cases:
(i) \( p \equiv 2 \pmod{4} \) and \( r \equiv 2 \pmod{8} \)

(ii) \( p \equiv 2 \pmod{4} \) and \( r \equiv 6 \pmod{8} \)

(iii) \( p \equiv 1 \pmod{4} \) and \( r \equiv 4 \pmod{8} \).

Result 1.6.17 If \( G \) is an \( d \)-regular graph, then \( T(G) \) is not felicitous if

(i) \( |E(G)| \equiv 1 \pmod{2} \) and \( d \equiv 0 \pmod{4} \)

(ii) \( |E(G)| \equiv 2 \pmod{4} \) and \( d \equiv 1 \pmod{2} \).

Result 1.6.18 The following graphs are not felicitous:

(i) \( C_{n,m} \)

(ii) \( C^+_{n,m} \)

(iii) \( C_n \vee K^c_m \) when \( n \equiv (2 \pmod{4}) \) and \( m \equiv 0 \pmod{2} \) and

(iv) \( S_m \vee C_n \) when \( n \equiv 3 \pmod{4}, m \equiv 1 \pmod{2} \).

Result 1.6.19 Suppose \( G_1 \) and \( G_2 \) are any two even graphs, then \( G_2[G_1] \) is not felicitous if

(i) \( |E(G_1)| \equiv 1 \pmod{4}, |V(G_2)| \equiv 1 \pmod{4} \) and \( |E(G_2)| \equiv 1 \pmod{4} \),

(ii) \( |E(G_1)| \equiv 1 \pmod{4}, |V(G_2)| \equiv 3 \pmod{4} \) and \( |E(G_2)| \equiv 3 \pmod{4} \).

Result 1.6.20 \( C_{2k+1} \odot K_{1,n} \) is harmonious.

Result 1.6.21 \( C_{2k} \odot K_{1,n} \) is felicitous.

Result 1.6.22 \( C_{2k} \odot K_{1,1} \) is felicitous.

Result 1.6.23 The generalized odd web \( W(m, 2k + 1) \) are harmonious.
**Result 1.6.24** The generalized even webs are harmonious.

**Result 1.6.25** All helms are felicitous.

**Result 1.6.26** The graph $G_a = C_{2k+1} \cup X$, where $X$ is the set of $r$ chords \{u_1u_{2k-1}, u_1u_4, u_1u_6, ..., u_1u_{2r}\}, 1 \leq r \leq k$ is felicitous.

**Definition 1.6.27 Elegant Labeling** (Chang et. al [39])

A graph $G$ with $p$ vertices and $q$ edges is said to have an elegant labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q\}$ is such that whenever an edge $uv \in E(G)$ is assigned the label $f(u) + f(v) \pmod{q+1}$ then the resulting edge labels are distinct and non-zero.

**Definition 1.6.28 Pseudo-odd-edge Labeling** (Yegnanarayanan et. al [151])

A graph $G$ with $p$ vertices and $q$ edges is said to have a pseudo odd edge labeling $f$, if $f : V(G) \rightarrow \{0, 1, ..., q\}$ is such that whenever an edge $uv \in E(G)$ is assigned the label $f(u) + f(v)$ then the resulting edge labels can be written as $f(E(G)) = \{1, 3, ..., q - 1, q, q + 3, q + 5, ..., 2q - 1\}$.

**Result 1.6.29** Let $G$ be any 2-regular graph with even number of edges. If $G$ is elegant then for any elegant labeling $f$ of $G$, $f(v) \neq 0$ for any $v \in V(G)$.

**Result 1.6.30** Every simple graph is a subgraph of an elegant graph.

**Result 1.6.31** Suppose $G$ is an elegant graph with odd number of edges. Then every even subgraph $H$ of $G$ contains an even number of odd edges.

**Result 1.6.32** Any even graph with $q \equiv 1 \pmod{4}$ is not elegant.

**Result 1.6.33** $C_{4m+1}$ is not elegant.
**Result 1.6.34** The graph $K_{n,m}$ is elegant.

**Result 1.6.35** The graph $T(P_n)$ is elegant for any positive integer $n$.

**Definition 1.6.36** By the bistar $B_{n,n}$, we mean the graph obtained by pasting the centre of $K_{1,n}$ at each end vertex of $K_2$. For a tree $T$ rooted at a vertex of maximum degree, we denote by $< T, m >$ the graph obtained by taking $m$ disjoint copies of $T$ and a new vertex which is made adjacent to the roots of all the $m$ copies of $T$. By $S_m(K_{1,n})$, we mean the $m$-th subdivision graph of $K_{1,n}$ that is, the graph obtained from $n$ disjoint copies of $P_{m+1}$ and a new vertex which is made adjacent to one of the end vertices of each copy of $P_{m+1}$.

**Result 1.6.37** The bistar $B_{n,n}$ is elegant when $n$ is even.

**Result 1.6.38** The graph $< K_{1,n}, 2t >$ is elegant for any $t \geq 1$.

**Result 1.6.39** The graph $S_{n-1}(K_{1,2m})$ is elegant for any positive integers $n$ and $m$.

**Result 1.6.40** The graph $< S_{m-2}(K_{1,n}), 2^k >$ is elegant for all $n, k \geq 1$ and $m \geq 2$.

**Definition 1.6.41** For any positive integer $k$, we define a complete binary tree with $k$ levels, denoted by $T(m, k)$ as follows: $T(m, k)$ contains a special vertex called its apex at level 0; the apex of $T(m, k)$ is adjacent to exactly $m$ vertices which are at level 1. In general, each vertex at level $i (0 \leq i \leq k - 1)$ is adjacent to exactly $m$ vertices at level $i + 1$. Each vertex at level $k - 1$, the last level of $T(m, k)$ is of degree 1. $T(2, k)$ is called the complete binary tree with $k$ levels. In $T(m, k)$, we denote the apex by $u$ and the $i$-th vertex at level $j$ by $x_{i}^{j}; 1 \leq j \leq k - 1, 1 \leq i \leq m^{j}$. 18
Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Fix a vertex, say, $v_1$ of $G$. For each integer $k \geq 1$, let $G_1, G_2, \ldots, G_{m^{k-1}}$ be the $m^{k-1}$ disjoint isomorphic copies of $G$, with $V(G_i) = \{v^i_1, \ldots, v^i_n\}$ where $1 \leq i \leq m^{k-1}$ and for each $j$, $1 \leq j \leq n$, let $v^i_j$ be the isomorphic image of $v_j$ in $G_i$. Now by $G(v_1, m, k)$ we mean the graph obtained by identifying the vertex $v^i_1$ of $G_i$ and the vertex $x^i_{k-1}$ of $T(m, k)$, for each $i$, $1 \leq i \leq m^{k-1}$.

**Result 1.6.42** If $G$ admits an odd-edge labeling $f : V(G) \to \{0, 1, \ldots, q\}$ with $g(v_l) = 0$ for some $v_l \in V(G)$. Then $G(v_l, 2, 2)$ has a pseudo odd-edge labeling.

**Result 1.6.43** If $G$ has an odd-edge labeling, then $G(v_l, 2, k)$ admits a pseudo odd-edge labeling for any $k \geq 2$.

**Result 1.6.44** If $G$ admits an odd edge labeling $f$ with $f(v_l) = 0$ for some $v_l \in V(G)$ then the graph $G(v_l, 2^k, 2)$ is elegant for $k \geq 1$.

### 1.7 Historical Background of Magic Labeling of Graphs

An interesting vertex/edge labeling with numbers is vertex/edge magic. Vertex/edge magic graphs are graphs labeled with numbers in which every vertex(edges) and its incident edges(vertices) add up to the same number. This number is called the common count. It is still unknown what types of graphs are vertex/edge magic and which are not. The following questions have interesting solutions regarding the labeling of vertex/edge-magic graphs: Can sharp bounds be found for the magic number of a given graph? What are some properties of vertex/edge-magic graphs?
with both an even and odd number of vertices?

Sedláček [121] coined a term called the magic labeling driven by the magic square notion in number theory. According to him a magic labeling is a function \( \alpha : E(G) \to R^+ \cup 0 \) with the sums of the edge labels around any vertex in \( G \) is the same. Stewart [128] called magic labeling a supermagic labeling if consecutive integers forms the set of edge labels of \( G \). Afterwards many new related terminologies have cropped up and new results were found. If the evaluation forms an arithmetic progression starting at \( a \) and with difference \( d \), \( d \) a non-negative integer, then the labeling is called an \((a, d)\)-antimagic labeling. If \( d = 0 \) then the labeling turns out to be a magic labeling. Thus, a magic labeling is a special case of an antimagic labeling. A recent survey of magic labeling can be found in the Gallian’s comprehensive dynamic survey [55].

While many researchers studied the properties of various magic labeling, other researchers examined their applications. Kalantari, Khosrovshahi and Mitchell [74, 106] tried to find applications of magic labeling in optimisation theory, especially for the travelling salesmen problem. Baskoro et al [27, 28] proposed a secret sharing scheme construction using edge-magic labeling. The results that have appeared in [32, 33, 140] have suggested that edge-magic total labeling can be thought of for assigning addresses of communications network and radar pulse codes. Lately in [67] a game based on vertex-magic labeling was introduced.

The notion of an antimagic graph was introduced by Hartsfield and Ringel in 1989 [68]. Many variations of antimagic labeling have been studied since this paper. More than 200 papers have been written on magic and antimagic labeling, and there
are still many open problems. In this thesis, we present some new results.

Generally, there are two main basic reasons for developing a theory. First, we may need a new theory to solve a problem. An example of this is graceful labeling that was developed to solve the problem of the decomposition of a complete graph into isomorphic subgraphs. Second, the development of a new theory comes from human curiosity. An example of this is magic labeling. In this thesis, we are mainly driven by the second reason. All those who are attracted by labeling problems in graph theory would aim for getting general results so that they can handle broad classes of graphs, but their actual experience is not very rosy. A reason that can be attributed is the popular “All trees are graceful” Conjecture alone has resisted all attempts over the past few decades. The same is case with other such conjectures also. The fact is they may be true but no clue regarding how to go about proving them. Naturally the third natural question addressed to applications can be answered partially once we realize the scope of applicability of edge magic labeling in the problem of creating a secret sharing scheme as done in [124]. A real motivation for it was derived from the work mentioned in [64]. One can also look to [1, 2, 5, 7, 8, 21, 25–28, 31, 33, 76, 79, 83, 85, 89, 91, 96, 102, 105, 111, 118, 120, 127–129, 134, 138–140, 143, 144, 147–149, 152, 153, 155] for more. As a prelude to Chapters 2, 3 and 4 we give here a few relevant definitions. We adopt the following notation: By \( c(v) \) we mean the common vertex count and by \( c(e) \) we mean the common edge count for all \( v \in V(G) \) and \( e \in E(G) \).
1.8 Different Types of Magic Labeling of Graphs

The following Definitions from 1.8.1 to 1.8.26 were due to Yegnanarayanan [147].

**Definition 1.8.1** A graph $G(p, q)$ is said to be (1,1) edge-magic with the common edge count $k_1$ if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, (p + q)\}$ such that $f(u) + f(v) + f(e) = k_1$ for all $e = (u, v) \in E(G)$.

**Definition 1.8.2** A graph $G(p, q)$ is said to be (1,1) edge-even magic with the common edge count $k_2$ if there exists a bijection $f : V(G) \cup E(G) \to \{0, 2, \ldots, 2(p + q - 1)\}$ such that $f(u) + f(v) + f(e) = k_2$ for all $e = (u, v) \in E(G)$.

**Definition 1.8.3** A graph $G(p, q)$ is said to be (1,1) edge-odd magic with the common edge count $k_3$ if there exists a bijection $f : V(G) \cup E(G) \to \{1, 3, \ldots, 2(p + q) - 1\}$ such that $f(u) + f(v) + f(e) = k_3$ for all $e = (u, v) \in E(G)$.

**Definition 1.8.4** A graph $G(p, q)$ is said to be nice (1,1) edge-magic if it is (1,1) edge magic and if $f(V(G)) = \{1, \ldots, p\}$.

**Definition 1.8.5** A graph $G(p, q)$ is said to be (1,0) edge-magic with the common edge count $k_4$ if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that $f(u) + f(v) = k_4$ for all $e = (u, v) \in E(G)$.

**Definition 1.8.6** A graph $G(p, q)$ is said to be (0,1) edge-magic with the common edge count $k_5$ if there exists a bijection $f : E(G) \to \{1, 2, \ldots, q\}$ such that for all $e \in E(G)$, $f(e) + f(e_0) = k_5$ for all $e_0 \in E(G)$ such that $e$ and $e_0$ are adjacent in $E(G)$.  

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Definition 1.8.7 A graph $G(p, q)$ is said to be (1,1) vertex-magic with the common edge count $k_6$ if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, \ldots, (p+q)\}$ such that for each $u \in V(G)$, $f(u) + \sum_e f(e) = k_6$ for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.8 A graph $G(p, q)$ is said to be (1,0) vertex-magic with the common edge count $k_7$ if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that for each $u \in V(G)$, $f(u) + f(v) = k_7$ for all $v \in V(G)$ and $(u, v) \in E(G)$.

Definition 1.8.9 A graph $G(p, q)$ is said to be (0,1) vertex-magic with the common edge count $k_8$ if there exists a bijection $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ such that for all $u \in V(G)$, $\sum_e f(e) = k_8$ for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.10 A graph $G(p, q)$ is said to be (1,1) edge-antimagic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, (p+q)\}$ such that $f(u) + f(v) + f(e)$ are distinct for all $e = (u, v) \in E(G)$.

Definition 1.8.11 A graph $G(p, q)$ is said to be (1,1) vertex-antimagic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, \ldots, (p+q)\}$ such that $f(u) + \sum_e f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.12 A graph $G(p, q)$ is said to be (1,0) edge-antimagic if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that $f(u) + f(v)$ are distinct for all $e = (u, v) \in E(G)$.

Definition 1.8.13 A graph $G(p, q)$ is said to be (1,0) vertex-antimagic if there exists a bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that for each $u \in V(G)$, $f(u) + f(v)$ are distinct for all $v \in V(G)$ and $(u, v) \in E(G)$. 23
Definition 1.8.14 A graph $G(p, q)$ is said to be (0,1) edge-antimagic if there exists a bijection $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ such that for all $e \in E(G)$, $f(e) + f(e_0)$ are distinct for all $e_0 \in E(G)$ such that $e$ and $e_0$ are adjacent in $E(G)$.

Definition 1.8.15 A graph $G(p, q)$ is said to be (0,1) vertex-antimagic if there exists a bijection $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ such that for all $u \in V(G)$, $\sum_{e} f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.16 A graph $G(p, q)$ is said to be (1,1) vertex-even antimagic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{0, 2, \ldots, 2(p + q - 1)\}$ such that for each $u \in V(G)$, $f(u) + \sum_{e} f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.17 A graph $G(p, q)$ is said to be (1,1) vertex-odd antimagic if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 3, \ldots, 2(p + q) - 1\}$ such that for each $u \in V(G)$, $f(u) + \sum_{e} f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.18 A graph $G(p, q)$ is said to be (0,1) edge-even antimagic if there exists a bijection $f : E(G) \rightarrow \{0, 2, \ldots, 2(q - 1)\}$ such that for each $e \in E(G)$, $f(e) + f(e_0)$ are distinct for all $e_0 \in E(G)$ where $e$ and $e_0$ are adjacent in $G$.

Definition 1.8.19 A graph $G(p, q)$ is said to be (0,1) edge-odd antimagic if there exists a bijection $f : E(G) \rightarrow \{1, 3, \ldots, 2(q - 1)\}$ such that for each $e \in E(G)$, $f(e) + f(e_0)$ are distinct for all $e_0 \in E(G)$ where $e$ and $e_0$ are adjacent in $G$.

Definition 1.8.20 A graph $G(p, q)$ is said to be (0,1) vertex-even antimagic if there exists a bijection $f : E(G) \rightarrow \{0, 2, \ldots, 2(q - 1)\}$ such that for each $u \in V(G)$, $\sum_{e} f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.
Definition 1.8.21  A graph $G(p, q)$ is said to be $(0,1)$ vertex-odd antimagic if there exists a bijection $f : E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ such that for each $u \in V(G)$, $\sum_e f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$.

Definition 1.8.22  A graph $G(p, q)$ is said to be $(1,0)$ vertex-even antimagic if there exists a bijection $f : V(G) \rightarrow \{0, 2, \ldots, 2(p-1)\}$ such that for each $u \in V(G)$, $f(u) + f(v)$ are distinct for all $v \in V(G)$ with $(u, v) \in E(G)$.

Definition 1.8.23  A graph $G(p, q)$ is said to be $(1,0)$ vertex-odd antimagic if there exists a bijection $f : V(G) \rightarrow \{1, 3, \ldots, 2p-1\}$ such that for each $u \in V(G)$, $f(u) + f(v)$ are distinct for all $v \in V(G)$ with $(u, v) \in E(G)$.

Definition 1.8.24  A graph $G(p, q)$ is said to be $(1, 0)$ edge-even antimagic if there exists a bijection $f : V(G) \rightarrow \{0, 2, \ldots, 2(p-1)\}$ such that for each $e = (u, v) \in E(G)$, $f(u) + f(v)$ are distinct.

Definition 1.8.25  A graph $G(p, q)$ is said to be $(1, 0)$ edge-odd antimagic if there exists a bijection $f : V(G) \rightarrow \{1, 3, \ldots, 2p-1\}$ such that for each $e = (u, v) \in E(G)$, $f(u) + f(v)$ are distinct.

Definition 1.8.26  Let $G(p, q)$ be a graph with $p$ vertices and $q$ edges. Let $f$ be a $(1, 1)$ edge-magic labeling of $G$. We call $f$ a nice $(1, 1)$ edge-magic labeling of $G$ if $f(V(G)) = \{1, 2, \ldots, p\}$. We then call $G$ a nice $(1, 1)$ edge-magic graph.

A motivation behind the idea of categorizing magic labeling of graphs into types like $(1,1)$, $(1,0)$ and $(0,1)$ arose as an answer to the inner voice of Yegnanarayanan for bringing uniformity in the nomenclature of different types of magic labeling.
of graphs. Inspired by the work of Sedláček [121], and Graham and Sloane [60], Yegnanarayanan chose to call a magic labeling that involves labeling of vertices and edges as (1,1) where the first 1 stands for the vertex and the second 1 stands for the edge. If any of these elements are not labeled then to indicate their absence he allot the number 0. That is, if the edges are not labeled then the corresponding magic labeling is referred as a (1,0) vertex-magic labeling. Similarly, if the vertices are not labeled then the corresponding magic labeling is referred as a (0,1) edge-magic labeling. Further as a generalization of this nomenclature he calls a magic labeling where all the three elements of a graph namely the vertices, edges and faces, are denoted by (1,1,1), where the first 1 stands for the vertex, the second 1 stands for the edge, and the third 1 stand for the face. For instance, if edges are not labeled then he choose to call such a labeling as a (1,0,1) vertex face-magic labeling. The above said nomenclature, the original idea of Yegnanarayanan was developed in the late 90’s and the same was well acknowledged by his contemporaries like Gallian [55]. Some of the results of this thesis which adopts the above nomenclature has appeared as research papers in journals like Utilitas Mathematica (2001, 2013), and Electronic Notes in Discrete Mathematics (2009) are also acknowledged by Gallian in his revised and latest survey in 2013. Refer [55].