CHAPTER 4
ESTIMATION OF THE PARAMETERS OF LOG-LOGISTIC DISTRIBUTION BASED ON MULTIPLY TYPE I CENSORED SAMPLES

4.1 INTRODUCTION

In survival analysis Type I censoring is very common because of the time restriction or the availability of the experimental materials. The Type I censoring was discussed by Miller (1981), Cox and Oakas (1984) Lawless (1981) and by many other research workers. Mann et al., (1974) discussed about Type I censoring and obtained maximum likelihood estimators (MLE's) of the parameters of the exponential and Weibull distribution. Schneider (1986) discussed about obtaining MLE's of parameters of normal distribution under this censoring. Persson and Rootzen (1977) obtained the estimators of the parameters of normal distribution under Type I censoring. Cohen (1991) discussed about obtaining MLE's of the parameters of normal, log-normal, exponential, extreme value Rayleigh, inverse Gaussian, gamma and Pareto distributions under Type I and progressive censoring. Klein and Moeschberger (1997) have also discussed the Type I censoring. Sirvanci and Yang (1984) have obtained estimates of Weibull parameters under Type I censoring. Piegorsch (1987) studied the interval estimates for two parameter exponential samples subject to Type I censoring. Thiagarajah and Paul (1997) obtained estimates of scale parameter of two parameter exponential distribution based on time censored data. Regal (1980) studied the estimates from Type I singly

It was noticed from the literature available that not much work was done on multiple Type I censoring in the case of log-logistic distribution. Therefore, in this Chapter we have obtained MLE's of the parameters of three parameter log-logistic distribution under multiple Type I censoring scheme.

In Section 2, we have obtained likelihood equations for $\mu$, $\sigma$ and $\alpha$. These likelihood equations cannot be solved explicitly. Therefore, we estimated the parameters numerically by iterative procedure. We also obtained bias, variances and covariances by simulation. For estimating the parameters we considered seven cases which are arising from the situation where one, two or none of the parameters are known.

In Section 3, we obtained second order partial derivatives for obtaining asymptotic covariance matrix of the MLE's ($\hat{\mu}, \hat{\sigma}, \hat{\alpha}$).
In Section 4, we considered numerical example. The sample of size $N = 20$ is generated from log-logistic distribution. Estimated the parameters $\mu$, $\sigma$ and $\alpha$ and also obtained the covariance matrix.

### 4.2 Maximum Likelihood Estimators

Let $N$ be the total sample size and $n$ be the number of items which failed and therefore resulted in completely determined life spans. Suppose the censoring occurred in $k$-stages at times $T_j > T_{j-1}$, $j=1, \ldots, k$ and that $r_j$ surviving items were removed (censored) from further observation at the $j$th stage. Then

$$N = n + \sum_{j=1}^{k} r_j$$

(4.2.1)

We consider the Type I multiple censoring in which $T_j$'s are fixed and the number of survivors at these times are random variables. Let $x_1 < x_2 < \ldots < x_n$ be the life times of the $n$ failed items. The likelihood function for this multiply Type I censored sample can be written as

$$L = C \prod_{j=1}^{k} f(x_j) \prod_{j=1}^{k} \left[ 1 - F(T_j) \right]$$

(4.2.2)

where $C$ is a constant, $f(x)$ and $F(x)$ are pdf and cdf of the underlying distribution (see Cohen (1975)). In the case of log-logistic distribution, we have

$$L = C \prod_{j=1}^{k} \left\{ \frac{\alpha \left( \frac{x_j - \mu}{\sigma} \right)^{\alpha-1}}{\sigma \left[ 1 + \left( \frac{x_j - \mu}{\sigma} \right)^{\alpha} \right]^2} \right\} \prod_{j=1}^{k} \left\{ \frac{1}{1 + \left( \frac{T_j - \mu}{\sigma} \right)^{\alpha}} \right\}$$

(4.2.3)
The log-likelihood function can be written as

\[
\log L = \log C + n \log \alpha - n \log \sigma + (\alpha - 1) \sum_{i=1}^{n} \log \left( \frac{x_i - \mu}{\sigma} \right)
\]

\[
-2 \sum_{i=1}^{n} \log \left[ 1 + \left( \frac{x_i - \mu}{\sigma} \right)^{\alpha} \right] - \sum_{j=1}^{k} r_j \log \left[ 1 + \left( \frac{T_j - \mu}{\sigma} \right)^{\alpha} \right].
\] (4.2.4)

As is seen in Chapter 2, when \( \alpha \leq 1 \), even the mean of the distribution does not exist. We are not considering this case in our study. We consider only the case where the shape parameter \( \alpha > 1 \). The log-likelihood function (4.2.4) can be rewritten as

\[
\log L = \log C + n \log \alpha + N \alpha \log \sigma + (\alpha - 1) \sum_{i=1}^{n} \log(x_i - \mu)
\]

\[
-2 \sum_{i=1}^{n} \log\left( (x_i - \mu)^\alpha + \sigma^\alpha \right) - \sum_{j=1}^{k} r_j \log\left( (T_j - \mu)^\alpha + \sigma^\alpha \right). 
\] (4.2.5)

In order to obtain the MLE's of \( \mu, \sigma \) and \( \alpha \), we first find the following three likelihood equations by differentiating the log-likelihood function (4.2.5) with respect to \( \mu, \sigma \) and \( \alpha \) in turn and equating the derivatives to zero.

\[
\frac{\partial \log L}{\partial \mu} = - (\alpha - 1) \sum_{i=1}^{n} \frac{1}{(x_i - \mu)} + 2\alpha \sum_{i=1}^{n} \frac{(x_i - \mu)^{\alpha-1}}{(x_i - \mu)^\alpha + \sigma^\alpha} + \alpha \sum_{j=1}^{k} \frac{r_j (T_j - \mu)^{\alpha-1}}{(T_j - \mu)^\alpha + \sigma^\alpha} = 0.
\] (4.2.6)

\[
\frac{\partial \log L}{\partial \sigma} = \frac{N \alpha}{\sigma} - 2 \sum_{i=1}^{n} \frac{\alpha \sigma^{\alpha-1}}{(x_i - \mu)^\alpha + \sigma^\alpha} - \alpha \sigma^{\alpha-1} \sum_{j=1}^{k} \frac{r_j}{(T_j - \mu)^\alpha + \sigma^\alpha} = 0.
\]
Equations (4.2.6), (4.2.7) and (4.2.8) can not be solved for $\mu$, $\sigma$ and $\alpha$ explicitly. So, these equations have to be solved numerically using an iterative procedure.

**Simulated Values of Bias, Variances and Covariances**

We estimated the parameters $\mu$, $\sigma$ and $\alpha$ using the Newton-Raphson iterative procedure based on 1000 simulated censored samples of size 20 each from the log-logistic distribution with the conceived values of the parameters $\mu = 0$, $\sigma = 1$ and $\alpha = 3$. For simulating a multiply Type I censored sample, we took $k=2$, $T_1=1$, $T_2=2$, $r_1=2$, $r_2=3$ and the sample size $N=20$. The estimating problem is considered in different cases arising from the situations where one, two or none of the three parameters are known. For estimating the unknown parameters appropriate equations from among the three equations viz. (4.2.6), (4.2.7) and (4.2.8) were used and they were solved numerically for each of the 1000 simulated samples as explained above. In all the cases bias, variances,
mean square errors (MSE's) and covariances of the estimators are computed and they are given in the following tables.

Case 1. $\sigma = 1$ and $\alpha = 3$

<table>
<thead>
<tr>
<th>bias ($\hat{\mu}$)</th>
<th>var ($\hat{\mu}$)</th>
<th>MSE ($\hat{\mu}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06607</td>
<td>0.00772</td>
<td>0.012085</td>
</tr>
</tbody>
</table>

Case 2. $\mu = 0$ and $\alpha = 3$.

<table>
<thead>
<tr>
<th>bias ($\hat{\sigma}$)</th>
<th>var ($\hat{\sigma}$)</th>
<th>MSE ($\hat{\sigma}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.17101</td>
<td>0.018259</td>
<td>0.047503</td>
</tr>
</tbody>
</table>

Case 3. $\mu = 0$ and $\sigma = 1$.

<table>
<thead>
<tr>
<th>bias ($\hat{\alpha}$)</th>
<th>var ($\hat{\alpha}$)</th>
<th>MSE ($\hat{\alpha}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.43921</td>
<td>0.177547</td>
<td>0.370452</td>
</tr>
</tbody>
</table>

Case 4. $\alpha = 3$.

<table>
<thead>
<tr>
<th>bias ($\hat{\mu}$)</th>
<th>bias ($\hat{\sigma}$)</th>
<th>var ($\hat{\mu}$)</th>
<th>var ($\hat{\sigma}$)</th>
<th>MSE ($\hat{\mu}$)</th>
<th>MSE ($\hat{\sigma}$)</th>
<th>cov ($\hat{\mu}, \hat{\sigma}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00382</td>
<td>0.25814</td>
<td>0.027493</td>
<td>0.044201</td>
<td>0.027508</td>
<td>0.110837</td>
<td>-0.00604</td>
</tr>
</tbody>
</table>

Case 5. $\sigma = 1$.

<table>
<thead>
<tr>
<th>bias ($\hat{\mu}$)</th>
<th>bias ($\hat{\alpha}$)</th>
<th>var ($\hat{\mu}$)</th>
<th>var ($\hat{\alpha}$)</th>
<th>MSE ($\hat{\mu}$)</th>
<th>MSE ($\hat{\alpha}$)</th>
<th>cov ($\hat{\mu}, \hat{\alpha}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1756</td>
<td>-0.36506</td>
<td>0.011774</td>
<td>0.196647</td>
<td>0.042609</td>
<td>0.329916</td>
<td>-0.02228</td>
</tr>
</tbody>
</table>
Case 6. $\mu = 0$

<table>
<thead>
<tr>
<th>bias ($\hat{\sigma}$)</th>
<th>bias ($\hat{\alpha}$)</th>
<th>var ($\hat{\sigma}$)</th>
<th>var ($\hat{\alpha}$)</th>
<th>MSE ($\hat{\sigma}$)</th>
<th>MSE ($\hat{\alpha}$)</th>
<th>cov ($\hat{\sigma}, \hat{\alpha}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.17123</td>
<td>-0.35093</td>
<td>0.029123</td>
<td>0.193573</td>
<td>0.058443</td>
<td>0.316725</td>
<td>0.006544</td>
</tr>
</tbody>
</table>

Case 7. All the three parameters are unknown

<table>
<thead>
<tr>
<th>bias ($\hat{\mu}$)</th>
<th>bias ($\hat{\sigma}$)</th>
<th>bias ($\hat{\alpha}$)</th>
<th>var ($\hat{\mu}$)</th>
<th>var ($\hat{\sigma}$)</th>
<th>var ($\hat{\alpha}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.038979</td>
<td>0.144797</td>
<td>-0.352681</td>
<td>0.071514</td>
<td>0.09063</td>
<td>0.528871</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MSE ($\hat{\mu}$)</th>
<th>MSE ($\hat{\sigma}$)</th>
<th>MSE ($\hat{\alpha}$)</th>
<th>cov ($\hat{\mu}, \hat{\sigma}$)</th>
<th>cov ($\hat{\sigma}, \hat{\alpha}$)</th>
<th>cov ($\hat{\mu}, \hat{\alpha}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.073033</td>
<td>0.111596</td>
<td>0.653255</td>
<td>-0.06528</td>
<td>0.175513</td>
<td>-0.14401</td>
</tr>
</tbody>
</table>

4.3 APPROXIMATE ASYMPTOTIC VARIANCES AND COVARIANCES OF THE MLE’S

For obtaining the asymptotic variances and covariances of the MLE’s we need the second order partial derivatives of the log-likelihood function (4.2.5), These derivatives are as given below

$$
\frac{\partial^2 \log L}{\partial \mu^2} = -\left(\alpha - 1\right) \sum_{i=1}^{n} \frac{1}{(x_i - \mu)^2} - 2\alpha \sum_{i=1}^{n} \frac{(x_i - \mu)^{\alpha-2}}{(x_i - \mu)\alpha + \sigma^\alpha} \left\{\alpha - 1\right\} \sigma^\alpha (x_i - \mu)^\alpha
$$

$$
- \alpha \sum_{i=1}^{n} r(T_i - \mu)^{\alpha-2} \left\{T_i - \mu\right\}^{\alpha} \left\{\alpha - 1\right\} \sigma^\alpha - (T_i - \mu)^\alpha
$$

$$
= h_{11} \quad \text{(say)} \quad (4.3.1)
$$
\[
\frac{\partial^2 \log L}{\partial \sigma^2} = -\frac{N \sigma}{\alpha^2} - 2\alpha \sigma^{a-2} \sum_{i=1}^{n} \left[ (\alpha - 1)(x_i - \mu)^a - \sigma^a \right] \left[ (x_i - \mu)^a + \sigma^a \right]^2
\]

\[= \beta_{22} \quad \text{(say)} \quad (4.3.2)\]

\[
\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - 2\sigma \sum_{i=1}^{n} (x_i - \mu)^a \left[ \log \left( \frac{(x_i - \mu)}{\sigma} \right) \right] \left[ (x_i - \mu)^a + \sigma^a \right]^2
\]

\[-\alpha \sigma^{a-2} \sum_{j=1}^{k} r_j (T_j - \mu)^a \left[ \log \left( \frac{(T_j - \mu)}{\sigma} \right) \right] \left[ (T_j - \mu)^a + \sigma^a \right]^2 \]

\[= \beta_{33} \quad \text{(say)} \quad (4.3.3)\]

\[
\frac{\partial^2 \log L}{\partial \sigma \partial \mu} = -2\alpha \sigma^{a-1} \sum_{i=1}^{n} (x_i - \mu)^a \left[ (x_i - \mu)^a + \sigma^a \right]^2
\]

\[-\alpha \sigma^{a-2} \sum_{j=1}^{k} r_j (T_j - \mu)^a \left[ (T_j - \mu)^a + \sigma^a \right]^2 \]

\[= \beta_{12} \quad \text{(say)} \quad (4.3.4)\]

\[
\frac{\partial^2 \log L}{\partial \alpha \partial \mu} = -\sum_{i=1}^{n} \frac{1}{(x_i - \mu)} + 2 \left[ \sum_{i=1}^{n} (x_i - \mu)^a \left[ (x_i - \mu)^a + \sigma^a \right] \right]
\]

\[+ \frac{\alpha \sigma^a}{\sum_{i=1}^{n} (x_i - \mu)^a \left[ \log \left( \frac{(x_i - \mu)}{\sigma} \right) \right] \left[ (x_i - \mu)^a + \sigma^a \right]^2} \]

\[+ \sum_{j=1}^{k} r_j (T_j - \mu)^a \left[ (T_j - \mu)^a + \sigma^a \right]^2 \]

\[= \beta_{13} \quad \text{(say)} \quad (4.3.5)\]
\[
\frac{\partial^2 \log L}{\partial \alpha \partial \sigma} = \frac{N}{\sigma} - 2\sigma^{-1} \sum_{i=1}^{n} \left( (x_i - \mu)^{a} + \sigma^a \right) \left( \sum_{j=1}^{k} r_j [T_j - \mu]^a + \sigma^a \right)^{-1} \\
+ 2 \alpha \sigma^{-1} \sum_{i=1}^{n} (x_i - \mu)^a \left( \log \left( \frac{(x_i - \mu)}{\sigma} \right) \right) \left( (x_i - \mu)^a + \sigma^a \right)^{-2} \\
+ \alpha \sigma^{-1} \sum_{j=1}^{k} r_j [T_j - \mu]^a \left( \log \left( \frac{(T_j - \mu)}{\sigma} \right) \right) \left( [T_j - \mu]^a + \sigma^a \right)^{-2} \\
= h_{23} \quad \text{(say)} \quad (4.3.6)
\]

The derivatives (4.3.1) – (4.3.6) can be written in the matrix form as

\[
H = \begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
& h_{22} & h_{23} \\
& & h_{33}
\end{bmatrix} \quad (4.3.7)
\]

Then, Fisher’s information matrix is given by

\[
I = -E(H) = \left( E(h_{ij}) \right), \quad (4.3.8)
\]

where \( (a_{ij}) \) denotes a matrix whose \((i, j)\)th element is \( a_{ij} \). The asymptotic covariance matrix of the MLE’s \( (\hat{\mu}, \hat{\sigma}, \hat{\alpha}) \) is given by the inverse matrix \( I^{-1} \).

But it is difficult to find the exception of \( H \). However as is suggested by Cohen (1975), for sufficiently large samples, we can use the approximation

\[
E(h_{ij}) \equiv \hat{h}_{ij} \quad (4.3.9)
\]

where, \( \hat{h}_{ij} \) denotes the \( h_{ij} \) with the parameters \( \mu, \sigma \) and \( \alpha \) replaced by their MLE’s \( \hat{\mu}, \hat{\sigma} \) and \( \hat{\alpha} \), respectively. Thus an approximate asymptotic covariance matrix is given by

65
Remark*: Under certain regularity conditions the vector of the MLE's $(\hat{\mu}, \hat{\sigma}, \hat{\alpha})$ has asymptotically a trivariate normal distribution with mean vector $(\hat{\mu}, \hat{\sigma}, \hat{\alpha})$ and covariance matrix $I$, where $I$ is the Fisher information matrix defined by (4.3.8). (see Rao (1975), Kendall and Stuart(1974)).

4.4 AN ILLUSTRATIVE EXAMPLE

A random sample of size 20 was generated from log-logistic distribution with the assumed values of the parameters $\mu = 0$, $\sigma = 1$, and $\alpha = 3$. The data is given below:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.569269</td>
<td>0.632384</td>
<td>0.775697</td>
<td>0.822089</td>
<td>0.879781</td>
</tr>
<tr>
<td>0.988328</td>
<td>1.002587</td>
<td>1.031368</td>
<td>1.039183</td>
<td>1.117048</td>
</tr>
<tr>
<td>1.123476</td>
<td>1.293044</td>
<td>1.327516</td>
<td>1.339086</td>
<td>1.446944</td>
</tr>
<tr>
<td>1.551937</td>
<td>1.936318</td>
<td>2.199744</td>
<td>2.938853</td>
<td>3.945368</td>
</tr>
</tbody>
</table>

This sample was subjected to Type I censoring at $k = 2$ stages with $T_1 = 1$, $r_1 = 2$ and $T_2 = 2$, $r_2 = 3$. Thus here $N = 20$ and $n = 15$. The maximum likelihood estimators (MLE's) of the parameters $\mu$, $\sigma$, and $\alpha$ computed by the Newton-Raphson iterative procedure are

$\hat{\mu} = 0.156$, $\hat{\sigma} = 1.079$, and $\hat{\alpha} = 3.518$. 

(4.3.10)
The second order partial derivatives evaluated at these MLE's are

\[
\hat{h}_{11} = \left( \frac{\partial^2 \log L}{\partial \mu^2} \right)_{\hat{\theta}} = -75.274, \quad \hat{h}_{22} = \left( \frac{\partial^2 \log L}{\partial \sigma^2} \right)_{\hat{\theta}} = -65.066,
\]

\[
\hat{h}_{33} = \left( \frac{\partial^2 \log L}{\partial \alpha^2} \right)_{\hat{\theta}} = -1.811, \quad \hat{h}_{12} = \left( \frac{\partial^2 \log L}{\partial \sigma \partial \mu} \right)_{\hat{\theta}} = -71.917,
\]

\[
\hat{h}_{13} = \left( \frac{\partial^2 \log L}{\partial \alpha \partial \mu} \right)_{\hat{\theta}} = -53.585, \quad \hat{h}_{23} = \left( \frac{\partial^2 \log L}{\partial \alpha \partial \sigma} \right)_{\hat{\theta}} = 11.122,
\]

where \( \theta = (\mu, \sigma, \alpha)' \) and \( \hat{\theta} \) is the corresponding vector of the MLE's. Thus from (4.3.10) an approximate asymptotic covariance matrix is given by

\[
(-\hat{H})^{-1} = \begin{bmatrix}
2.077 \times 10^{-5} & 2.572 \times 10^{-3} & 0.015 \\
0.002572 & 9.686 \times 10^{-3} & -0.017 \\
0.015 & 9.714 \times 10^{-4} & 0.00097
\end{bmatrix}
\]

Table 4.4.1

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( \hat{\mu} )</th>
<th>( \hat{\sigma} )</th>
<th>( \hat{\alpha} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population values</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>MLE</td>
<td>0.156</td>
<td>1.079</td>
<td>3.518</td>
</tr>
<tr>
<td>Variance</td>
<td>0.000021</td>
<td>0.009686</td>
<td>0.00097</td>
</tr>
</tbody>
</table>

\[ \text{cov} (\hat{\mu}, \hat{\sigma}) = 0.002572 \quad \text{cov} (\hat{\mu}, \hat{\alpha}) = 0.015 \quad \text{cov} (\hat{\sigma}, \hat{\alpha}) = -0.017 \]

67