Chapter 1

INTRODUCTION
1.1. Introduction

The decomposition, synthesis and analysis of functions, in various function spaces, is one of the most useful activities of mathematics, mathematical physics and lastly of Data/signal processing in engineering. It is one of the most intensively studied topics in mathematics since the beginning of 19th century. The bold declaration of Fourier (Pinsky, (2002)), about expressing any periodic function in the form of trigonometric series kept the entire scientific community on alert for nearly two centuries-especially in establishing convergence, uniform convergence etc of Fourier Series. In this process one finds blossoming of rich topics like Lebesgue integral, functions of striking theories in Banach space, Hilbert space etc.

A function has power series representation if it is smooth and gives its local structure accurately. Fourier series as well series of functions with classical orthogonal functions are useful for the global analysis of functions. These can be used for representing functions with no smoothness properties but they are inefficient for analyzing the detail behavior of a function near a point. A pure Fourier basis diagonalises translation invariant linear operators. We look for a basis (of function spaces) that is well localized in frequency and nearly diagonalizes the operator i.e. their matrix entries decay rapidly away from the diagonal. Also, it is desirable a basis to be well localized in space for effective local analysis. What is missing is a method for analyzing the local irregular behavior of functions that aren't smooth and this is where wavelets come into
play. Haar’s (1910) research led to the simplest orthonormal wavelet basis, (a set of rectangular basis functions of $L^2(0, 1)$ with compact support and localized in space/time but not continuous). The classical uncertainty principle stood as an obstruction in achieving the best of the both (localization in both time – frequency or space – frequency). The work of early mathematicians like Calderon-Zygmund, Paley-Littlewood, Balian-Low and Stein laid the groundwork for future development of the theory. Later, it was Gabor (1946) whose idea was to break the signal into segments/slices and then analyse the individual segments each of which will have well defined frequency band and position in time which enabled for data analysis.

Wavelet has oscillatory behavior in some region and decay rapidly to zero outside this region. A remarkable discovery of recent years is that the translations and dilations of certain wavelets can be used to form set of basic (basis) functions for representing general function into infinite series which have been playing dominant role in mathematics and applications. In fact wavelets were implicit in several scientific fields but it is the greatest tribute to the efforts of many since last 10-15 years to know, for instance Littlewood-Paley theory in mathematics and pyramidal algorithm of DSP have the same roles. The synthesis of these theories leading to Wavelet analysis and the impetus to great interest was the result of research by Grossman and Morlet (1984) who called Littlewood–Paley theory as Wavelet theory. Meyer (1993) and his coworkers noticed that Calderon – Zygmund theory, in particular the Littlewood – Paley representation having
discrete analog gave unified view of research in harmonic analysis. Lamarie and Meyer (1986) constructed new orthogonal Wavelet expansion. Mallat (1989) introduced the most useful functional analysis of multiresolution concept. Daubechies (1988), using Mallat's ideas, constructed discrete orthonormal smooth wavelets with compact support. From this stage – from continuous signal processing to discrete signal processing opened the flood gates in its theory and applications and since then several others have been contributing substantially (Daubechies, 1988, 1992), Jawerth and Sweldens, (1994), Strang, (1989), Benedetto and Frazier, (1994)), and Wavelet theory became microscope of Mathematics.

Wavelets in numerical analysis are found to have the same successful role which they are enjoying in signal processing. Resnikoff and Wells (1998), Benedetto and Frazier (1994) list some of the recent contributions of Wavelet theory in solving b.v.p. with periodic or unbounded domain and also those with finite domain. The accuracy and ease with which these wavelet based techniques can be implemented are illustrated by taking model equations: Helmholtz equation in two dimension (Qian and Weiss, (1993)), Burger equation (Chen et al. (1996)). Capacitance matrix method (Priskurowski and Widlund, (1976)), Green function formulation, periodized wavelet methods (Beylkin, (1992)), are also used successfully for the solution of model examples. Wavelet – Galerkin method has proved to be a success in its applications in solving second kind of integral equations (Micchelli et al. (1997)) and large class of p.d.e. Wavelet – capacitance
matrix method enables to handle boundary geometry (especially non separable domain etc.). The discretization of linear equations results into linear (sparse for differential equations and full for Integral equations) system of equations. Condition numbers of such systems are large for differential operator and small for integral operator. In wavelet bases there is a diagonal preconditioner for differential operator leading to O(N) algorithms for their solution.

Wavelet-Galkerkin methods involving discretisation of equations using wavelet bases require integration of integrals with integrands being products of wavelet bases and their derivatives (Connection coefficients). Suitable algorithms for their evaluations can be found in Chen et al. (1996). It is quite interesting and fascinating to note that the evaluation of these integrals lead to algorithms which depend only on wavelet matrix defining wavelet bases. With Daubechies wavelet - system connection coefficients are obtained in terms of rational numbers resulting from recursive relations. In recent developments wavelet packets are used for unveiling smoothness property or singularity, if any, of P.D.E. Recently the wavelet analysis is gaining considerable attention in the numerical solution of Integral equations (variational problems(Hsiao, (2004), etc), Integro-differential equations (Razzaghi & Ordokhani, (2001), Avudainayagam & Vani, (2000)), Stiff-differential equations (Hsiao, (2004), Sepehrian & Razzaghi, (2003), etc) and Partial differential equations( Williams & Amaratunga, (1994), Chen et al. (1996), Rathish Kumar, (2005), Avudainayagam & Vani, (2004), etc), which are current topics of interest.
Mathematical modeling of many problems of interest are (in general) expressed in variational formulations, Integro-differential equations, stiff-differential equations and partial differential equations. In this class of problems, nonlinear problems and particularly elliptic partial differential equations have attracted the attention of best brains in mathematics. This has motivated us to study the wavelet series solution of equations. Hence, the thesis is devoted to study the Haar wavelet series solution to different class of variational problems, Rationalized Haar wavelet series solution to nonlinear Integro-differential equations, Single term Haar wavelet series solution to nonlinear stiff-differential equations and Dabechies wavelet series Multigrid Method to elliptic partial differential equations arising in fluid dynamics.

Solving Navier-Stokes equations numerically is important since, many applications in computational science and engineering depend on it and only few simple cases have analytical solutions. Extensive research is done on finding fast and reliable methods for solving Navier-Stokes equations on very fine meshes for a wide range of parameters involved. Numerical (finite difference based Daubechies wavelet series multigrid method) study of Reynolds equations describing mathematical modeling of hydrodynamic lubrication characteristics of isotropic, poroelastic squeeze film bearings is one of the current topics of interest.
1.2. Hilbert Spaces and Bases for function spaces

The theory of Hilbert spaces (Bachman et al. (2000)), plays an important role in the development of wavelet series analysis. One of the nice features of normed spaces is that their geometry is very much similar to the familiar two-and three-dimensional Euclidean geometry. Inner product spaces and Hilbert spaces are even nicer because their geometry is even closer to Euclidean geometry. In fact, the geometry of Hilbert spaces is more or less a generalization of Euclidean geometry to infinite dimensional spaces. The main reason for this simplicity is that the concept of orthogonality can be introduced in any inner product space so that the familiar Pythagorean formula holds. Thus, the structure of Hilbert spaces is more simple and beautiful, and hence, a large number of problems in mathematics, science, and engineering can be successfully treated with geometric methods in Hilbert spaces (Bachman et al. (2000)).

Bases for function spaces: An orthonormal basis is a natural representation and an effective tool / language for function spaces. Given an orthonormal basis \( \{f_i\} \) for \( L^2(\mathbb{R}) \) any \( f \in L^2(\mathbb{R}) \) has representation

\[
f(x) = \sum_{n=1}^{\infty} a_n f_n(x)
\]

(1.2.1)

where

\[
a_n = \langle f_n, f \rangle = \int f_n(x) f(x) dx, \quad n \in \mathbb{N}.
\]
Without orthogonality the co-efficients are to be determined by solving system of equations. Generally, the above summation is truncated and only finite terms are retained

\[ f(x) = \sum_{k=1}^{n} a_k f_k(x) + E_{n+1}(x), \quad (1.2.2) \]

where the truncation error \( E_{n+1}(x) \) is given by

\[ E_{n+1}(x) = \sum_{j=n+1}^{\infty} a_j f_j(x). \quad (1.2.3) \]

For the accurate representation of \( f \) by (1.2.2) it is desirable that the error decays with increasing \( n \) as rapidly as possible. If \( k \) is the largest integer such that \( n^k \|E_{n+1}(x)\| \) is bounded as \( n \to \infty \), for a given \( x \), then the expansion (1.2.1) of \( f \) is \( k^{th} \) order convergent. If \( \sup_n n^k \|E_{n+1}(x)\| < \infty \), for all \( k \), the expansion is said to converge super-algebraically.

Classical bases: \( \{e^{inx}\}, \forall x \in \mathbb{Z} \) is an orthonormal basis for \( L^2[-\pi, \pi] \) under the inner product

\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx. \quad (1.2.4) \]

Many properties of the Fourier basis make it well suited to computation. Differentiation and integration of functions represented in Fourier basis is easy. Transformation of a function tabulated at \( n \) equispaced points on \([-\pi, \pi]\) into an \( n \) terms Fourier expansion or its inverse is obtained rapidly using Fast Fourier transform. Also, convolutions are diagonal operators in Fourier basis.
Another most common class of bases are orthogonal polynomials. Given an integral \( I \subseteq \mathbb{C} \) and a positive weight function

\[
w : I \to \mathbb{C}
\]

we can define an inner product

\[
\langle f, g \rangle_w = \int_I f(x)g(x)w(x) \, dx
\]

There is a sequence of polynomials \( \{p_j(x)\} \) with \( p_j(x) \) of degree \( j \) which is orthogonal w.r.t \( w \) i.e. \( \langle p_m(x), p_n(x) \rangle \equiv 0 \) if \( m \neq n \). The sequence \( \{p_j(x)\} \) forms a basis for the function space defined on \( I \) which are square integrable with weight \( w \). The sequence is uniquely determined upto leading coefficient and can be computed by the Gram-Schmidt orthogonalization process. For \( I = [-1,1] \), \( w(x) = 1 \), the sequence is Legendre polynomials – they form orthogonal basis for \( L^2[-1,1] \). The transformation of tabulated function to expansion in terms of orthogonal polynomials is easy to perform. Differentiation and integration of such representations are also easy.

Time frequency localization: Classical bases have global nature and not suitable to local analysis to account for sharp/sudden variations in parts of the domain. These are associated with high frequency components. Sincere efforts, using short term Fourier transform, Gabor transform etc to accommodate these features were of limited success/utility. Whereas wavelet bases are developed to analyse time frequency (or other similar effect) localisation effectively. They have built in zooming structure to accomplish global as well as local analysis.
1.3. Multiresolution Analysis

In 1986, Stephane Mallat and Yves Meyer first formulated the idea of multiresolution analysis (MRA). This is a new and a remarkable idea which deals with a general formulation for construction of an orthogonal basis of wavelets. Indeed, multiresolution analysis is central to all constructions of wavelet bases. Mallat's brilliant work (1989a,b,c) has been the major source of many new developments in wavelet analysis and its wide variety of applications.

Mathematically, the fundamental idea of multiresolution analysis is to represent a function (or signal) $f$ as a limit of successive approximations, each of which is a finer version of the function $f$. These successive approximations correspond to different levels of resolutions. Thus, multiresolution analysis is a formal approach in constructing orthogonal wavelet bases using a definite set of rules and procedures. The key feature of this analysis is to describe mathematically the process of studying signals or images at different scales. The basic principle of the MRA deals with the decomposition of the whole function space into individual subspaces $V_n \subset V_{n+1}$ so that the space $V_{n+1}$ consists of all rescaled functions in $V_n$. This essentially means a decomposition of each function (or signal) into components of different scales (or frequencies) so that an individual component of the original function $f$ occurs in each subspace. These components can describe finer versions of the original function $f$. In audio signals; these scales are basically octaves which represent higher and higher frequency
components. For images and, indeed, for all signals, the simultaneous existence of a multiscale may also be referred to as multiresolution. From the point of view of practical application, MRA is really an effective mathematical framework for hierarchical decomposition of an image (or signal) into components of different scales (of frequencies).

**Multiresolution Analysis:** A multiresolution analysis (MRA) consists of a sequence \( \{ V_m : m \in \mathbb{Z} \} \) of embedded closed subspaces of \( L^2(\mathbb{R}) \) that satisfy the following conditions:

(i) \( \ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m \subset V_{m+1} \ldots \),

(ii) \( \bigcup_{m = -\infty}^{\infty} V_m \) is dense in \( L^2(\mathbb{R}) \), that is, \( \bigcup_{m = -\infty}^{\infty} V_m = L^2(\mathbb{R}) \),

(iii) \( \bigcap_{m = -\infty}^{\infty} V_m = \{0\} \),

(iv) \( f(x) \in V_m \) if and only if \( f(2x) \in V_{m+1} \) for all \( m \in \mathbb{Z} \),

(v) there exists a function \( \phi \in V_0 \) such that \( \{ \phi_{0,n} = \phi(x-n), n \in \mathbb{Z} \} \) is an orthonormal basis for \( V_0 \), that is,

\[
\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{n = -\infty}^{\infty} |(f, \phi_{0,n})|^2 \text{ for all } f \in V_0
\]

The function \( \phi \) is called the **scaling function or father wavelet**. If \( \{ V_m \} \) is a multiresolution of \( L^2(\mathbb{R}) \) and if \( V_0 \) is the closed subspace generated by the integer
translates of a single function $\phi$, then we say that $\phi$ generates the multiresolution analysis.

Sometimes, condition (v) is relaxed by assuming that $\{\phi(x - n), n \in \mathbb{Z}\}$ is a Riesz basis for $V_0$, that is, for every $f \in V_0$, there exists a unique sequence $\{c_n\}_{n=-\infty}^{\infty} \in l^2(\mathbb{Z})$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi(x - n)$$

with convergence in $L^2(\mathbb{R})$ and there exist two positive constants $A$ and $B$ independent of $f \in V_0$ such that

$$A \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \|f\|^2 \leq B \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where $0 < A < B < \infty$. In this case, we have a multiresolution analysis with a Riesz basis.

Note that condition (v) implies that $\{\phi(x - n), n \in \mathbb{Z}\}$ is a Riesz basis for $V_0$ with $A = B = 1$, and we get orthonormal basis, since $\phi_{0,n}(x) \in V_0$ for all $n \in \mathbb{Z}$. Further, if $n \in \mathbb{Z}$, it follows from (iv) that

$$\phi_{m,n}(x) = 2^{n/2} \phi(2^m x - n), \ m \in \mathbb{Z}, \quad (1.3.1)$$

is an orthonormal basis for $V_m$.

In general, frames have many of the properties of bases, but they lack a very important property of orthogonality. If the condition of orthogonality $\langle \phi_k, \phi_{m,n} \rangle = 0$ for all $(k, l) \neq (m, n)$ is satisfied, the reconstruction of the
function $f$ from $\langle f, \phi_{m,n} \rangle$ is much simpler and for any $f \in L^2(R)$, we have the following representation

$$f = \sum_{m,n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n},$$

where $\phi_{m,n}$ is given by (1.3.1).

The real importance of a multiresolution analysis lies in the simple fact that it enables us to construct an orthonormal basis for $L^2(R)$. In order to prove this statement, we first assume $\{V_m\}$ as a multiresolution analysis. Since $V_m \subset V_{m+1}$, we define $W_m$ as the orthogonal complement of $V_m$ in $V_{m+1}$ for every $m \in \mathbb{Z}$ so that we have

$$V_{m+1} = V_m \oplus W_m = (V_{m-1} \oplus W_{m-1}) \oplus W_m = \ldots = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_m = V_0 \oplus \bigoplus_{n=0}^{m} W_n$$

(1.3.2)

and $V_n \perp W_m$ for $n \neq m$. Since $\bigcup_{m=-\infty}^{\infty} W_m$ is dense in $L^2(R)$, we may take the limit as $m \to \infty$ to obtain $V_0 \oplus \bigoplus_{m=0}^{\infty} W_m = L^2(R)$.

Similarly, we may go in the other direction to write

$$V_0 = V_{-1} \oplus W_{-1} = (V_{-2} \oplus W_{-2}) \oplus W_{-1} = \ldots = V_{-m} \oplus W_{-m} \ldots \oplus W_{-1}.$$  

We may again take the limit as $m \to \infty$. since $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$, it follows that $V_{-m} = \{0\}$. Consequently, it turns out that

$$\bigoplus_{m=-\infty}^{\infty} W_m = L^2(R).$$  

(1.3.3)
Finally, the difference between the two successive approximations $p_m f$ and $p_{m+1} f$ is given by the orthogonal projection $Q_m f$ of $f$ onto the orthogonal complement $W_m$ of $V_m$ in $V_{m+1}$, so that

$$Q_m f = p_{m+1} f - p_m f.$$  

It follows from conditions (i)-(v) in MRA that the space $W_m$ are also scaled versions of $W_0$ and, for $f \in L^2(R)$,

$$f \in W_m \text{ if and only if } f(2^{-m} x) \in W_0 \text{ for all } m \in \mathbb{Z},$$  \hspace{1cm} (1.3.4)

and they are translation-invariant for the discrete translations $n \in \mathbb{Z}$, that is, $f \in W_0$ if and only if $f(x-n) \in W_0$, and they are mutually orthogonal spaces generating all of $L^2(R)$,

$$W_m \perp W_k \text{ for } m \neq k, \hspace{1cm} (1.3.5a)$$

$$\bigoplus_{m \in \mathbb{Z}} W_m = L^2(R). \hspace{1cm} (1.3.5b)$$

Moreover, there exists a function $\psi \in W_0$ such that $\psi_{0,n}(x) = \psi(x-n)$ constitutes an orthonormal basis for $W_0$. It follows from (1.3.4) that

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n), \text{ for } n \in \mathbb{Z} \hspace{1cm} (1.3.6)$$

constitute an orthonormal basis for $W_m$. Thus, the family $\psi_{m,n}(x)$ represents an orthonormal basis of wavelets for $L^2(R)$.

Normally, MRA enables to select suitable scaling function $\phi$ and generates required bases. Detail and interesting analysis about existence of given $\phi$ and (i)-(v) of MRA can be found in Daubechies (1992).
1.4. Construction of Haar Wavelets

The spaces $V_m$ of piecewise constant functions in the intervals $[2^{-m} n, 2^{-m}(n+1)]$, where $n \in \mathbb{Z}$, constitutes a MRA with the scaling function $\phi = \chi_{[0,1]}$, must generate a subspace $V_0$ and $V_0 \subset V_1$.

Moreover, $\phi$ satisfies the dilation equation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \phi(2x - n),$$

(1.4.1)

where the coefficients $c_n$ are given by

$$c_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(x) \phi(2x - n) dx.$$  

(1.4.2)

Evaluating this integral with $\phi = \chi_{[0,1]}$ gives $c_n$ as follows:

$$c_0 = c_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad c_n = 0 \quad n \neq 0, 1.$$

Consequently, the dilation equation becomes

$$\phi(x) = \phi(2x) + \phi(2x - 1).$$

(1.4.3)

This means that $\phi(x)$ is a linear combination of the even and odd translates of $\phi(2x)$ and satisfies a very simple two-scale relation (1.4.3).

Scaling function also satisfies

$$\int_{-\infty}^{\infty} \phi(x) dx = 1, \quad \int_{-\infty}^{\infty} [\phi(x)]^2 dx = 1.$$

A wavelet $\psi(x)$ for which $\{\psi(x-k)\}$ spans the wavelet subspace $W_0$ and can be defined by
\[ \psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} d_n \phi(2x-n), \]  
\hfill (1.4.4)

where \( d_n = (-1)^n c_{1-n} \). We obtain
\[ d_0 = c_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad d_1 = -c_0 = -\frac{1}{\sqrt{2}}. \]

Thus, the Haar mother wavelet is obtained from (1.4.4) as a simple two-scale relation
\[ \psi(x) = \phi(2x) - \phi(2x-1) \]
\[ = \chi_{[0,0.5]}(x) - \chi_{[0.5,1]}(x) \]
\hfill (1.4.5)

\[ \psi(x) = \begin{cases} 
+1, & 0 \leq x < \frac{1}{2} \\
-1, & \frac{1}{2} \leq x < 1 
\end{cases}. \]  
\hfill (1.4.6)

It is zero outside \([0, 1)\) and well localized in time. Besides this it satisfies
\[ \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1. \]

1.5. Construction of Daubechies Wavelets

The major defect with Haar wavelets is its discontinuity. Daubechies wavelets have smoothness of varying degree, continuous and are also compactly supported (Daubechies, 1988, 1992)).

Let us show how the program of the multiresolution analysis works in practice when applied to the problem of finding out \( c_n \) and \( d_n \) which are often referred to as scaling and wavelet coefficients or filter coefficients in signal processing. These coefficients are defined by the relations (1.4.1) and (1.4.4)
\begin{align}
\phi(x) &= \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \phi(2x-n); \quad \psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} d_n \phi(2x-n), \tag{1.5.1}
\end{align}

where \( \sum_n |c_n|^2 < \infty \). The orthogonality of the scaling functions defined by the relation

\[ \int \phi(x)\phi(x-l)dx = 0 \tag{1.5.2} \]

leads to the following equation for the coefficients:

\[ \sum_n c_n c_{n+2l} = \delta_{01}. \tag{1.5.3} \]

The orthogonality of wavelets to the scaling functions

\[ \int \psi(x)\phi(x-l)dx = 0 \tag{1.5.4} \]

gives the equation

\[ \sum_n c_n d_{n+2l} = 0, \tag{1.5.5} \]

having a solution of the form

\[ d_n = (-1)^n c_{2L-1-n}. \tag{1.5.6} \]

Another condition of the orthogonality of wavelets to all polynomials up to the power (L-1), defining its regularity and oscillatory behavior

\[ \int x^k \psi(x)dx, \quad k=0,\ldots,(L-1), \tag{1.5.7} \]

provides the relation

\[ \sum_n n^k d_n = 0, \tag{1.5.8} \]

which in turn gives
\[ \sum_n (-1)^n n^4 c_n = 0, \]  
when the formula (1.5.6) is taken into account.

The normalization condition
\[ \int \phi(x) dx = 1 \]
(1.5.10)
can be rewritten as another equation for \( c_n \)
\[ \sum_n c_n = \sqrt{2}. \]  
(1.5.11)

Let us write down the equations (1.5.11), (1.5.9), (1.5.3) for \( L = 2 \) explicitly
\[
\begin{align*}
    c_0 + c_1 + c_2 + c_3 &= \sqrt{2}, \\
    c_0 - c_1 + c_2 - c_3 &= 0, \\
    -c_1 + 2c_2 - 3c_3 &= 0, \\
    c_0c_2 + c_1c_3 &= 0.
\end{align*}
\]

The solution of this system is
\[ c_i = \frac{1}{4\sqrt{2}}(1 \pm \sqrt{3}), \quad c_2 = \frac{1}{2\sqrt{2}} + c_3, \quad c_1 = \frac{1}{\sqrt{2}} - c_3, \quad c_0 = \frac{1}{2\sqrt{2}} - c_3, \]  
(1.5.12)
that, in the case of the minus sign for \( c_3 \), they corresponds to the well known filter coefficients
\[ c_0 = \frac{1}{4\sqrt{2}}(1 + \sqrt{3}), \quad c_1 = \frac{1}{4\sqrt{2}}(3 + \sqrt{3}), \quad c_2 = \frac{1}{4\sqrt{2}}(3 - \sqrt{3}), \quad c_3 = \frac{1}{4\sqrt{2}}(1 - \sqrt{3}). \]  
(1.5.13)

These coefficients define the simplest \( D_4 \) wavelet from the famous family of orthonormal Daubechies wavelets with finite support \([0, 3]\).

The choice of the plus sign in the expression for \( c_3 \) would not change the general shapes of the scaling function and wavelet \( D_4 \). It results in their mirror
symmetrical forms obtained by a simple reversal of the signs on the horizontal and vertical axes, correspondingly. However, for higher rank wavelets different choices of signs would correspond to different forms of the wavelet. After the signs are chosen, it is clear that compactly supported wavelets are unique, for a given multiresolution analysis up to a shift in the argument (translation) which is inherently there. The dilation factor must be rational within the framework of the multiresolution analysis. Typically wavelets are more regular in some points than in others.

For the filters of higher order in $L$, i.e., for higher rank Daubechies wavelets, the coefficients can be obtained in an analogous manner. It is however necessary to solve the equation of the $L^{th}$ power in this case. Therefore the numerical values of the coefficients can be found only approximately, but with any predefined accuracy. The wavelet support is equal to $2L-1$. It is wider than for the Haar wavelets. However, the regularity properties are better. The higher order wavelets are smoother compared to $D4$. For sufficiently regular functions, Daubechies wavelet coefficients are much smaller ($2^{-L}$ times) than the Haar wavelet coefficients, i.e., the signal can be compressed much better with Daubechies wavelets. Since they are more regular, the synthesis is more efficient also.

Except for the Haar basis, all real orthonormal wavelet bases with compact support are asymmetric, i.e., they have neither symmetry nor antisymmetry axis. Better symmetry for a wavelet necessarily implies better symmetry for the coefficients $c_n$ but the converse statement is not always true (Valens, (1999)).
The above description about the construction of wavelets is in time domain. The most general way of construction of various types of wavelets is normally in frequency domain (Debnath, (2002), Hardle, (1998), Daubechies, (1992)), which is not repeated here, the whole analysis is quite elegant and most of the arguments are recursive in nature requiring less effort ($O(n)$), and enable computing of coefficients rapidly.

1.6. Fluid Mechanics

The fluid mechanics theory is based on the continuum hypothesis concept that the fluid mass is distributed throughout the space such that the field theories become applicable mathematical tools in the description of fluid motions. Mathematical modeling is useful to describe physical chemical, biological or engineering phenomena. One of the most common features of the modeling process is the one resulting into systems of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations is the central part of the applied mathematics.

The governing equations for conservation of mass, momentum and energy for the flow region (Serrin, (1959)), without sources and sinks etc. are

\[
\frac{D \rho}{Dt} + \rho V_{i,j} = 0, 
\]

\[\rho \frac{DV_i}{Dt} = \rho F_i + T_{ij}, \quad (1.6.2)\]

and \[\rho \frac{De}{Dt} = \phi' - divq, \quad (1.6.3)\]
respectively, where \( \frac{D}{Dt} \) is the material time derivative, comma denotes the covariant derivative, \( \rho \) is the density of the fluid, \( V_i \) are the components of velocity vector, \( F_i \) the components of body force per unit mass, \( T \) the components of stress vector, \( e \) the internal energy, \( \bar{q} \) is the heat flux vector given by

\[
\bar{q} = -k' \frac{\partial T}{\partial n} \bar{q}
\]

(1.6.4)

where \( k' \) is the thermal conductivity of the fluid, \( T \) the temperature and \( n \) the normal to the area element across which the heat flux is considered. \( \phi' \) denotes the heat flux in unit volume due to the heating during fluid motion, which depends upon the properties of the fluid material and internal heat sources if present in the flow field.

In the analysis of the problems considered in this thesis, it is assumed that, the lubricant is an incompressible fluid and for such fluids \( \rho \) is constant. Then the continuity equation (1.6.1) takes the form

\[
V_{\alpha} = 0.
\]

(1.6.5)

1.7. Hydrodynamic Lubrication

The theoretical study of hydrodynamic lubrication is the study of a particular form of the Navier-Stokes equations governing the pressure in the fluid film, which was first derived by Reynolds (1886) in the wake of the experiment of Beauchamp Tower. The experimental results of Tower were analyzed and interpreted by Reynolds. Tower's experiment showed the formulation of a thin
film between the two lubricating surfaces for the first time. A bearing with reference to a machine or a structure, means contacting surfaces through which load is transmitted when surfaces are in relative motion, one desires to minimize friction and wear. Any substance when introduced between the moving surfaces that reduces the friction and wear is called lubricant. The objective of the lubrication is to reduce friction, wear and heating of machine parts, which are in relative motion. In fluid film bearings, a very thin layer of fluid completely separates two solid surfaces, which are in relative motion. With the fluid film, this motion causes a shearing action that requires relatively small effort in the direction of motion. The surfaces are usually part of a bearing, which locate and support a shaft and its loads. The pressure must be developed in the fluid film to support a normal load. This pressure developed in the fluid depends on the type of lubrication. The types of lubrication, hydrodynamic lubrication, mixed lubrication, boundary lubrication, hydrostatic lubrication, elastohydrodynamic lubrication etc. are based on the degree with which lubricant separates the surfaces (Cameron, (1981), Pinkus and Sternlicht, (1961) and Majumdar, (1999)).

In addition to this, there is another class of fluid film lubrication known as squeeze-film bearing, which supports load due to relative normal motion. In the design process of hydrodynamic bearings, some of the important characteristics of bearings such as load carrying capacity, flow requirement and power loss due to viscous friction are to be predicted accurately. These bearing parameters can be obtained only when the pressure in the fluid film is known. To determine the
pressure in the film region, one has to solve a particular form of the Navier-Stokes equation along with the continuity equation after making the appropriate lubrication approximation (Pinkus and Sternlicht, (1961)). Such an equation is generally called Reynolds equation in the tribology literature. Normally, the parameters representing various aspects of bearings, such as bearing geometry, operating conditions and film shape, influence of the film forces under steady state conditions of laminar incompressible films etc. are useful for detail study.

1.8. Synovial Joint Lubrication

With the increased interest by both Engineers and Orthopedic surgeons into biomechanics of degenerative joint disease, different lubrication modes are of immense important. Synovial joint acts as a biomechanical bearing and its lubricating mechanisms are extremely complex. Thus the theory of lubrication has developed from the simple dry friction of two sliding solids to elaborate non-Newtonian, porous, elastic and temperature dependent formulations. Boundary lubrication is governed by the chemical composition and physical properties of the lubricant and articular cartilage. Boundary layer (Barwell, 1967), occurs when the surfaces are separated by film of thickness of few molecules. Friction develops due to the shearing of the lubricant. The friction coefficient is independent of lubricant viscosity. The main function of the lubricant is to reduce the friction and carry part of the applied load.

In fluid film lubrication, the opposing surfaces are separated by a relatively thick film of lubricant. The developed fluid film thickness is greater than bearing
surface roughness and wear due to surface-surface contact is eliminated. Due to the relative surface motion the lubricant is drawn between the opposing surfaces forming a wedge of lubricant which acts to separate the surface and carry some of the bearing loads (Dowson, 1967). For a constant load, the bearings surface deformation aids in producing a greater lubricant film thickness than that of hydrodynamic theory (Crook, 1958). The lubrication theory of elastic deformation of the bearing surfaces considered together with hydrodynamic thin film theory is called elastohydrodynamic lubrication (EHL). Because of bearing deformation, bearings are able to carry extreme loads.

Dintenfass (1963) has shown that the deformation of the cartilage and the non-Newtonian behavior of the synovial fluid are important in the consideration of joint lubrication. He suggested the use of the concepts of the EHL theory to describe joint lubrication and then it was confirmed by Tanner (1966). These studies postulated that the lubricant fluid is squeezed out of the tissue under the loaded region where the tissue deformation takes place and reabsorbed when unloaded. McCutchen (1962) hypothesized weeping lubrication in which the lubricant film is supplied by the exudation of interstitial fluid from the compressed cartilage. Maroudas (1967) and Walker et al. (1968) proposed independently an alternate model called boosted lubrication of joints in which the hyaluronic acid-protein complex of synovial fluid flow into loaded region helps to increase its concentration during squeeze film action. A fixed synovial film is formed between the surfaces preventing their intimate contact. In order to show the joints low
friction, Dowson (1967) proposed a combination of boundary, weeping and EHL lubrication. This study suggests that due to the hyaluronic acid complex present in the synovial fluid, a thick dense substance was being formed on the cartilage surface during the squeezing process.

1.9. Couple-stress Fluids

Number of theories have been proposed to explain the peculiar behavior of fluids, which contain microstructures such as additives, suspensions or granular matter. Eringen’s (1966) micropolar fluid theory defines the rotational field in terms of kinematically independent rotation vector called micro rotation vector for setting up of stress-strain rate constitutive equations. However, the theory of couple-stress fluid given by Stokes (1966) defines the rotational field in terms of the velocity field, thereby reducing considerably the number of material constants in the constitutive equations characterizing fluid material. This theory introduces a second order gradient of velocity vector, instead of kinematically independent rotation vector in the constitutive relationship between stress rates. Stokes theory of couple-stress fluids is the simplest generalization of the classical theory of fluids which allows for the polar effects such as the presence of a non-symmetric stress tensor, couple stresses and body couples. The constitutive equations for couple-stress fluids proposed by Stokes are

\[
T_{(i,j)} = -p\delta_{ij} + \lambda_c \delta_{ij} + 2\mu \ n_{ij},
\]

(1.9.1)

\[
T_{(\eta)} = -2\eta \ n_{ij} - \frac{\rho}{2} \ v_{ij} G_z,
\]

(1.9.2)
and \( M_y = 4\eta w_{,j,j} + 4\eta' w_{,j,j}, \) \hspace{1cm} (1.9.3)

where

\[ d_y = \frac{1}{2} [V_{,i,j} + V_{,j,i}], \] \hspace{1cm} (1.9.4)

\[ W_y = -\frac{1}{2} [V_{,i,j} + V_{,j,i}], \] \hspace{1cm} (1.9.5)

\[ w_i = \frac{1}{2} \varepsilon_{ikl} V_{k,j}, \] \hspace{1cm} (1.9.6)

\( T_{(i,j)} \) and \( T_{[i,j]} \) are the symmetric and anti-symmetric parts of stress \( T_y \), \( M_y \) the couple-stress tensor, \( W_y \) the vorticity tensor, \( d_y \) the deformation rate tensor, \( \varepsilon_{ikl} \) the alternating unit tensor, \( V_i \) the components of velocity vector, \( G_s \) the body couple \( \delta_y \) the Kronecker delta, \( w_i \) the velocity vector, \( \rho \) the density, \( p \) the pressure, \( \lambda \) and \( \mu \) are the material constants of the dimension of viscosity, \( \eta \) and \( \eta' \) are the material constants having the dimensions of length squared and characterize the microstructure size.

When a natural synovial fluid fails to work properly, appropriate remedy has to be taken. The long chain hyaluronic acid molecules are found as polar additives in the synovial fluid which are characterized by two material constants \( \mu \) and \( \eta \). These two material constants are also found in Stokes couple-stress fluid as polar additives. This has motivated us to study the effect of couple stresses by modeling it as synovial fluid on the squeeze film lubrication of synovial joints. These aspects are extensively analyzed in part of the thesis.
1.10. Surface Roughness

The effect of surface roughness plays an important role in the development of science and technology of tribology. Since all bearing surfaces are rough to some extent on the microscopic scale, the effect of surface topography must be taken into account in dealing with any contact situation of real surfaces. In many cases, the roughness asperity heights are of the same order as the mean separation between the lubricated contacts. Several approaches have been proposed to study the surface roughness effects on the hydrodynamic lubrication of various bearings. Mitchell (1953) used the surface roughness by a high frequency sine curve. Perhaps it was the first published work on the study of surface roughness. It has been shown by Tzeng and Seibel (1967) that the surface roughness of the bearing surfaces significantly affects the bearing performance. Davies (1963) employed a saw-tooth curve to mathematically model the roughness asperities on the bearing surfaces. The Fourier series type approximation to model the large number of roughness asperities on the bearing surfaces is used by Burton (1963). The random character of the surface roughness was recognized by several investigators and have used a stochastic model to study surface roughness effects on bearing surfaces. Tzeng and Seibel (1967) have used the beta probability density function for the random variable to characterize the surface roughness. This distribution approximates the Gaussian distribution with good degree of accuracy and having symmetrical in nature with zero mean. Tonder (1972) studied lubrication of slider
bearing considering roughness striations in transverse and longitudinal direction and found that roughness effects result into increase of load, friction and side flow.

Christensen (1969) developed a stochastic theory for hydrodynamic lubrication of rough surfaces. This approach has formed the basis for the analysis of rough bearings. Rohde and Whicker (1977) have shown that using perturbation approach, the Christensen model can be viewed as an asymptotic limit in texture frequency and predicts some accurate results even for low texture frequency. Phan-Thein (1981) used Keller’s method to obtain Reynolds equation which is correct up to second order in amplitude of the surface roughness and shown that the theory of Christensen gives the load enhancement in two-dimensional slider bearing with exponential film thickness. Thus, Christensen stochastic theory has enabled to conjecture that the assumptions made above are correct up to second order for two-dimensional bearings with longitudinal or transverse roughness.

Electronic microscopic study of Sayles et al. (1979) revealed that the surfaces of articular cartilage are rough and roughness height distribution is Gaussian in nature. This has motivated us to study the effect of roughness on cartilage surfaces. Hence part of the thesis is devoted to study the effect of roughness on the articular surfaces by adopting the Christensen stochastic theory using Daubechies wavelet series Multigrid method.