5.1 INTRODUCTION

In Chapters 2 and 3, we have obtained MLE and AMLE respectively, based on the multiply Type II censored sample. In this chapter we consider the best linear unbiased estimators (BLUE's) and best linear invariant estimators (BLIE's) of the location and scale parameters of the log-logistic distribution, based on multiply Type II censored sample, under the assumption that the shape parameter is known.


In Section 2, we considered the problem of obtaining the BLUE’s of the location and scale parameters, based on the multiply Type II censored sample, under the assumption that the shape parameter is known (David (1981)). We obtained the BLUE of $\mu$ when $\sigma$ is known and the BLUE of $\sigma$ when $\mu$ is known.

In Section 3, we considered the simplified linear unbiased estimators of Gupta (1952). It was noticed in the previous section that determining the product moments and obtaining the inverse of the covariance matrix of order statistics was difficult. So, to overcome this difficulty, we considered in this section simplified linear unbiased estimators. We estimated the parameters by the different special cases of the multiple Type II censoring schemes.

In Section 4, we obtained the BLIE’s of the location and scale parameters following the procedure given by Mann (1969).
In Section 5, we obtained the simplified linear invariant estimators of $\mu$ and $\sigma$. Here also we considered the different special cases of the multiple Type II censoring schemes and estimated the parameters.

### 5.2 BEST LINEAR UNBIASED ESTIMATORS OF LOCATION AND SCALE PARAMETERS

Suppose the following multiply Type II censored sample of size $n$

$$x_{r_1} < ... < x_{r_2} < x_{r_3} < ... < x_{r_n} < ... < x_{r_{n+1}} < ... < x_{r_k} < x_n$$

is available from the log-logistic distribution with pdf

$$f(x;\mu,\sigma) = \frac{\alpha \left( \frac{x - \mu}{\sigma} \right)^{\alpha-1}}{\sigma \left[ 1 + \left( \frac{x - \mu}{\sigma} \right)^{\alpha} \right]^{\gamma+1}}, \quad x > \mu$$

where $\alpha$ is assumed to be known.

Let

$$Y_i = \frac{X_i - \mu}{\sigma}, \quad i \in J = \{(r_{i-1} + 1, ..., r_i), i = 1, ..., k - 1\}$$

The total number of observations $X_i$ available is

$$N = \sum_{i=1}^{k-1} (r_i - r_{i-1})$$

From (5.2.3) we have

$$X_i = \mu + \sigma Y_i, \quad i \in J$$

The moments of the order statistics $Y_i$ do not depend on $\mu$ and $\sigma$.

Let

$$E(Y_i) = \alpha_i \quad \text{and} \quad \text{Cov}(Y_i, Y_j) = \beta_{ij}, \quad i, j \in J$$
Then
\[ E(X_i) = \mu + \sigma \alpha_i \quad (5.2.5) \]
and
\[ \text{Cov}(X_i, X_j) = \sigma^2 \beta_{ij}. \quad (5.2.6) \]

Thus \( E(X_i) \) is a linear function of the parameters \( \mu \) and \( \sigma \). When the co-efficients \( \alpha_i \) and the covariance, \( \text{Cov}(X_i, X_j) \) are known apart from \( \sigma^2 \), a generalized version of the Gauss–Markov least squares (LS) theorem (Lloyd (1952)) may be applied to obtain the BLUE’s of \( \mu \) and \( \sigma \). The equation (5.2.5) can be written in the matrix notation as
\[ \theta' = A \alpha \]
where
\[ \begin{align*}
    \theta' &= (\mu, \sigma) \\
    A &= (1, \alpha) \\
    B &= \beta_{11}, \text{ the covariance matrix of } Y \\
    \Omega &= B^{-1}
\end{align*} \]

In these notations we have
The covariance matrix of $X$ is

$$\text{cov}(X) = \sigma^2 B.$$ 

The BLUE of $\theta$, based on the multiply Type II censored sample in (5.2.1) is obtained by minimizing

$$((X - A\theta)' \Omega (X - A\theta)) \quad (5.2.7)$$

with respect to $\theta$. Differentiating this equation with respect to $\theta$ and equating to zero, we have

$$-2A'\Omega X + 2A'\Omega A\theta = 0.$$ 

Thus the BLUE of $\theta$ is obtained as

$$\theta^* = (A'\Omega A)^{-1} A'\Omega X. \quad (5.2.8)$$

The covariance matrix of $\theta^*$ is

$$E[(\theta^* - \theta)(\theta^* - \theta)'] = E[(A'\Omega A)^{-1} A'\Omega (X - A\theta)(X - A\theta)' \Omega A (A'\Omega A)^{-1}]$$

$$= \sigma^2 (A'\Omega A)^{-1} A' \Omega B \Omega A (A'\Omega A)^{-1}$$

$$= \sigma^2 (A'\Omega A)^{-1}. \quad (5.2.9)$$

Now,

$$A'\Omega A = \begin{bmatrix} 1' \\ \alpha' \end{bmatrix} \Omega \begin{bmatrix} 1 & \alpha \end{bmatrix} = \begin{bmatrix} 1' \Omega 1 & 1' \Omega \alpha \\ \alpha' \Omega 1 & \alpha' \Omega \alpha \end{bmatrix} = \Delta, \text{ say,} \quad (5.2.10)$$

with all the elements of the matrix being scalars (David, (1981)). The inverse of the matrix $\Delta = A'\Omega A$ is
From the (5.2.8) we obtain

\[ \begin{vmatrix} \alpha' \Omega & -\alpha' \Omega \\ -\Omega \alpha & \Omega \end{vmatrix} \]

\[ (A'\Omega A)^{-1} = \frac{1}{|A|} \begin{vmatrix} \alpha' \Omega & -\alpha' \Omega \\ -\Omega \alpha & \Omega \end{vmatrix} \]

(5.2.11)

with \( |A| = |A'\Omega A| \).

From the (5.2.8) we obtain

\[ g' = \frac{1}{|A|} \begin{vmatrix} \alpha' \Omega & -\alpha' \Omega \\ -\Omega \alpha & \Omega \end{vmatrix} \begin{vmatrix} l' \Omega \\ \alpha' \Omega \end{vmatrix} X \]

\[ = \frac{1}{|A|} \begin{vmatrix} \alpha' \Omega \alpha l' \Omega - \alpha' \Omega \alpha' \Omega \\ -\Omega \alpha l' \Omega + \Omega \alpha' \Omega \end{vmatrix} X. \]

(5.2.12)

So,

\[ \mu' = \frac{1}{|A|} \alpha' \Omega (\alpha l' - \alpha') \Omega X \]

\[ = -\alpha' \Gamma X, \]

(5.2.13)

where

\[ \Gamma = \frac{\Omega (\alpha l' - \alpha') \Omega}{|A|} \]

(5.2.14)

is a skew symmetric matrix, and

\[ \sigma' = l' \Gamma X. \]

(5.2.15)

It can be easily seen that

\[ E \mu' = \mu \quad \text{and} \quad E \sigma' = \sigma. \]

Also, from (5.2.9) and (5.2.11)
\[ \text{Var } \mu^* = \frac{\alpha' \Omega \alpha}{|\Delta|} \sigma^2, \quad (5.2.16) \]

\[ \text{Var } \sigma^* = \frac{1'}{\Omega} \frac{1}{|\Delta|} \sigma^2, \quad (5.2.17) \]

and

\[ \text{Cov}(\mu^*, \sigma^*) = -\frac{1'}{\Omega} \frac{\alpha}{|\Delta|} \sigma^2. \quad (5.2.18) \]

Thus \( \mu^* \) and \( \sigma^* \) can be expressed as linear functions of the order statistics namely

\[ \mu^* = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{r_{2i}} a_j X_j, \]

\[ \sigma^* = \sum_{i=1}^{k-1} \sum_{j=r_{2i-1}+1}^{r_{2i}} b_j X_j. \]

The coefficients \( a_j \) and \( b_j \) may be tabulated once for all.

**Special Cases:**

Here we consider two special cases:

1. \( \sigma \) is known when \( \mu \) is unknown

2. \( \mu \) is known when \( \sigma \) is unknown

1. BLUE of \( \mu \) when \( \sigma \) is known

Let \( \sigma = \sigma_0 \). As earlier it is assumed that the shape parameter \( \alpha \) is known. Here we have

\[ X = \mu_1 + \sigma_0 Y \]

\[ E(X) = \mu_1 + \sigma_0 \alpha \quad \text{and} \quad \mathcal{V}(X) = \sigma_0^2 B. \]
We have to minimize

\[(X - \mu | - \sigma \alpha)' \Omega (X - \mu | - \sigma \alpha)\]  \hspace{1cm} (5.2.19)

with respect to \( \mu \).

Differentiating (5.2.19) with respect to \( \mu \) and equating the derivative to zero, we get the BLUE of \( \mu \), when \( \sigma \) is known, as

\[\mu^* = \frac{(X - \mu | - \sigma \alpha) \Omega \Omega'}{\Omega \Omega'},\]  \hspace{1cm} (5.2.20)

with the variance

\[\text{Var}(\mu^*) = \frac{\sigma^2 \Omega}{\Omega \Omega'}\]  \hspace{1cm} (5.2.21)

2. BLUE of \( \sigma \) when \( \mu \) is known

Let \( \mu = \mu_0 \). Then

\[E(X) = \mu_0 | + \sigma \alpha,\]

\[\text{Cov}(X) = \sigma^2 \Omega.\]

To find the BLUE of \( \sigma \) we have to minimize

\[(X - \mu_0 | - \sigma \alpha)' \Omega (X - \mu_0 | - \sigma \alpha)\]  \hspace{1cm} (5.2.22)

with respect to \( \sigma \).

Differentiating (5.2.22) with respect to \( \sigma \) and equating the derivative to zero, we get.

\[-2(X - \mu_0 |)' \Omega \alpha + 2 \sigma \Omega \alpha \sigma = 0.\]
So, the BLUE of $\sigma$, when $\mu$ is known is

$$\sigma^* = \frac{(X - \mu P)^\prime \Omega \sigma}{\alpha' \Omega \alpha} \tag{5.2.23}$$

and

$$\text{Var}(\sigma^*) = \frac{\sigma^2}{\alpha' \Omega \alpha}. \tag{5.2.24}$$

### 5.3 SIMPLIFIED LINEAR UNBIASED ESTIMATORS OF GUPTA

Lloyd’s procedure requires full knowledge of the expectations $\alpha_i$ and the covariance matrix $B$ of the order statistics $Y_i$. Since it is difficult to determine the covariances, we consider the simple method proposed by Gupta (1952) for estimating $\mu$ and $\sigma$. This method consists in taking $B = I$, the unit matrix and is applicable when only the expectations $\alpha_i$’s are known. Here

$$\alpha_i = \frac{\Gamma(i + \frac{1}{\alpha}) \Gamma(n - i + 1 - \frac{1}{\alpha})}{\Gamma(i) \Gamma(n - i + 1)} \tag{5.3.1}$$

(see David (1981) and Balakrishnan and Cohen (1991)). We call the estimators of $\mu$ and $\sigma$, obtained by the method of Gupta (1952) as simplified linear unbiased estimators (SLUE) and denote them by $\mu^{**}$ and $\sigma^{**}$ respectively. In this case we have

$$\Delta = A'A = \begin{bmatrix} \alpha' 1 & \alpha' \alpha \\ \alpha' \alpha & \alpha' \alpha \end{bmatrix}$$

and

$$|\Delta| = \alpha' 1 - (\alpha' 1)^2$$

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\[ \begin{align*}
&= N \left[ \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_i^2 \right] - \left[ \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_j \right]^2, \\
\text{where } N \text{ is as given in (5.2.4).}
\end{align*} \] 

Also,
\[ (A'A)^{-1} = \frac{1}{|A|} \begin{bmatrix} \alpha'\alpha & -\alpha' \\ -\alpha' & 1 \end{bmatrix} \]  

and \( \Gamma \) of (5.2.14) in this case is
\[ \Gamma = \frac{1}{|A|} (1 \alpha' - \alpha 1'). \]  

From (5.2.13) and (5.3.3) we obtain the SLUE of \( \mu \) as
\[ \mu'' = -\alpha' \Gamma X \]
\[ = \frac{1}{|A|} \left[ \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_i^2 \right) \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} x_i \right) - \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_j \right) \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_j x_i \right) \right]. \]  

Similarly from (5.2.15) and (5.3.4) we obtain the SLUE of \( \sigma \) as
\[ \sigma'' = 1' \Gamma X \]
\[ = \frac{1}{|A|} \left[ N \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_j x_i \right) - \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} \alpha_j \right) \left( \sum_{i=1}^{k-l} \sum_{j=r_2}^{n} x_i \right) \right]. \]  

It can be easily seen that
\[ E \mu'' = \mu \quad \text{and} \quad E \sigma'' = \sigma. \]
Also from (5.2.16) to (5.2.18) we obtain

\[
\operatorname{Var} \mu^* = \frac{\alpha' \alpha}{|\Delta|} \sigma^2 = \frac{\sigma^2}{|\Delta|} \sum_{i=1}^{k} \sum_{j=j_{p_0+1}}^{k} \alpha_j^2, \quad (5.3.7)
\]

\[
\operatorname{Var} \sigma^* = \left| \frac{1}{|\Delta|} \right| \sigma^2 = \frac{N \sigma^2}{|\Delta|}, \quad (5.3.8)
\]

and

\[
\operatorname{Cov} (\mu^*, \sigma^*) = -\frac{\sigma^2}{|\Delta|} \frac{1}{|\Delta|} \alpha = -\frac{\sigma^2}{|\Delta|} \sum_{i=1}^{k} \sum_{j=j_{p_0+1}}^{k} \alpha_j. \quad (5.3.9)
\]

**Special cases (a)**

As in Section 5.2, we consider here the two special cases, viz

1. \( \sigma \) is known and \( \mu \) is unknown

2. \( \mu \) is known and \( \sigma \) is unknown

1. SLUE of \( \mu \) when \( \sigma = \sigma_0 \) is known.

Letting \( B = \Omega = I \) in (5.2.20) and (5.2.21), we get the SLUE \( \mu^* \) of \( \mu \) and its variance as

\[
\mu^* = \frac{(X - \sigma_0 \mathbf{a}^')}{N}, \quad (5.3.10)
\]

and

\[
\operatorname{Var} (\mu^*) = \frac{\sigma_0^2}{N}. \quad (5.3.11)
\]
2. SLUE of $\sigma$ when $\mu = \mu_0$ is known.

Letting $B = \Omega = I$ in (5.2.23) and (5.2.24), we get the SLUE $\sigma^\star\star$ of $\sigma$, when $\mu = \mu_0$ known and its variance as

$$\sigma^\star\star = \frac{\alpha'(X-\mu_0)}{\alpha'\alpha} \quad (5.3.12)$$

and

$$\text{Var}(\sigma^\star\star) = \frac{\sigma^2}{\alpha'\alpha}. \quad (5.3.13)$$

Special cases (b)

We consider here the four important special cases of the multiply Type II censoring schemes given in Section 2.4. We obtain the SLUE's of $\mu$ and $\sigma$, under each of these censoring schemes and also in three cases:

(i) both $\mu$ and $\sigma$ are unknown

(ii) $\sigma = \sigma_0$ known and $\mu$ is unknown

(iii) $\mu = \mu_0$ known and $\sigma$ in unknown

1. Double censoring

(i) Both the parameters $\mu$ and $\sigma$ are unknown

Letting $k = 2$ in (5.3.5) and (5.3.6), we get the expressions for $\mu^\star\star$ and $\sigma^\star\star$, respectively, as

$$\mu^\star\star = \frac{1}{|\Delta|} \left[ \left( \sum_{j=r+1}^{s} \alpha_j \right) \left( \sum_{j=r+1}^{s} x_j \right) - \left( \sum_{j=r+1}^{s} \alpha_j \right) \left( \sum_{j=r+1}^{s} \alpha_j x_j \right) \right]. \quad (5.3.14)$$
\[
\sigma^{**} = \frac{1}{|\Delta|} \left[ (r_2 - r_i) \sum_{j=r+1} \alpha_j x_j - \left( \sum_{j=r+1} \alpha_j \right) \left( \sum_{j=r+1} x_j \right) \right]. \tag{5.3.15}
\]

with
\[
|\Delta| = (r_2 - r_i) \sum_{j=r+1} \alpha_j^2 - \left( \sum_{j=r+1} \alpha_j \right)^2. \tag{5.3.16}
\]

Similarly the variances and covariances of \( \mu^{**} \) and \( \sigma^{**} \) are obtained from the
(5.3.7) – (5.3.9) as
\[
\text{Var} \mu^{**} = \frac{\sigma^2}{|\Delta|} \sum_{j=r+1} \alpha_j^2, \tag{5.3.17}
\]
\[
\text{Var} \sigma^{**} = \frac{(r_2 - r_i) \sigma^2}{|\Delta|}, \tag{5.3.18}
\]
and
\[
\text{Cov}(\mu^{**}, \sigma^{**}) = -\frac{\sigma^2}{|\Delta|} \sum_{j=r+1} \alpha_j. \tag{5.3.19}
\]

ii) \( \sigma = \sigma_0 \) known and \( \mu \) is unknown

In this case the estimate \( \mu^{**} \) and its variance are estimated by letting \( k = 2 \)
in (5.3.10) and (5.3.11)

iii) \( \mu = \mu_0 \) known and \( \sigma \) in unknown

In this case the estimate \( \sigma^{**} \) and its variance are obtained by letting \( k = 2 \) in
(5.3.12) and (5.3.13)
2. Right censoring

(i) Both the parameters are unknown

Letting \(r_1 = 0\) in (5.3.14) to (5.3.19) we get the corresponding results for right censored data.

The results for other two cases can be obtained similarly.

3. Left censoring

(i) Both the parameters are unknown

Letting \(r_2 = n\) in (5.3.14) to (5.3.19) we get the corresponding results for left censoring.

The results for other two cases can be obtained similarly.

4. Middle censoring

(i) Both the parameters are unknown.

Letting \(k = 3\), \(r_1 = 0\) and \(r_4 = n\) in (5.3.5) and (5.3.6) we get \(\mu^{*}\) and \(\sigma^{*}\).

\[
\mu^{*} = \frac{1}{\Delta} \left[ \left( \sum_{i=1}^{2} \sum_{j=p_{r_1+1}}^{b_n} \alpha_i \right) \left( \sum_{i=1}^{2} \sum_{j=p_{r_2+1}}^{b_n} x_j \right) - \left( \sum_{i=1}^{2} \sum_{j=p_{r_3+1}}^{b_n} \alpha_i \right) \left( \sum_{i=1}^{2} \sum_{j=p_{r_4+1}}^{b_n} x_j \right) \right],
\]

(5.3.20)

\[
\sigma^{*} = \frac{1}{\Delta} \left[ \left( \sum_{i=1}^{2} (r_{2i} - r_{2i-1}) \right) \left( \sum_{i=1}^{2} \sum_{j=p_{r_2+1}}^{b_n} \alpha_i x_j \right) - \left( \sum_{i=1}^{2} \sum_{j=p_{r_1+1}}^{b_n} \alpha_i x_j \right) \right],
\]

(5.3.21)
with

\[ |\Delta| = \left( \sum_{i=1}^{2} (r_{2i} - r_{2i-1}) \right) \left( \sum_{j=\ell_{2i-1}+1}^{2i} \sum_{j=\ell_{2i-1}+1}^{2i} \alpha_j \right)^2. \]  \hspace{1cm} (5.3.22)

The variances and covariances of \( \mu^{**} \) and \( \sigma^{**} \) are obtained from (5.3.7) to (5.3.9) as

\[ \text{Var} \mu^{**} = \frac{\sigma^2}{|\Delta|} \sum_{i=1}^{2} \sum_{j=\ell_{2i-1}+1}^{2i} \alpha_j, \]  \hspace{1cm} (5.3.23)

\[ \text{Var} \sigma^{**} = \frac{\sigma^2}{|\Delta|} \sum_{i=1}^{2} (r_{2i} - r_{2i-1}), \]  \hspace{1cm} (5.3.24)

and

\[ \text{Cov} \mu^{**}, \sigma^{**} = \frac{\sigma^2}{|\Delta|} \sum_{i=1}^{2} \sum_{j=\ell_{2i-1}+1}^{2i} \alpha_j. \]  \hspace{1cm} (5.3.25)

ii) \( \sigma = \sigma_0 \) known and \( \mu \) is unknown

The results for \( \mu^{**} \) and its variance are obtained from (5.3.10) and (5.3.11) by substituting the above values for \( k, r_1 \) and \( r_4 \).

iii) \( \mu = \mu_0 \) known and \( \sigma \) is unknown

In this case the estimator \( \sigma^{**} \) and its variance are obtained from (5.3.12) and (5.3.13) by substituting the above values for \( k, r_1 \) and \( r_4 \).
5.4 BEST LINEAR INVARIANT ESTIMATORS OF LOCATION AND SCALE PARAMETERS

Here we consider best linear invariant estimators (BLIE's) of the location and scale parameters. These invariant estimators were proposed and studied in detail by Mann (1969). An invariant estimator is defined to be the one with risk (expected loss) invariant under transformations of location and scale and it is said to be best if it has minimum risk. The loss is taken to be weighted squared error with the weight equal to the square of the inverse of a scale parameter.

Let $\mu^*$ and $\sigma^*$ be the BLUE's of $\mu$ and $\sigma$ given by (5.2.13) and (5.2.15) respectively. Then using Theorem 1 of Mann (1969) we obtained the BLIE's of the location parameter $\mu$ and the scale parameter $\sigma$ as

$$\bar{\mu} = \mu^* - \left( \frac{B'}{1 + C'} \right) \sigma^*$$

(5.4.1)

and

$$\bar{\sigma} = \frac{\sigma^*}{1 + C^*}$$

(5.4.2)

where

$$B'\sigma^2 = \text{Cov}(\mu^*, \sigma^*)$$

and

$$C^*\sigma^2 = \text{Var}(\sigma^*)$$

which are given by (5.2.18) and (5.2.17) respectively.
The mean square errors (MSE's) of $\bar{\mu}$ and $\bar{\sigma}$ are

$$\text{MSE}((\bar{\mu}) = A^* B^* \sigma^2$$

and

$$\text{MSE}((\bar{\sigma}) = \left( C^* \right)^2 \left( \frac{1 + C^*}{1 + C^*} \right)^2$$

respectively, and

$$E[(\bar{\mu} - \mu)(\bar{\sigma} - \sigma)] = \left( B^* \frac{1}{1 + C^*} \right)^2 \sigma^2,$$

where

$$A^* \sigma^2 = \text{Var}(\mu^*)$$

which is given by (5.2.16) and $B^*$ and $C^*$ are as defined above.

**Special cases**

1) BLIE of $\mu$ when $\sigma = \sigma_0$ known

From Theorem 3, of Mann (1969) the BLIE of $\mu$ when $\sigma = \sigma_0$ known is given by

$$\bar{\mu}_o = \mu^* - \frac{B^*}{C^*} (\sigma^* - \sigma_0)$$

It can be easily seen that the estimate $\bar{\mu}_o$ is unbiased. The variance of $\bar{\mu}_o$ is

$$\text{Var}(\bar{\mu}_o) = \left( A^* \right)^2 \left( \frac{B^*}{C^*} \right)^2 \sigma_0^2.$$
where $A^*$, $B^*$ and $C^*$ are as defined above.

2) BLIE of $\sigma$ when $\mu = \mu_0$ known

Let $\sigma^*_\mu$ be the BLUE of $\sigma$ when $\mu = \mu_0$ known with the variance $C_\mu \sigma^2$.

Then under assumptions 1 and 2' of Mann (1969), when the loss function is squared error divided by $\sigma^2$, the BLIE of $\sigma$ is given by

$$\bar{\sigma}_\mu = \frac{\sigma^*_\mu}{1 + C_\mu}, \quad (5.4.8)$$

with

$$\text{MSE} (\bar{\sigma}_\mu) = \left( \frac{C_\mu}{1 + C_\mu} \right) \sigma^2,$$

where, $\sigma^*_\mu$ as given in (5.2.23) and $C_\mu = (\alpha' \Omega \alpha)^{-1}$ from (5.2.24)

5.5 Simplified Linear Invariant Estimators of $\mu$ and $\sigma$

Using simplified linear unbiased estimators of $\mu$ and $\sigma$ given by (5.3.5) and (5.3.6) respectively, we obtain the corresponding simplified linear invariant estimators (SLIE's) of $\mu$ and $\sigma$ as

$$\tilde{\mu}' = \mu^* - \left( \frac{B'}{1 + C'} \right) \sigma^{**} \quad (5.5.1)$$

and

$$\tilde{\sigma}' = \frac{\sigma^{**}}{(1 + C')}. \quad (5.5.2)$$
The MSE's of $\bar{\mu}'$ and $\bar{\sigma}'$ are given by

$$MSE(\bar{\mu}') = \left( A' - \frac{B' \sigma'}{1 + C'} \right) \sigma^2,$$

and

$$MSE(\bar{\sigma}') = \left( \frac{C'}{1 + C'} \right) \sigma^2,$$

where

$$A' = \frac{\text{Var}(\mu'' \sigma^2)}{\sigma^2} = \frac{\alpha' \alpha}{|\Lambda|},$$

$$B' = \frac{\text{Cov}(\mu'', \sigma'')}{\sigma^2} = -\frac{\bar{1}' \alpha}{|\Lambda|},$$

$$C' = \frac{\text{Var}(\sigma'')}{\sigma^2} = \frac{\bar{1}' \bar{1}}{|\Lambda|}$$

and

$$|\Lambda| = \bar{1}' \alpha' \alpha - (\alpha' \bar{1})^2.$$  

Also,

$$E[(\bar{\mu}' - \mu) (\bar{\sigma}' - \sigma)] = \left( \frac{B'}{1 + C'} \right) \sigma^2$$

with $B'$ and $C'$ as defined above.
Special cases (a)

1) SLIE of $\mu$ when $\sigma = \sigma_0$ known.

Using the SLUE's $\mu^{**}$ and $\sigma^{**}$ of $\mu$ and $\sigma$ given by (5.3.5) and (5.3.6), in place of BLUE's of $\mu$ and $\sigma$ in (5.4.6), we obtain the SLIE of $\mu$ when $\sigma$ is known as

$$\hat{\mu}_\sigma = \mu^{**} - \frac{B'}{C'}(\sigma^{**} - \sigma_0)$$  \hspace{1cm} (5.5.10)

which is unbiased.

The variance of $\hat{\mu}_\sigma$ is given by

$$\text{Var}(\hat{\mu}_\sigma) = \left( A' - \frac{B'^2}{C'} \right) \sigma_0^2,$$  \hspace{1cm} (5.5.11)

where $A'$, $B'$ and $C'$ are as given by (5.5.5), (5.5.6) and (5.5.7) respectively.

2) SLIE of $\sigma$ when $\mu = \mu_0$ known.

Using $\sigma^{**}$ given by (5.3.6) in place of $\sigma^*$ in (5.4.8), we get SLIE of $\sigma$ when $\mu$ is known, as

$$\hat{\sigma}_\mu = \frac{\sigma^{**}}{1 + C'_\mu}. \hspace{1cm} (5.5.12)$$

Also,

$$\text{MSE}(\hat{\sigma}_\mu) = \left( \frac{C'_\mu}{1 + C'_\mu} \right) \sigma^2. \hspace{1cm} (5.5.13)$$

where

$$C'_\mu = (\alpha' \alpha)^{-1} = \left( \sum_{i=1}^{k-l} \sum_{j=i+1}^{l} \alpha_i^2 \right)^{-1}$$
Special cases (b)

We consider here the four important special cases of the multiply Type II censoring schemes as given in Section 2.4. We obtained the SLIE’s of $\mu$ and $\sigma$ under each of these censoring schemes and also in the three cases as explained in Section 5.3 special cases (b).

1) Double censoring

(i) Both the parameters $\mu$ and $\sigma$ are unknown

Letting $k = 2$ in (5.5.1) to (5.5.4) and (5.5.9) we get the results for $\bar{\mu}'$, $\bar{\sigma}'$, $\text{MSE}(\bar{\mu}')$, $\text{MSE}(\bar{\sigma}')$ and $\text{Cov}(\bar{\mu}', \bar{\sigma}')$ respectively.

(ii) $\sigma = \sigma_0$ known and $\mu$ is unknown

In this case the SLIE $\bar{\mu}'$ and its variance are obtained from (5.5.10) and (5.5.11) respectively, by letting $k = 2$ in them.

(iii) $\mu = \mu_0$ known and $\sigma$ is unknown

The SLIE $\bar{\sigma}'$ and its variance are obtained from (5.5.12) and (5.5.13) respectively, by letting $k = 2$ in them.

2. Right censoring

i) Both the parameters are unknown

Letting $r_1 = 0$ in the results for double censoring, we get the corresponding results for right censoring.

The results for the other two cases can be obtained similarly.
3. **Left censoring**

Letting \( r_2 = n \) in the results for double censoring, we get the corresponding results for left censoring.

The results for the other two cases can be obtained similarly.

4. **Middle censoring**

i) Both the parameters are unknown

Letting \( k = 3, r_1 = 0 \) and \( r_4 = n \) in (5.5.1) to (5.5.4) and (5.5.9) we get the results for double censoring. Letting \( k = 2 \) in (5.5.1) to (5.5.4) and (5.5.9) we get the \( \mu', \sigma', \text{MSE}(\mu'), \text{MSE}(\sigma') \) and \( \text{Cov}(\mu', \sigma') \) respectively.

(ii) \( \sigma = \sigma_0 \) known and \( \mu \) is unknown

When we assume \( \sigma = \sigma_0 \) known, the \( \mu' \) and the variance of \( \mu' \) are obtained from (5.5.10) and (5.5.11) respectively, by letting the above values for \( k, r_1 \) and \( r_4 \).

(iii) \( \mu = \mu_0 \) known and \( \sigma \) is unknown

When we assume \( \mu = \mu_0 \), the \( \sigma' \) and the variance \( \text{Var}(\sigma') \) are obtained from (5.5.12) and (5.5.13) respectively, by substituting the above values for \( k, r_1 \) and \( r_4 \).