3.1 INTRODUCTION

The maximum Likelihood estimation of the parameters of log-logistic distribution, based on multiply Type-II censored sample was considered in the previous chapter. It was noticed that, it was not possible to get an explicit expression of MLE even for a single parameter. Hence it is desirable to develop approximations to the maximum likelihood method of estimation, which would provide explicit solutions, at least for one or two parameters of the distribution.

The approximate maximum likelihood estimation method has been developed by Balakrishnan(1989 a, b, and 1990 a, b), Balakrishnan and Varadan (1991), Balakrishnan and Wong (1991). They have explained this method of estimation by considering the Rayleigh, normal, logistic, extreme value, Type I generalized logistic and half-logistic distributions. They have considered for all these distributions Type II censored sample. These are also discussed in Balakrishnan and Cohen(1991).

Balakrishnan, et. al., (1995) have obtained approximate maximum likelihood estimators (AMLE's) of the location and scale parameters of extreme value distribution, based on multiply Type II censored samples. Fei, et. al., (1995) have also considered the same problem and have obtained
similar estimators for parameters of the Weibull distribution and extreme value distribution under multiple Type II censoring.

In this Chapter, we obtained the AMLE's of the parameters of the log-logistic distribution, under multiple Type II censoring scheme, by approximating the likelihood equations appropriately.

Let us consider an experiment in which $n$-components are put on test simultaneously and the failure times of these components are recorded. Here we have considered Type II multiple censoring with odd number $(2k-1)$, $k \geq 2$, of subsets of order statistics, out of which all odd numbered subsets are censored. We assume that the first $r_1$ observations are censored, $r_1 + 1$ to $r_2$ observed, $r_2 + 1$ to $r_3$ censored, $r_3 + 1$ to $r_4$ observed and so on and also we assume that the last subgroup $r_{2k-2} + 1$ to $r_{2k-1} = n$ is censored. The multiply Type II censored sample as explained in (2.1.3) be available from log-logistic distribution with pdf and cdf given by (2.1.1) and (2.1.2) respectively.

In Section 2 we have obtained expressions for the AMLE's of $\mu$ and $\sigma$ in explicit forms, when $\alpha$ is known, under the multiple Type II censoring scheme. It was noticed that only in the case where the shape parameter $\alpha$ is known, we get the explicit solutions of the approximate likelihood equations for $\mu$ and $\sigma$. However, the approximate likelihood equations are given for all the three parameters, for obtaining the AMLE's numerically by an iterative procedure.
Section 3, is concerned with an approximate asymptotic covariance matrix of the AMLE's for large samples. We have obtained the second order partial derivatives from the first order partial derivatives used in finding approximate likelihood equations in Section 2, in order to determine approximate information matrix. The inverse of this matrix is used to obtain an approximate asymptotic covariance matrix in the case of large sample (see Cohen (1975)).

In Section 4, we have obtained the AMLE's under four important censoring schemes, which are the special cases of the multiple Type II censoring scheme, and which are oftenly used in practice. These censoring schemes are: i) double censoring, ii) right censoring, iii) left censoring and iv) middle censoring. If it is assumed that out of the three parameters μ, σ and α, only one, only two or all the three parameters are unknown, then there will be seven different cases of estimation problem. All these seven cases are considered under each of the above mentioned four censoring schemes. As it is noted earlier, only when the shape parameter α is known, it is possible to get the AMLE's of μ and σ in explicit forms; so, this case is considered separately, other cases the AMLE's are determined by solving the approximate likelihood equations numerically, using Newton-Raphson iterative procedure.

There does not exist any analytical method for finding the bias, variances and covariances of these AMLE's in the case of small samples. So,
Monte-Carlo simulation was carried out to find these characteristics of the estimators, and these results are presented in Chapter 6.

3.2. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATORS OF LOCATION, SCALE AND SHAPE PARAMETERS

Here we want to estimate the location parameter \( \mu \), scale parameter \( \sigma \) and shape parameter \( \alpha \) by approximate maximum likelihood method, under multiply Type II censored sample. The log-likelihood function for the standard log-logistic distribution is given by (2.2.5). The partial derivatives of log \( L \) with respect to \( \mu \), \( \sigma \) and \( \alpha \) are given respectively, by

\[
\frac{\partial \log L}{\partial \mu} = - \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=r_{i-1}+1}^{r_{i}} \left[ \frac{g(y_j; \alpha)}{g(y_j; \alpha)} \right] - \frac{1}{\sigma} \sum_{i=2}^{r_k} \left( r_{2i-1} - r_{2i-2} \right)
\]

\[
\frac{\partial \log L}{\partial \sigma} = \frac{1}{\sigma} \sum_{i=1}^{k} \sum_{j=r_{i-1}+1}^{r_{i}} \left[ \frac{y_j g(y_j; \alpha)}{g(y_j; \alpha)} \right] - \frac{A}{\sigma} - \frac{1}{\sigma} \sum_{i=2}^{r_k} \left( r_{2i-1} - r_{2i-2} \right) \left[ \frac{y_{r_{2i-1}, i} g(y_{r_{2i-1}, i}; \alpha) - y_{r_{2i-2}, i} g(y_{r_{2i-2}, i}; \alpha)}{G(y_{r_{2i-1}, i}; \alpha) - G(y_{r_{2i-2}, i}; \alpha)} \right]
\]

\[
\frac{\partial \log L}{\partial \alpha} \left[ \frac{y_{r_{2i}, i} g(y_{r_{2i}, i}; \alpha)}{G(y_{r_{2i}, i}; \alpha)} \right] + \frac{(n-r_{2k-2})}{\sigma} \left[ \frac{y_{r_{2k-2}, i} g(y_{r_{2k-2}, i}; \alpha)}{G(y_{r_{2k-2}, i}; \alpha)} \right]
\]

(3.2.1)

(3.2.2)
\[ \frac{\partial \log L}{\partial \alpha} = \frac{A}{\alpha} + \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{r_{2i}} \left( 1 - 2G(y_j; \alpha) \right) \log y_j \]

\[ + \frac{1}{\alpha} \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \left[ g(y_{b_{2i-1}}; \alpha) y_{b_{2i}} \log y_{b_{2i+1}} \right. \]

\[ - g(y_{b_{2i-2}}; \alpha) y_{b_{2i-2}} \log y_{b_{2i-2}} \left[ \left( G(y_{b_{2i-1}}; \alpha) - G(y_{b_{2i-2}}; \alpha) \right) \right]^2 \]

\[ + r_1 G(y_{n+1}; \alpha) \log y_{n+1} - (n - r_{2k-2}) G(y_{t_{1k-2}}; \alpha) \log y_{t_{1k-2}}, \quad (3.2.3) \]

where \( A = \sum_{i=1}^{k-1} \left( r_{2i} - r_{2i-1} \right) \).

As it is noted in Chapter 2, none of these likelihood equations can be solved explicitly; so, we go for AMLE's by finding the approximate likelihood equations. To obtain the approximate likelihood equations first we approximate the following functions by expanding them in Taylor's series (see Balakrishnan et. al., (1995)):

\[ k_1(y_j) = \frac{g'(y_j; \alpha)}{g(y_j; \alpha)}, \quad k_2(y_{n+1}) = \frac{g(y_{n+2}; \alpha)}{G(y_{n+2}; \alpha)} \]

\[ k_1(y_{n+1}) = \frac{g(y_{n+1}; \alpha)}{G(y_{n+1}; \alpha)}, \quad G(y_j; \alpha) \]

\[ h_1(y_{b_{2i-1}}, y_{b_{2i+1}}) = \frac{g(y_{b_{2i+1}}; \alpha)}{G(y_{b_{2i+1}}; \alpha) - G(y_{b_{2i-2}}; \alpha)} \]

and

\[ h_2(y_{b_{2i}}, y_{b_{2i+1}}) = \frac{g(y_{b_{2i+1}}; \alpha)}{G(y_{b_{2i+1}}; \alpha) - G(y_{b_{2i-2}}; \alpha)} . \]
In this connection we note the following

\[ k_i(y_i) = \frac{g(y_i; \alpha)}{g(y_i; \alpha)} = \frac{1}{y_i} \left[ \alpha - 1 - 2\alpha G(y_i; \alpha) \right], \]  \hspace{1cm} (3.2.4)

\[ k'_i(y_i) = \frac{1}{y_i} \left[ -2\alpha^2 G(y_i; \alpha) G(y_i; \alpha) - \alpha + 1 + 2\alpha G(y_i; \alpha) \right]. \]  \hspace{1cm} (3.2.5)

Since

\[ y_i g(y_i; \alpha) = \alpha G(y_i; \alpha) \overline{G}(y_i; \alpha), \]  \hspace{1cm} (3.2.6)

we have

\[ k_2(y_i) = \frac{g(y_i; \alpha)}{G(y_i; \alpha)} = \frac{\alpha G(y_i; \alpha)}{y_i}. \]  \hspace{1cm} (3.2.7)

Hence,

\[ k'_2(y_i) = \frac{\alpha}{y_i^2} \left[ y_i g(y_i; \alpha) - G(y_i; \alpha) \right] \]

\[ = \frac{\alpha G(y_i; \alpha)}{y_i^2} \left[ \alpha \overline{G}(y_i; \alpha) - 1 \right]. \]  \hspace{1cm} (3.2.8)

From (3.2.6), we have

\[ k_3(y_i) = \frac{g(y_i; \alpha)}{G(y_i; \alpha)} = \frac{\alpha \overline{G}(y_i; \alpha)}{y_i}. \]  \hspace{1cm} (3.2.9)

Hence,

\[ k'_3(y_i) = \frac{\alpha}{y_i^2} \left[ y_i g(y_i; \alpha) - \overline{G}(y_i; \alpha) \right] \]

\[ = -\frac{\alpha \overline{G}(y_i; \alpha)}{y_i^2} \left[ \alpha G(y_i; \alpha) + 1 \right]. \]  \hspace{1cm} (3.2.10)

Now,

\[ h_i(y_i, y_j) = \frac{g(y_j; \alpha)}{G(y_j; \alpha) - G(y_i; \alpha)}, \quad i < j \]
\[
\frac{\partial}{\partial y_i} h_i(y_i, y_j) = \frac{g(y_i; \alpha)g(y_j; \alpha)}{[G(y_i; \alpha) - G(y_j; \alpha)]^2} \tag{3.2.11}
\]

\[
\frac{\partial}{\partial y_j} h_i(y_i, y_j) = \frac{g'(y_i; \alpha) [G(y_i; \alpha) - G(y_j; \alpha)] - g'(y_j; \alpha)}{[G(y_j; \alpha) - G(y_i; \alpha)]^2} \tag{3.2.12}
\]

\[
h_2(y_i, y_j) = \frac{g(y_i; \alpha)}{G(y_i; \alpha) - G(y_j; \alpha)} \tag{3.2.13}
\]

\[
\frac{\partial}{\partial y_i} h_2(y_i, y_j) = \frac{g'(y_i; \alpha) [G(y_i; \alpha) - G(y_j; \alpha)] + g^2(y_i; \alpha)}{[G(y_j; \alpha) - G(y_i; \alpha)]^3} \tag{3.2.14}
\]

\[
\frac{\partial}{\partial y_j} h_2(y_i, y_j) = \frac{g^2(y_i; \alpha)}{[G(y_j; \alpha) - G(y_i; \alpha)]^3} \tag{3.2.15}
\]

Let \( p_j = \frac{j}{n+1}, \quad q_j = 1 - p_j. \)

Then, it can be easily seen that

\[
G^{-1}(p_j) = \xi_j = (p_j/q_j)^\alpha \tag{3.2.16}
\]

is quantile of order \( p_j \) of the cdf \( G \), where \( G \) is standard log-logistic cdf with the
shape parameter \( \alpha \). Expanding the functions \( k_1(y_j), k_2(y_j), k_3(y_j) \) and \( G(y_j; \alpha) \) in
Taylor’s series around the point \( \xi_j = G^{-1}(p_j) \), we get the following
approximations of these functions.

\[
k_1(y_j) = \frac{g'(y_j; \alpha)}{g(y_j; \alpha)}
\]

\[
= \frac{1}{\xi_j} (\alpha - 1 - 2\alpha p_j) + (y_j - \xi_j) \frac{1}{\xi_j^2} (-2\alpha^2 p_j q_j - \alpha - 1 + 2\alpha p_j),
\]

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which follows from (3.2.4) and (3.2.5),

\[ \frac{2}{\xi_j} (\alpha - 1 - 2\alpha p_j + \alpha^2 q_j) - \frac{1}{\xi_j^2} (\alpha - 1 - 2\alpha p_j + 2\alpha^2 q_j)y_j \]

\[ = \frac{2}{\xi_j} (\alpha p_j + 1)(\alpha q_j - 1) - \frac{1}{\xi_j^2} [(\alpha p_j + 1)(\alpha q_j - 1) + \alpha^2 q_j y_j] \]

\[ \alpha_j = \frac{a}{\xi_j} - \beta_j y_j, \quad (3.2.17) \]

where

\[ \alpha_j = \frac{2}{\xi_j} (\alpha p_j + 1)(\alpha q_j - 1) \quad (3.2.18) \]

and

\[ \beta_j = \frac{1}{\xi_j^2} [\alpha p_j + 1)(\alpha q_j - 1) + \alpha^2 q_j y_j]. \quad (3.2.19) \]

From (3.2.7) and (3.2.8) we have

\[ k_2(y_j) = \frac{g(y_j; \alpha)}{G(y_j; \alpha)} \]

\[ = \frac{\alpha p_j}{\xi_j} + (y_j - \xi_j) \frac{\alpha p_j}{\xi_j^2} (\alpha q_j - 1) \]

\[ = \frac{\alpha p_j}{\xi_j} (2 - \alpha q_j) + \frac{\alpha p_j}{\xi_j^2} (\alpha q_j - 1) y_j \]

\[ = \epsilon_j + \delta_j y_j, \quad (3.2.20) \]

where

\[ \epsilon_j = \frac{\alpha p_j}{\xi_j} (2 - \alpha q_j) \quad (3.2.21) \]

and

\[ \delta_j = \frac{\alpha p_j}{\xi_j} (\alpha q_j - 1). \quad (3.2.22) \]
Similarly, from (3.2.9) and (3.2.10)

\[ k_3(y_j) = \frac{g(y_j; \alpha)}{G(y_j; \alpha)} \]

\[ \equiv \frac{\alpha q_j}{\xi_j} - \frac{\alpha q_j}{\xi_j} (\alpha q_j + 1) \]

\[ = \frac{\alpha q_j}{\xi_j} (\alpha p_j + 2) - \frac{\alpha q_j}{\xi_j} (\alpha p_j + 1) y_j \]

\[ = \epsilon_j + \delta_j y_j \quad (3.2.23) \]

where

\[ \epsilon_j = \frac{\alpha q_j}{\xi_j} (\alpha p_j + 2) \quad (3.2.24) \]

and

\[ \delta_j = -\frac{\alpha q_j}{\xi_j} (\alpha p_j + 1). \quad (3.2.25) \]

And from (3.2.6)

\[ G(y_j; \alpha) \equiv G(\xi_j) + (y_j - \xi_j) g(\xi_j) \]

\[ = p_j + (y_j - \xi_j) \frac{\alpha p_j q_j}{\xi_j} \]

\[ = p_j (1 + \alpha q_j) + \left( \frac{\alpha p_j q_j}{\xi_j} \right) y_j \]

\[ = A_j + B_j y_j, \quad (3.2.26) \]

where

\[ A_j = p_j (1 - \alpha q_j) \quad (3.2.27) \]
and

\[ B_j = \frac{\alpha p_j q_j}{\xi_j}. \]  

(3.2.28)

Expanding the functions \( h_1(y_i, y_j) \) and \( h_2(y_i, y_j) \) around the point \((\xi_i, \xi_j)\) in bivariate Taylor's series, we obtain the following approximations of these functions from (3.2.4), (3.2.6), (3.2.11) and (3.2.12),

\[
\begin{align*}
    h_1(y_i, y_j) &= \frac{g(y_j; \alpha)}{G(y_j; \alpha) - G(y_i; \alpha)}, \quad i < j \\
    &= h_1(\xi_i, \xi_j) + (y_i - \xi_i) \left( \frac{\partial h_1(y_i, y_j)}{\partial y_i} \right)_{\xi_i, \xi_j} + (y_j - \xi_j) \left( \frac{\partial h_1(y_i, y_j)}{\partial y_j} \right)_{\xi_i, \xi_j} \\
    &= \frac{\alpha p_i q_i}{\xi_j(p_j - p_i)} + (y_i - \xi_i) \frac{\alpha^2 p_i q_i q_j}{\xi_i \xi_j (p_j - p_i)^2} \\
    &\quad + (y_j - \xi_j) \frac{\alpha p_i q_i}{\xi_j (p_j - p_i)} \left[ \frac{1}{\xi_j} (\alpha - 1 - 2\alpha p_j)(p_j - p_i) - \frac{\alpha p_i q_j}{\xi_j} \right] \\
    &= \left[ \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)} - \frac{\alpha^2 p_i q_i q_j}{\xi_j (p_j - p_i)^2} + \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)} \left\{ \alpha p_j + (1 + \alpha p_j)(p_j - p_i) \right\} \right] \\
    &\quad + \left[ \frac{\alpha^2 p_i q_i q_j}{\xi_j (p_j - p_i)^2} \right] y_j - \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)^2} \left[ \alpha p_j + (1 + \alpha p_j)(p_j - p_i) \right] y_j. \\
\end{align*}
\]

(3.2.29)

Similarly from (3.2.4), (3.2.6), (3.2.13) and (3.2.14) we have

\[
\begin{align*}
    h_2(y_i, y_j) &= \frac{g(y_i; \alpha)}{G(y_j; \alpha) - G(y_i; \alpha)}, \quad i < j \\
    &= \frac{\alpha p_i q_i}{\xi_j (p_j - p_i)} + (y_i - \xi_i) \frac{\alpha^2 p_i q_i q_j}{\xi_i \xi_j (p_j - p_i)^2} \\
    &\quad + (y_j - \xi_j) \frac{\alpha p_i q_i}{\xi_j (p_j - p_i)} \left[ \frac{1}{\xi_j} (\alpha - 1 - 2\alpha p_j)(p_j - p_i) - \frac{\alpha p_i q_j}{\xi_j} \right] \\
    &= \left[ \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)} - \frac{\alpha^2 p_i q_i q_j}{\xi_j (p_j - p_i)^2} + \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)} \left\{ \alpha p_j + (1 + \alpha p_j)(p_j - p_i) \right\} \right] \\
    &\quad + \left[ \frac{\alpha^2 p_i q_i q_j}{\xi_j (p_j - p_i)^2} \right] y_j - \frac{\alpha p_i q_j}{\xi_j (p_j - p_i)^2} \left[ \alpha p_j + (1 + \alpha p_j)(p_j - p_i) \right] y_j. \\
\end{align*}
\]
\equiv h_2(\xi_i, \xi_j) + (y_i - \xi_i) \left( \frac{\partial h_2(y_i, y_j)}{\partial y_i} \right)_\xi_i \\
+ (y_j - \xi_j) \left( \frac{\partial h_2(y_i, y_j)}{\partial y_j} \right)_\xi_j \\
= \left[ \frac{\alpha p_i q_i}{\xi_i(p_j - p_i)^2} - \frac{\alpha p_i q_i}{\xi_i(p_j - p_i)^2} \{ \alpha p_i q_i - (1 + \alpha p_i)(p_j - p_i) \} \right] y_i \\
+ \frac{\alpha^2 p_i p_i q_i q_j}{\xi_j(p_j - p_i)^2} + \frac{\alpha p_i q_i}{\xi_j(p_j - p_i)^2} \{ \alpha p_i q_i - (1 + \alpha p_i)(p_j - p_i) \} y_i \\
- \left[ \frac{\alpha^2 p_i q_i q_i q_j}{\xi_i \xi_j(p_j - p_i)^2} \right] y_j . \quad (3.2.30)

Using these approximations in the likelihood equations (3.2.1), (3.2.2) and (3.2.3), we obtain the corresponding approximate likelihood equations. For brevity we use the following notations.

\eta_{ij} = \frac{\alpha^2 p_{ij} q_{ij} q_{ij+1}}{\xi_{ij}(p_{ij+1} - p_{ij})^2} \quad (3.2.31)

\eta_{2i} = -\frac{\alpha p_{ij} q_{ij+1}}{\xi_{ij}(p_{ij+1} - p_{ij})^2} \left[ \alpha p_{ij} q_{ij+1} + (1 + \alpha p_{ij})(p_{ij+1} - p_{ij}) \right] \quad (3.2.32)

\eta_{n1} = \frac{\alpha p_{n1} q_{n1+1}}{\xi_{n1}(p_{n1+1} - p_{n1})} - \eta_{11} - \eta_{n2} \xi_{n1+1} \quad (3.2.33)

\eta_{n1} = \frac{\alpha p_{n1} q_{n1+1}}{\xi_{n1}(p_{n1+1} - p_{n1})} \left[ \alpha p_{n1} q_{n1+1} - (1 + \alpha p_{n1})(p_{n1+1} - p_{n1}) \right] \quad (3.2.34)

\eta_{22} = -\frac{\alpha p_{n2} q_{n2+2}}{\xi_{n2}(p_{n2+1} - p_{n2+2})} \quad (3.2.35)
\[ \eta_{ni}^* = \frac{\alpha_{pi_2}^* e_{q_{i_2}}^*}{\xi_{r_{i_1}}^* (p_{r_{i_1,i_1}}^* - p_{r_{i_1,i_2}}^* - \eta_{ni}^* \xi_{r_{i_1,i_1}}^* - \eta_{ni}^* \xi_{r_{i_1,i_2}}^*)} \]  

(3.2.36)

It may be noted that

\[ \eta_{2i}^* = -\eta_{ni} . \]  

(3.2.37)

Then we have

\[ -\sigma \left( \frac{\partial \log \mu}{\partial \mu} \right) = -\sigma \left( \frac{\partial \log \mu^*}{\partial \mu} \right) \]

\[ = \sum_{i=1}^{k-1} \sum_{j=1}^{p_{i_1,i_1}} (\alpha_j - \beta_j y_j) + \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) \]

\[ \cdot \left[ (\eta_{ni} - \eta_{ni}^*) + (\eta_{ni} - \eta_{ni}^*) y_{2i-2} + (\eta_{ni} - \eta_{ni}^*) y_{r_{i_1,i_1}^*} \right] \]

\[ + r_i (\epsilon_{i_1}^* + \delta_{i_1}^* y_{i_1}^*) - (n - r_{2k-2}) (\epsilon_{i_2}^* + \delta_{i_2}^* y_{i_2}) \]

\[ = 0. \]  

(3.2.38)

\[ -\sigma \left( \frac{\partial \log L}{\partial \sigma} \right) = -\sigma \left( \frac{\partial \log L^*}{\partial \sigma} \right) \]

\[ = A + \sum_{i=1}^{k-1} \sum_{j=1}^{p_{i_1,i_1}} y_j (\alpha_j - \beta_j y_j) \]

\[ + \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) [y_{r_{i_1,i_1}^*} (\eta_{ni}^* + \eta_{ni}^* y_{2i-2} + \eta_{ni}^* y_{r_{i_1,i_1}^*}) \]

\[ - y_{r_{i_1,i_1}^*} (\eta_{ni}^* + \eta_{ni}^* y_{2i-2} + \eta_{ni}^* y_{r_{i_1,i_1}^*})] \]

\[ + \epsilon_{i_1}^* y_{i_1}^* (\epsilon_{i_1}^* + \delta_{i_1}^* y_{i_1}^*) \]

\[ - (n - r_{2k-2}) y_{i_2} (\epsilon_{i_2} + \delta_{i_2} y_{i_2}) \]

\[ = 0 , \]  

(3.2.39)
with \[ A = \sum_{i=1}^{k-1} (r_{2i} - r_{2i-1}) . \]

\[
\alpha \left( \frac{\partial \log L}{\partial \alpha} \right) = \alpha \left( \frac{\partial \log L^*}{\partial \alpha} \right)
\]

\[
= A + \alpha \sum_{i=1}^{k-1} \sum_{j=r_{2i+1}}^{r_{2i+2}} (\log y_j) \left[ 1 - 2(A_j + B_j y_j) \right]
\]

\[
+ \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) \left[ (\log y_{r_{2i-1}+1}) (\eta_{2i} + \eta_{2i-2} y_{2i-2}) + \eta_{2i+1} y_{2i+1} ) \right]
\]

\[
- \eta_{2i-2} (\log y_{r_{2i-1}}) (\eta_{2i-2} + \eta_{2i-1} y_{2i-1} ) ) \frac{1}{2} \left( A_{r_{2i-1}} - B_{r_{2i-1}} y_{2i-1} \right)
\]

\[
+ \frac{1}{2} \left( \eta_{2i+1} (\log y_{r_{2i+1}}) (A_{r_{2i+1}} + B_{r_{2i+1}} y_{2i+1} ) \right)
\]

\[
= 0 , \quad (3.2.40)
\]

where \( \bar{A}_i = 1 - A_i \). \quad (3.2.41)

The three simultaneous equations (3.2.38), (3.2.39) and (3.2.40) can not be solved explicitly. They have to be solved numerically by iterative procedure. However, if shape parameter \( \alpha \) is assumed to be known, then the equations (3.2.38) and (3.2.39) can be solved explicitly for \( \mu \) and \( \sigma \).

**AMLE's of location and scale parameters when shape parameter \( \alpha \) is known.**

Here we have obtained AMLE's of \( \mu \) and \( \sigma \) when \( \alpha = \alpha_0 \) known.

Letting

\[
\delta_{1a} = \eta_{1a} - \eta_{1a}^* , \; \delta_{2a} = \eta_{2a} - \eta_{2a}^* , \; \delta_{3a} = \eta_{3a} - \eta_{3a}^* 
\]

and \( y_j = (x_j - \mu)/\sigma \) in (3.2.38) and then multiplying it by \( \sigma \), we obtain
\[-\sigma\left(\frac{\partial \log L^*}{\partial \mu}\right) = \sigma \left[ \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \alpha_j + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \delta_{\theta_0} + r_i \epsilon_{\theta_{i+1}} \right. \]

\[-(n - r_{2k-2}) \epsilon_{\theta_{k-2}} \bigg] + \left[ -\sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \beta_j x_j + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \right] \]

\[\left( \delta_{\theta_{k-1}} \right) + r_i \delta_{\theta_{i+1}} x_{i+1} - (n - r_{2k-2}) \delta_{\theta_{k-2}} x_{\theta_{k-2}} \bigg] \]

\[+ \mu \left[ \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \beta_j x_j - \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \left( \delta_{\theta_{k-1}} + \delta_{\theta_i} \right) \right] \]

\[-r_i \delta_{\theta_{i+1}} x_{i+1} + (n - r_{2k-2}) \delta_{\theta_{k-2}} x_{\theta_{k-2}} \bigg] \]

\[= m(C\sigma - \beta + \mu) \]

\[= 0, \quad \text{(3.2.44)}\]

where

\[m = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \beta_j - \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \left( \delta_{\theta_{k-1}} + \delta_{\theta_i} \right) - (n - r_{2k-2}) \delta_{\theta_{k-2}} \quad \text{(3.2.45)}\]

\[mB = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \beta_j x_j - \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \left( \delta_{\theta_{k-1}} x_{\theta_{k-1}} + \delta_{\theta_i} x_{\theta_i} \right) \]

\[= -r_i \delta_{\theta_{i+1}} x_{i+1} + (n - r_{2k-2}) \delta_{\theta_{k-2}} x_{\theta_{k-2}} \quad \text{(3.2.46)}\]

\[mC = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{k} \alpha_j + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \delta_{\theta_0} + r_i \epsilon_{\theta_{i+1}} - (n - r_{2k-2}) \epsilon_{\theta_{k-2}} \quad \text{(3.2.47)}\]

From (3.2.44) we obtain

\[\mu = B - C \sigma. \quad \text{(3.2.48)}\]
Letting $y_i = \frac{x_i - \mu}{\sigma}$ in (3.2.39) and then multiplying it by $\sigma^2$, we obtain

$$-\sigma^2 \left( \frac{\partial \log L^*}{\partial \sigma} \right) = \Lambda \sigma^2 + \sum_{i=1}^{k-1} \sum_{j=r_{2i}+1}^{r_{2i+1}} (x_j - \mu) \left[ \alpha_i \sigma - \beta_i (x_j - \mu) \right]$$

$$+ \sum_{i=2}^{k-2} \left( r_{2i-1} - r_{2i-2} \right) \left\{ (x_{r_{2i-1}} - \mu) \left[ \eta_{0i} \sigma + \eta_{0i} (x_{r_{2i-1}} - \mu) + \eta_{0i} (x_{r_{2i-1}} - \mu) \right] \right\} + r_i (x_{r_i+1} - \mu)$$

$$- (x_{r_{2k-3}} - \mu) \left[ \eta_{0i} \sigma + \eta_{0i} (x_{r_{2k-3}} - \mu) + \eta_{0i} (x_{r_{2k-3}} - \mu) \right] - r_{2k-2} (x_{r_{2k-2}} - \mu)$$

$$- (\varepsilon_{r_{k-1}} + \delta_{r_{k-2}} (x_{r_{k-2}} - \mu)) (n - r_{2k-2}) (x_{r_{2k-2}} - \mu)$$

$$= A \sigma^2 + C_1 \sigma + C_2 \quad \text{(say)} \quad (3.2.49)$$

$$= 0. \quad (3.2.50)$$

Here

$C_1$ = coefficient of $\sigma$ in (3.2.49)

$$= \sum_{i=1}^{k-1} \sum_{j=r_{2i}+1}^{r_{2i+1}} (x_j - \mu) \alpha_i + \sum_{i=2}^{k-2} \left( r_{2i-1} - r_{2i-2} \right) \left[ \eta_{0i} (x_{r_{2i-1}} - \mu) + \eta_{0i} (x_{r_{2i-1}} - \mu) \right]$$

$$+ r_i (x_{r_i+1} - \mu) \varepsilon_{r_{i+1}} - (n - r_{2k-2}) \varepsilon_{r_{2k-2}} (x_{r_{2k-2}} - \mu)$$

$$= m (C^* - C \mu), \quad (3.2.51)$$

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where, $m$ and $C$ are as defined in (3.2.45) and (3.2.47), respectively, and

$$mC^* = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{n} \alpha_i x_j + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) (\eta_{0i} x_{\eta_{r_{2i-1}+1}} - \eta_{0i} x_{\eta_{r_{2i+1}}})$$

$$+ r_{1i} e_{r_{2i+1}}^* x_{\eta_{r_{2i+1}}} - (n - r_{2k-2}) e_{r_{2k-2}} x_{\eta_{r_{2k-2}}}.$$

(3.2.52)

Also, $C_2$ of (3.2.50) which is the expression in (3.2.49) not involving $\sigma$, can be written as a quadratic function of $\mu$ as shown below.

$$C_2 = \left[ \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{n} (-\beta_j) + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) (3 \eta_{0i} + 3 \eta_{0i}) + r_{1i} \delta_{r_{2i+1}}^* - (n - r_{2k-2}) \delta_{r_{2k-2}} \right] \mu^2$$

$$+ \left[ 2 \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{n} \beta_j x_j + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) \left\{ - \eta_{0i} (x_{\eta_{r_{2i-1}+1}} + x_{\eta_{r_{2i+1}}}) ight\} - 2 r_{1i} e_{r_{2i+1}} x_{\eta_{r_{2i+1}}}$$

$$+ 2(n - r_{2k-2}) \delta_{r_{2k-1}} x_{\eta_{r_{2k-1}}} \right] \mu + \left[ \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{n} (-\beta_j x_j^2) + \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right)$$

$$\left( \eta_{0i} x_{\eta_{r_{2i-1}+1}} x_{\eta_{r_{2i+1}}} + \eta_{0i} x_{\eta_{r_{2i-1}+1}}^2 - \eta_{0i} x_{\eta_{r_{2i+1}}}^2 - \eta_{0i} x_{\eta_{r_{2i-1}+1}} x_{\eta_{r_{2i+1}}} \right) + r_{1i} x_{\eta_{r_{2i+1}}}^2 \delta_{r_{2i+1}}^*$$

$$- (n - r_{2k-2}) \delta_{r_{2k-1}} x_{\eta_{r_{2k-1}}}^2 \right] \mu$$

$$= -m(\mu^2 - 2B\mu + B^*),$$

(3.2.53)

where, $m$ and $B$ are as defined in (3.2.45) and (3.2.46) respectively, and

$$mB^* = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{n} \beta_j x_j^2 - \sum_{i=2}^{k-1} \left( r_{2i-1} - r_{2i-2} \right) (\eta_{0i} x_{\eta_{r_{2i-1}+1}} - \eta_{2i} x_{\eta_{r_{2i+1}+1}})$$

$$- x_{\eta_{r_{2i+1}}}(\eta_{0i} x_{\eta_{r_{2i-1}+1}} + \eta_{2i} x_{\eta_{r_{2i+1}+1}+1}) - r_{1i} x_{\eta_{r_{2i+1}}}^2 \delta_{r_{2i+1}}^*$$

$$+ (n - r_{2k-2}) \delta_{r_{2k-1}} x_{\eta_{r_{2k-1}}}^2.$$

(3.2.54)
Substituting the value of \( \mu \) from (3.2.48) in \( C_1 \) and \( C_2 \), given by (3.2.51) and (3.2.53), respectively, we get

\[
C_1 = m \left( C^* - BC + C^2 \sigma \right) \tag{3.2.55}
\]
and

\[
C_2 = m \left( B^2 - C^2 \sigma^2 - B^* \right). \tag{3.2.56}
\]

Substituting these values of \( C_1 \) and \( C_2 \) in (3.2.50) we get

\[
-\sigma \left( \frac{\partial \log L^*}{\partial \sigma} \right) = A\sigma^2 + m(C^* - BC + C^2\sigma)\sigma + m(B^2 - C^2\sigma^2 - B^*)
\]

\[
= A\sigma^2 + m(C^* - BC)\sigma + m(B^2 - B^*) = 0. \tag{3.2.57}
\]

Letting

\[
D = m(C^* - BC) \tag{3.2.58}
\]
and

\[
E = m(B^2 - B^*), \tag{3.2.59}
\]
the equation (3.2.57) reduces to

\[
A\sigma^2 + D\sigma - E = 0. \tag{3.2.60}
\]

Since \( \sigma > 0 \), the solution of the quadratic equation (3.2.60) which is admissible, is given by

\[
\sigma = \frac{-1}{2A} \left[ -D + \sqrt{D^2 + 4AE} \right] \tag{3.2.61}
\]

From (3.2.48) and (3.2.61) we get

\[
\hat{\mu} = \frac{1}{2A} \left[ 2AB + CD - C\sqrt{D^2 + 4AE} \right]. \tag{3.2.62}
\]
3.3 APPROXIMATE VARIANCES AND COVARIANCE OF THE ESTIMATORS FOR LARGE SAMPLES

We obtain here the second order partial derivatives of $\log L^*$ in order to find an approximate asymptotic covariance matrix of the estimators $\hat{\mu}_s, \hat{\sigma}_s$ and $\hat{\alpha}_s$.

From (3.2.44) we have

$$\frac{\partial^2 \log L^*}{\partial \mu^2} = \frac{m}{\sigma^2}. \quad (3.3.1)$$

Again from (3.2.44), we obtain

$$-2\sigma \left( \frac{\partial \log L^*}{\partial \mu} \right) - \sigma^2 \left( \frac{\partial^2 \log L^*}{\partial \sigma \partial \mu} \right) = mC. \quad (3.3.2)$$

That is

$$\frac{\partial^2 \log L^*}{\partial \sigma \partial \mu} = -\frac{1}{\sigma^2} \left[ mC + 2\sigma \left( \frac{\partial \log L^*}{\partial \mu} \right) \right]$$

$$= -\frac{1}{\sigma^2} \left[ mC - \frac{2m}{\sigma} (C\sigma - B + \mu) \right]$$

$$= \frac{m}{\sigma^2} [C\sigma - 2(B - \mu)]. \quad (3.3.2)$$

From (3.2.57) we have

$$-\sigma^2 \left( \frac{\partial \log L^*}{\partial \sigma} \right) = A\sigma^2 + C_1\sigma + C_2$$

$$= A\sigma^2 + m(C^* - C\mu)\sigma - m(\mu^2 - 2B\mu + B^*)$$

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by substituting the values of $C_1$ and $C_2$ from (3.2.55) and (3.2.56).

Differentiating the above equation with respect to $\sigma$, we get

$$-3\sigma^2 \left( \frac{\partial \log L'}{\partial \sigma} \right) - \sigma^3 \left( \frac{\partial^2 \log L'}{\partial \sigma^2} \right) = 2A\sigma + m(C' - C_\mu).$$

So,

$$-\sigma^3 \left( \frac{\partial^2 \log L'}{\partial \sigma^2} \right) = \left[ 2A\sigma + m(C' - C_\mu) \right]$$

$$-\frac{3}{\sigma} \left[ A\sigma^2 + m(C' - C_\mu)\sigma - m(\mu^2 - 2B\mu + B^*) \right]$$

$$= -A\sigma - 2m(C' - C_\mu) + \frac{3m}{\sigma}(\mu^2 - 2B\mu + B^*)$$

$$= -A\sigma - 2mC' + \frac{3mB^*}{\sigma} + 2m \left( \frac{C - 3B}{\sigma} \right) \mu + \frac{3m}{\sigma} \mu^2$$

$$= -A\sigma - 2mC' + \frac{3mB^*}{\sigma} + 2m \left( \frac{C - 3B}{\sigma} \right) \mu + \frac{3m}{\sigma} \mu^2$$

$$+ \frac{3m}{\sigma}(B^2 - 2BC\sigma + C^2\sigma^2).$$

by substituting the value of $\mu$ from (3.2.48),

$$= -A\sigma - 2mC' + \frac{3mB^*}{\sigma} + 2m(BC - C'\sigma - \frac{3B^2}{\sigma} + 3BC)$$

$$+ \frac{3m}{\sigma}(B^2 - 2BC\sigma + C^2\sigma^2)$$

$$= -A\sigma + mC^2\sigma - 2mC' + 2mBC + \frac{3B}{\sigma}(B^* - B^3)$$

$$= (mC^2 - A)\sigma - 2m(C' - BC) + \frac{3m}{\sigma}(B^* - B^3) \quad (3.3.3)$$
Differentiating the derivative in (3.2.40) with respect to \( \mu \) we get

\[
\alpha \left( \frac{\partial^2 \log L}{\partial \mu \partial \alpha} \right) = \frac{\alpha}{\sigma} \sum_{i=1}^{k} \sum_{j=2_{i-1}+1}^{k} \left[ 2B_j (1 + \log y_j) - (1 - 2A_j) \frac{1}{y_j} \right]
\]

\[
+ \frac{r \alpha}{\sigma} \left[ B_{r+i} (1 + \log y_{r+i}) - \frac{A_{r+i}}{y_{r+i}} \right]
\]

\[
- \frac{r \alpha}{\sigma} \sum_{i=2}^{k} \left( r_{2i-2} - r_{2i-3} \right) \left[ 1 + \log y_{2_{i-1}+1} \right] \left[ \eta_{0,i} + \eta_{0,i} y_{2_{i-1}+1} + (\eta_{0,i} - 2) \eta_{0,i} y_{2_{i-1}+1} \right]
\]

\[
+ y_{0_{i-1}+1} \log y_{0_{i-1}+1} \left( \eta_{0,i} + \eta_{0,i} \right) \left[ 1 + \log y_{0_{i-1}+1} \right] \left[ \eta_{0,i} + \eta_{0,i} y_{0_{i-1}+1} + (\eta_{0,i} - 2) \eta_{0,i} y_{0_{i-1}+1} \right]
\]

Differentiating the derivative in (3.2.40) with respect to \( \sigma \), we get

\[
\alpha \left( \frac{\partial^2 \log L}{\partial \sigma \partial \alpha} \right) = -\frac{\alpha}{\sigma} \sum_{i=1}^{k} \sum_{j=2_{i-1}+1}^{k} \left[ A_j + B_j (1 + \log y_j) \right]
\]

\[
+ \frac{r \alpha}{\sigma} \left[ B_{r+i} (1 + y_{r+i}) \log y_{r+i} - \frac{\overline{A}_{r+i}}{y_{r+i}} \right]
\]

\[
+ \frac{\alpha}{\sigma} \left( n - 2_{k-2} \right) \left[ A_{2_{k-2}} + B_{2_{k-2}} y_{2_{k-2}} (1 + \log y_{2_{k-2}}) \right]
\]

\[
- \frac{\alpha}{\sigma} \sum_{i=2}^{k} \left( r_{2i-2} - r_{2i-3} \right) \left[ y_{2_{i-1}+1} \log y_{2_{i-1}+1} \right] \left[ \eta_{0,i} + 2(\eta_{0,i} y_{2_{i-1}+1} + \eta_{2,i} y_{2_{i-1}+1}) \right]
\]

\[
+ y_{0_{i-1}+1} (\eta_{0,i} + \eta_{0,i} y_{0_{i-1}+1} + \eta_{2,i} y_{0_{i-1}+1}) - y_{2_{i-1}+1} (\log y_{2_{i-1}+1})
\]

\[
+ \left[ \eta_{0,i} + 2(\eta_{0,i} y_{2_{i-1}+1} + \eta_{2,i} y_{2_{i-1}+1}) \right]
\]

\[
- y_{0_{i-1}+1} \left( \eta_{0,i} + \eta_{0,i} y_{0_{i-1}+1} + \eta_{2,i} y_{0_{i-1}+1} \right)
\]

\[
\left( 3.3.5 \right)
\]
Differentiation of the derivative in (3.2.40) with respect to $\alpha$ and a lengthy algebraic simplification yield the following second order partial derivative

$$
\alpha \left( \frac{\partial^2 \log L^*}{\partial \alpha^2} \right) = -\frac{A}{\alpha} + 2 \sum_{i=1}^{k-1} \sum_{j=1}^{n_{i+1}} (B_j \log y_j) \left[ \xi_j - y_j (1 + \log y_j) \right] 
$$

\[
+ r_i \log y_{n_{i+1}} \left[ q_{n_{i+1}} (1 + 2\alpha p_{n_{i+1}}) - B_{n_{i+1}} y_{n_{i+1}} (2 + \log \xi_{n_{i+1}}) \right] 
\]

\[
+ (n - r_{n_{i+1}}) B_{n_{i+1}} \log y_{n_{i+1}} \left[ \xi_{n_{i+1}} - y_{n_{i+1}} (1 + \log \xi_{n_{i+1}}) \right] 
\]

\[
- \frac{1}{\alpha} \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) \left[ y_{2i} \log y_{2i+1} \left\{ \eta_0 (1 - \log \xi_{2i+1}) \right\} 
\]

\[
+ 2(\eta_i^* \xi_{2i+1} + \eta_{2i}^* \xi_{2i+1}) \right\} \cdot y_{2i} \log y_{2i+1} \left\{ \eta_0 (1 - \log \xi_{2i+1}) \right\} 
\]

\[
+ 2(\eta_i^* \xi_{2i+1} + \eta_{2i}^* \xi_{2i+1}) \right\} \cdot \frac{1}{\alpha} \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) 
\]

\[
\cdot \left\{ y_{2i} \log(y_{2i+1} y_{2i+1}) \left\{ 1 + \log(\xi_{2i} \xi_{2i+1}) \right\} \right\} 
\]

\[
+ \frac{1}{\alpha} \sum_{i=2}^{k-1} (r_{2i-1} - r_{2i-2}) \left[ 2(\eta_0^* y_{2i+1}) \log y_{2i+1} (1 + \log \xi_{2i+1}) \right] 
\]

\[
- \eta_i^* y_{2i} \log y_{2i+1} (1 + \log \xi_{2i+1}) \right\} \right\} 
+ \left\{ \frac{1}{\xi_{2i+1}^*} (\eta_0^* + \eta_i^* \xi_{2i+1}) \right\} \cdot y_{2i} \log y_{2i+1} 
\]

\[
- \frac{1}{\xi_{2i+1}^*} (\eta_0^* + \eta_i^* \xi_{2i+1}) \right\} \cdot \left\{ y_{2i+1} \log y_{2i+1} \right\} \right\}. \quad (3.3.6)
\]
In this case an approximate information matrix is the matrix $I'$ whose elements are negatives of expected values of the second order partial derivatives given by (3.3.1) to (3.3.6). Asymptotic covariance matrix of the AMLE's $\hat{\mu}_s$, $\hat{\sigma}_s$ and $\hat{\alpha}_s$ is given by the inverse of the information matrix $I'$. Since AMLE's $\hat{\mu}_s$, $\hat{\sigma}_s$ and $\hat{\alpha}_s$ are the solutions of approximate likelihood equation, the following remarks holds.

Remark: Under certain regularity conditions the vectors of AMLE's $(\hat{\mu}_s, \hat{\sigma}_s, \hat{\alpha}_s)$ has asymptotic trivariate normal distribution with mean vector $(\mu, \sigma, \alpha)$ and covariance matrix $I'^{-1}$. (See Rao (1975), Kendall and Stuart (1973)).

3.4 SPECIAL CASES OF MULTIPLE TYPE II CENSORING

In this section we consider the same four important special cases of multiple Type II censoring, namely, i) double censoring ii) right censoring iii) left censoring and iv) middle censoring, mentioned in Section 2.4.

1. Double censoring

i) All the three parameters are unknown.

As in Section 2.4, letting $k = 2$ and equating the sum $\sum_{i=2}^1 (-)$ to zero in (3.2.38) to (3.2.40), we get the approximate likelihood equations for $\mu$, $\sigma$ and $\alpha$ for double censoring.
\[-\sigma \left( \frac{\partial \log L^*}{\partial \mu} \right) = \sum_{j=r_1+1}^{r_2} (\alpha_j - \beta_j y_j) + r_j(\epsilon_j^* + \delta^*_j \cdot y_{j+1}) + (n-r_2)(\epsilon_{n+1} + \delta_{n+1} y_{n+1}) = 0 \quad (3.4.1)\]

\[-\sigma \left( \frac{\partial \log L^*}{\partial \sigma} \right) = A + \sum_{j=r_1+1}^{r_2} y_j(\alpha_j - \beta_j y_j) + r_j y_{j+1}(\epsilon_j^* + \delta^*_j \cdot y_{j+1}) = 0 \quad (3.4.2)\]

\[
\left( \frac{\partial \log L^*}{\partial \alpha} \right) = \frac{A \alpha}{\alpha} + \sum_{j=r_1+1}^{r_2} (\log y_j)[1 - 2(A_j + B_j y_j)]
- r_j (\log y_{j+1})(\tilde{A}_{n+1} - B_{n+1} y_{n+1})
-(n-r_2)(\log y_{n+1})(A_{n+1} + B_{n+1} y_{n+1}) = 0, \quad (3.4.3)\]

where, \(A = r_2 - r_1\) and \(\frac{\mu(x_j - \mu)}{\sigma}\) as earlier.

For finding approximate variances and covariances we need the six second order partial derivatives, given in the previous section. The following three of these derivatives,

\[
\frac{\partial^2 \log L^*}{\partial \mu^2}, \quad \frac{\partial^2 \log L^*}{\partial \sigma^2} \quad \text{and} \quad \frac{\partial^2 \log L^*}{\partial \sigma \partial \mu},
\]

are given by (3.3.1), (3.3.2) and (3.3.3) respectively, with the values of \(m, B, C, B^*\) and \(C^*\) are given below

\[
m = \sum_{j=r_1+1}^{r_2} \beta_j - r_2 \delta^*_{n+1} + (n-r_2)\delta_{n+1} \quad (3.4.4)
\]
\[ m B = \sum_{j=r+1}^{n} \beta_j x_j - r_j \delta_{h+1} x_{h+1} + (n-r_j) \delta_{h} x_{h} \tag{3.4.5} \]

\[ m C = \sum_{j=r+1}^{n} \alpha_j + r_j e^*_{h+1} - (n-r_j) e_{h} \tag{3.4.6} \]

\[ m B' = \sum_{j=r+1}^{n} \beta_j x_j^2 - r_j x_j^2 \delta_{h+1}^* + (n-r_j) \delta_{h} x_j^2 \tag{3.4.7} \]

\[ m C' = \sum_{j=r+1}^{n} \alpha_j x_j + r_j e^*_{h+1} x_{h+1} - (n-r_j) e_{h} x_{h} \tag{3.4.8} \]

The other three second order partial derivatives, obtained from (3.3.4), (3.3.6), are

\[
\sigma \left( \frac{\partial^2 \log L^*}{\partial \mu \partial \alpha} \right) = \sum_{j=r+1}^{n} \left[ 2B_j (1 + \log y_j) - (1 - 2A_j) \frac{1}{y_j} \right] \\
+ r_j \left[ B_{h+1} (1 + \log y_{h+1}) \frac{A_{h+1}}{y_{h+1}} \right] \\
+ (n-r_j) \left[ B_{h} (1 + \log y_{h}) + \frac{A_{h}}{y_{h}} \right] \tag{3.4.9} \\
\sigma \left( \frac{\partial^2 \log L^*}{\partial \sigma \partial \alpha} \right) = -A + 2 \sum_{j=r+1}^{n} \left[ A_j + B_j (1 + \log y_j) \right] \\
+ r_j \left[ B_{h+1} (1 + y_{h+1}) \log y_{h+1} - A_{h+1} \right] \\
+ (n-r_j) \left[ A_{h} + B_{h} y_{h} (1 + \log y_{h}) \right] \tag{3.4.10} 
\]
\[
\alpha \left( \frac{\partial^2 \log L^*}{\partial \alpha^2} \right) = -\frac{A}{\alpha} + 2 \sum_{j=r+1}^{n} (B_j \log y_j) \left[ \xi_j - y_j(1 + \log y_j) \right] \\
+ r_1 \log y_{r+1} \left[ q_{r+1}(1 + 2\alpha p_{r+1}) - B_{r+1} y_{r+1}(2 + \log \xi_{r+1}) \right] \\
+ (n - r_2) B_{r_2} \log y_{n_2} \left( \xi_{n_2} - y_{n_2}(1 + \log \xi_{n_2}) \right)
\] (3.4.11)

(ii) \( \alpha = \alpha_0 \) known and \( \mu \) and \( \sigma \) are unknown

In this case, we estimated the parameters \( \mu \) and \( \sigma \). Using the three second order, partial derivatives corresponding to \( \mu \) and \( \sigma \) given in (i) with \( \alpha = \alpha_0 \), we find an approximate asymptotic covariance matrix of the AMLE's \((\hat{\mu}, \alpha)\) and \((\hat{\sigma}, \alpha)\).

The remaining five cases, out of the seven given in Section 2.4, are also considered for estimation purpose.

2. Right censoring

In the results for double censoring, if we take \( r_1 = 0 \) we get the corresponding results for right censored data.

3. Left censoring

Letting \( r_2 = n \) in the results for double censoring, we obtain the corresponding results for left censoring.
4. Middle censoring

(i) All the three parameters are unknown.

As in Section 2.4, letting $k = 3$, $r_1 = 0$ and $r_4 = n$ in the results for the multiple Type II censoring we get the corresponding results for middle censoring. The approximate likelihood equations are,

$$-\sigma \left( \frac{\partial \log L^*}{\partial \mu} \right) = \sum_{i=1}^{3} \sum_{j=r_2}^{n_2} (\alpha_j - \beta_j y_j) + (r_3 - r_2) \left( \beta_{o2} + \beta_{r2} + \beta_{v22} y_{n+1} \right)$$

$$= 0 \quad (3.4.12)$$

$$-\alpha \left( \frac{\partial \log L^*}{\partial \sigma} \right) = A + \sum_{i=1}^{3} \sum_{j=r_2}^{n_2} y_j (\alpha_j - \beta_j y_j) + (r_3 - r_2) \left[ (\eta_{o2} + \eta_{r2} y_{r-1} + \eta_{v22} y_{n+1}) - y_{r-1} (\eta_{o2} + \eta_{r2} y_{r-1} + \eta_{v22} y_{n+1}) \right]$$

$$= 0 \quad (3.4.13)$$

$$\alpha \left( \frac{\partial \log L^*}{\partial \alpha} \right) = A + \alpha \sum_{i=1}^{3} \sum_{j=r_2}^{n_2} (\log y_j) \left[ 2(A_j + B_j y_j) \right]$$

$$+ (r_3 - r_2) \left[ (\log y_{n+1}) (\eta_{o2} + \eta_{r2} y_{r-1} + \eta_{v22} y_{n+1}) \right. \left. - y_{r-1} (\log y_{r-1}) (\eta_{o2} + \eta_{r2} y_{r-1} + \eta_{v22} y_{n+1}) \right]$$

$$= 0 \quad (3.4.14)$$
For finding approximate asymptotic variances and covariances the six second order partial derivatives, given in Section 3, are used. The derivatives,

\[ \frac{\partial^2 \log L^*}{\partial \mu^2}, \frac{\partial^2 \log L^*}{\partial \sigma^2} \text{ and } \frac{\partial^2 \log L^*}{\partial \sigma \partial \mu}, \]

are given by (3.3.1), (3.3.2) and (3.3.3) respectively, with the following values for \( m, B, C, B^* \) and \( C^* \)

\[ m = \sum_{i=1}^{2} \sum_{j=r_2+1}^{n}\beta_j - (r_3 - r_2)(\mathcal{I}_{12} + \mathcal{J}_{22}) \]  

(3.4.15)

\[ mB = \sum_{i=1}^{2} \sum_{j=r_2+1}^{n}\beta_j x_j - (r_3 - r_2)(\mathcal{I}_{12}x_{r_2} + \mathcal{J}_{22}x_{n+1}) \]  

(3.4.16)

\[ mC = \sum_{i=1}^{2} \sum_{j=r_2+1}^{n}\alpha_j + (r_3 - r_2)\mathcal{J}_{02} \]  

(3.4.17)

\[ mB^* = \sum_{i=1}^{2} \sum_{j=r_2+1}^{n}\beta_j x_j^2 - (r_3 - r_2)\left[x_{r_2}(n_{12}x_{r_2} + n_{22}x_{n+1}) \right] \]  

\[-x_{r_2}(n_{12}^* x_{r_2} + n_{22}^* x_{n+1}) \]  

(3.4.18)

\[ mC^* = \sum_{i=1}^{2} \sum_{j=r_2+1}^{n}\alpha_j x_j + (r_3 - r_2)(n_{02}x_{n+1} - n_{02}^* x_{n}) \]  

(3.4.19)

The other three second order partial derivatives, obtained from (3.3.4) to (3.3.6), are

\[ \alpha \left( \frac{\partial^2 \log L^*}{\partial \mu \partial \alpha} \right) = \frac{\alpha}{\sigma} \sum_{i=1}^{2} \sum_{j=r_2+1}^{n} \left[ 2B_j (1 + \log y_j) - (1 - 2A_j) \frac{1}{y_j} \right] \]  

(54)
\[ -\frac{1}{\sigma} (r_3 - r_2) \left\{ (1 + \log y_{n,t}) (\eta_{00} + \eta_{12} y_t + \eta_{22} y_{n,t}) \right\} \]

\[ + y_{n,t} (\log y_{n,t}) (\eta_{02} + \eta_{12}^* y_t + \eta_{22}^* y_{n,t}) \]

\[ + y_n (\log y_n) (\eta_{02} + \eta_{12}^* y_t + \eta_{22}^* y_{n,t}) \] (3.4.20)

\[ \alpha \left( \frac{\partial^2 \log L^*}{\partial \sigma \partial \alpha} \right) = -\frac{\Lambda \alpha}{\sigma} + 2 \alpha \sum_{i=1}^{2} \sum_{j=\tau_t+1}^{\infty} \left[ A_i + B_i (1 + \log y_j) \right] \]

\[ -\frac{1}{\sigma} (r_3 - r_2) \left\{ y_{n,t} (\log y_{n,t}) \left\{ \eta_{02} + 2 (\eta_{12} y_t + \eta_{22} y_{n,t}) \right\} \right\} \]

\[ + y_{n,t} (\eta_{02} + \eta_{12} y_t + \eta_{22} y_{n,t}) - y_n (\log y_t) \]

\[ \cdot \left\{ \eta_{02} + 2 (\eta_{12} y_t + \eta_{22} y_{n,t}) - y_n^* (\eta_{02} + \eta_{12}^* y_t + \eta_{22}^* y_{n,t}) \right\} \] (3.4.21)

\[ \alpha \left( \frac{\partial^2 \log L^*}{\partial \alpha^2} \right) = -\frac{\Lambda}{\alpha} + 2 \sum_{i=1}^{2} \sum_{j=\tau_t+1}^{\infty} (B_i \log y_j) \left[ \xi_j - y_j (1 + \log y_j) \right] \]

\[ -\frac{1}{\alpha} (r_3 - r_2) \left\{ y_{n,t} (\log y_{n,t}) \left\{ \eta_{02} (1 - \log \xi_{n,t}) \right\} \right\} \]

\[ + 2 (\eta_{12} \xi_{n,t} + \eta_{22} \xi_{n,t}) \left\{ \eta_{02} (1 - \log \xi_{n,t}) \right\} \]

\[ - y_n (\log y_t) \left\{ - \eta_{02} (1 - \log \xi_{n,t}) \right\} \]

\[ + 2 (\eta_{12} \xi_{n,t} + \eta_{22} \xi_{n,t}) \left\{ \eta_{12} y_t y_{n,t} (\log (y_t y_{n,t})) \right\} \]

\[ \cdot \left\{ 1 + \log (\xi_{n,t} \xi_{n,t}) \right\} - \left\{ 2 \eta_{22} y_{n,t} (\log y_{n,t}) \right\} \]

\[ \cdot \left\{ (1 + \log \xi_{n,t}) - \eta_{12}^* y_t^2 \log y_t (1 + \log \xi_{n,t}) \right\} \]
The other parameter related cases are treated as in double censoring with the above values of \( k, r_1 \) and \( r_4 \).

Numerical results for all the above cases are given in Chapter 6.