Chapter 5*

DAUBECHIES WAVELET SERIES MULTIGRID ANALYSIS OF SQUEEZE FILM CHARACTERISTICS OF POROELASTIC BEARINGS

* Part of this chapter is accepted for publication in the journal "Journal of computational and applied mathematics", (2006), Elsevier Publication (article in press).
5.1. Introduction

In previous chapters we have used Haar wavelet series methods for the solution of Variational problems, Integro-differential equations and stiff-differential equations. In this Chapter, we present a wavelet-based method using Daubechies wavelet series for the solution of partial differential equations. We study the combined effects of poroelasticity and couple stresses on the performance of lubrication aspects of poroelastic bearings in general and that of synovial joints in particular. The modified form of Reynolds equation which incorporates the elastic nature of cartilage and Stokes couple-stress fluid as lubricant is derived and solved using a recently proposed Daubechies wavelet series multigrid method. This method has greatest advantage of minimizing the errors using wavelet transforms in obtaining accurate solution as grid size tends to zero. The system of equations arising from finite difference discretization is represented in wavelet-basis. These equations are solved using multiresolution properties of wavelets characterized by sparse matrices having condition number $O(1)$ together with a multigrid strategy for accelerating convergence. The filter coefficients of $D_4$ from Daubechies family of wavelet series are used to demonstrate the effectiveness and efficiency of the method. The distinguishing feature of the method is that it works as both solver and preconditioner. As a consequence, it avoids instability, minimizes error and speeds up convergence. It is found that, 6-7 cycles are required to obtain a reasonably accurate solution in classical multigrid scheme, whereas, only one cycle is required to obtain the solution in the Daubechies wavelet series-multigrid
method. Also, matrix of DWT (Discrete wavelet transform) acts as a natural preconditioner producing rapid convergence.

Daubechies wavelets are already recognized as a powerful new mathematical tool in signal and image processing, time-series analysis, geophysics and approximation theory etc. The last few years have witnessed an intense activity and interest in the application of wavelet theory and its associated multiresolution analysis. Another area in which Daubechies wavelets are gaining importance is in the numerical analysis of partial differential equations. The most popular numerical methods such as Galerkin and collocation methods share one common drawback. They require solution of linear systems with matrices that are dense; requiring large storage requirements and at the same time, increasing the computational complexity drastically. An attractive alternative to this is an iterative solver called multigrid scheme. For regular elliptic equations, it is well known that multigrid method is practically efficient. However, for the elliptic differential equations with highly oscillatory coefficients standard multigrid method is not efficient. There are several papers that have probed the use of wavelets in Galerkin, Collocation and other classical methods (Xu & Shann (1992), Vasilyev & Paolucci (1996), Vasilyev et al. (1995)). There has also been some development in finite element and multigrid methods using wavelets (Sudarshan et al. (2003)). But, orthogonal wavelets, particularly, Daubechies wavelets have great potential with regard to multiresolution approximation methods that turn out to be adaptive in nature. Furthermore, the hierarchical nature
of wavelets makes them particularly appealing to the techniques for solving partial
differential equations. Due to hierarchical nature of wavelets, it was felt by the
scientific community that they were naturally suited to those multilevel methods,
which contain multigrid algorithms (Briggs & Henson, (1993)). A wavelet method
which is closely related to multigrid algorithm was developed by Zhu & Wang
(1997). But, the proposed method uses a slightly different approach. This is quite
similar in spirit to the classical field of application of wavelets to image
compression. Moreover, this algorithm fully exploits the wavelet matrix structures,
sparsity and multiscale representations. A built in feature of this approach is that,
it incorporates the boundary conditions in the physical domain, as it is based on
finite difference discretization. It should, however, be noted that non-periodic
boundary conditions are hard to impose in wavelet methods (Zhu & Wang,
(1997)). We also present a simple example that illustrates, perhaps, the surprising
fact that the matrix derived from the finite difference discretization, can be
preconditioned trivially to have a uniform bounded condition number. The issue of
wavelet preconditioning has been addressed extensively in (Dahmen & Kunoth,
(1992)) and we adopt some of these ideas. Basically, our approach consists of
solving a linear system $Ax = y$, obtained from finite difference method (FDM),
where $A$ itself does not have a low condition number. In fact, the condition number
of $A$ increases quadratically, with its size. But, we can replace it by $W(Ax) =
(WA)x = Wy$, $W$ being matrix of the DWT. This pre-conditioning technique
applied recursively produces a system, having condition number bounded
independently of the step size. So, as the step size decreases, which achieves more accuracy, the condition number remaining bounded all the while. The boundedness of the condition number ensures the fast convergence of the iterative solver in the wavelet algorithm (Hackbusch, 1989). In general, it enables one to use iterative methods, such as the Gauss-Seidel method, conjugate gradient method etc. to solve linear systems with great efficiency. There are many very efficient methods such as multigrid that exploit the special structure of these linear systems to solve with high accuracy and with low computational cost. Wavelets and multiresolution analysis can be used to improve the multigrid ideas in terms of easy implementability and rapidity of its convergence (Hackbusch, 1989). Even though the fundamental similarities between wavelets and the ideas underlying multigrid schemes have been recognized by many authors (Wesseling, 1992, Geodecker & Chauvin, 2003), the similarities between the intergrid operators of the multigrid scheme and approximation and detail operators arising from multiresolution analysis of wavelet theory were first brought out by Briggs and Henson (1993). They also predicted that accuracy comparable to multigrid algorithms could be achieved by using compactly supported Daubechies family of wavelets on boundary value problems. These ideas motivated Bujurke et al. (2006a) to develop a Daubechies wavelet series multigrid method to solve elliptic partial differential equations.

With the advent of the increased interest by both Engineers and Orthopedic surgeons into biomechanics of degenerative joint disease, different modes of
lubrication in movable joints are studied as they exhibit low friction and almost negligible wear properties. Also, articular cartilage serves as a bearing and shock absorbing material of synovial joints. But, for various reasons, a long term process of cartilage degeneration (osteoarthritis) often arises. This generative process leads to a deterioration of the normal function of the joint and this genuine joint must often be replaced by an artificial one.

A successful model for cartilage with interstitial fluid has been developed by Mow and his co-workers Mow et al. (1980). This simplest linear version of biphasic mixture includes the small deformation of the porous elastic matrix which corresponds to Biot's model for soil consolidation (Biot (1941)). Mow and Ling (1969) and Mansour et al. (1973) have modeled the normal synovial joint as a single layer of homogenous fluid filled, porous permeable, deformable elastic material (articular cartilage). The governing equations for the tissue deformation and interstitial fluid motion were formulated using Biot's soil consolidation theory. Collins (1982) modeled articular cartilage as poroelastic material which is assumed to satisfy generalized form of Darcy's law for unsteady flow. Various aspects of articular cartilage and non-Newtonian characteristics of the synovial fluid are presented by Torzilli and Mow (1976). Recently, Mercer and Barry (1999) gave a numerical method on finite difference approximations for the calculation of deformation, pressure and flow within a finite two-dimensional poroelastic medium.
The long chain hyaluronic acid (HA) molecules found as polar additives in synovial fluid are characterized by two material constants $\mu$ and $\eta$. Breakdown of the HA component reduces the viscosity of synovial fluid to approximately that of water and in certain pathological cases it becomes Newtonian. The similar material constants are also found in Stokes (1966) couple-stress fluid. The simplest generalization of the classical theory of fluids by Stokes provides the macroscopic description of the behavior of fluids containing a substructure such as lubricants with polymer additives. This has motivated us to model synovial fluid as couple-stress fluid in human knee joint. Recently, Lin (1996) and Walicki and Walicka (2000) have modeled synovial fluid as couple-stress fluid for finding squeeze film characteristics of hemispherical bearings without including poroelasticity of the cartilage. So far no attempt has been made to investigate the effect of elasticity and couple stresses on poroelastic bearings in synovial knee joint.

5.2. Multiresolution Properties of Wavelets and Multigrid schemes

Multiresolution Decomposition and Reconstruction:

Any function $f \in L^2(R)$, may be approximated by the multiresolution apparatus by its projection $P_j f$ onto the subspace $V_j$:

$$P_j f = \sum_{m} c_{j,k} \phi_{j,k}(x)$$

and in fact $P_j f \to f$ as $j \to \infty$ (5.2.1)

If in addition, the projection of $f$ onto the subspace $W_j$ is denoted by $Q_j f$, then from Linear algebra,
\[ P_j f = P_{j-1} f + Q_{j-1} f \] (5.2.2)

The projections P and Q are referred to as approximation and detail operators.

Multiresolution analysis takes the expansion coefficients \( c_{j,k} \) of (5.2.1) to the function \( f \in L_2(R) \) at scale \( j \) and breaks them into

i. Approximation (scaling) coefficients \( c_{j-1,k} \) of \( P_{j-1} f \) at the next coarser level \( j-1 \)

ii. Detail (wavelet) coefficients \( d_{j-1,k} \) of \( f = P_j f - P_{j-1} f \).

This process is repeated recursively, to find \( c_{j-2,k} \) and \( d_{j-2,k} \) until the desired level is reached (Strang, 1989). The discrete wavelet transform (DWT) is an algorithm for computing \( c_{j,k} \) and \( d_{j,k} \) when a function is sampled at equally spaced intervals over a finite interval. The transform was first introduced by Mallat (1989a) and hence is known as Mallat's pyramid algorithm which provides a simple means of transforming data from one level of resolution \( m \) to the next coarser level of resolution \( m-1 \). The inverse Mallat transform is a transformation from coarser level \( m-1 \) back to the finer level \( m \). Further details can be found in Debnath, (2002).

The matrix formulation of DWT by using Daubechies wavelets:

The matrix formulation of the discrete signals and DWT, which play a pivotal role in the proposed method, requires a brief discussion. This is highly convenient and instructive, especially for numerical computations. Since we deal with only finite signals in practice, this raises a question: how to represent and take the DWT of a finite signal. There are many ways of dealing with this. One is
to extend the signal sufficiently beyond the segment of interest. A more efficient representation is achieved if the finite signal is viewed as a periodic signal repeated over the entire domain (periodization). Next step is to split a vector $f$ say of length $n$ (in the discrete setting vectors will replace functions) into components at different scales (resolutions). This is achieved by a DWT represented by an $n \times n$ matrix $W$, in the form $\hat{f} = Wf$. Thus, DWT can be carried out by multiplying the signal with an appropriate matrix. Because of the orthogonal properties of scaling and wavelet filters, $W$ is an orthogonal matrix. The reconstruction can also be done by a single matrix-vector multiplication. Another advantage of the matrix $W$ is that, it lowers the condition number of the matrix obtained by finite difference discretization of $Lu = f$. This preconditioning is made possible by the orthogonality property of wavelets. It is clear that, the condition number measures the stability of the linear system $Ax = y$, under small perturbation of $A$ as well as $y$. In applications, therefore, a small condition number (i.e. near 1) is desirable. If the condition number of $A$ is high, it becomes necessary to replace the linear system $Ax = y$ by an equivalent system whose matrix has a low condition number, for example, multiplying by a preconditioning matrix $W$ to obtain $WAx = Wy$, leading to an equivalent linear system, whose matrix has a lower condition number i.e. $C^*(WA) < C^*(A)$. Here is an illustration of 8x8 matrix taking filter coefficients from $D_4$ (Daubechies wavelet of order 4). The assumption that the data is periodic is incorporated in $W$ to produce "wrap around" effect (Abbate et al. (2002)).
1-D and 2-D discrete signal representations using DWT:

A 1-D DWT is a sum of products of samples of the input signal and samples of the wavelet basis. The sum can be represented as a matrix product, of matrix $W$ and a vector $x$. The wavelet matrix $W$, like other transformation matrices is unitary. Given a $n \times n$ wavelet matrix $W$ and vector $x$ of length $n$ forward and inverse transformations can be computed using matrix-vector multiplications:

$$\hat{x} = Wx, \quad x = W^T \hat{x}.$$ 

The 2-D DWT comes in two forms. The first is called separable and the second non-separable. In the first case, one-dimensional technique is used by interpreting a 2-D signal as 1-D signal. This can be done by using a nice strategy i.e. arranging the rows of $x$ in lexicographic order; which is referred to as ‘vectorization’ of a matrix.
Multigrid schemes:

It is well known that multigrid methods are among the fastest solution methods. Especially for elliptic equations, they have been proved to be highly efficient. In classical multigrid scheme pioneered by Wesseling (1992), a solution to

\[ Lu = f, \]  

(5.2.3)

where \( L \) is self-adjoint operator on a fine grid \( \Omega^h \) (with grid spacing \( h \)) by using standard relaxation (such as Gauss-Seidel) methods, to approximate errors on the coarse grids \( \Omega^{2h}, \Omega^{4h}, \ldots \), a hierarchy of grids with grid spacing that is increased by a factor of two is introduced. The details of the various cycles are not important, for our discussion, except to say that, it is the residue that is passed from the fine grids to the coarser grids. Vectors are transferred from coarse grids to the finer grids with a Prolongation operator \( (PO) I^{h}_{2h} \), while vectors from fine grids are transferred to coarser grids with Restriction operator \( (RO) I^{2h}_h \).
Fig. 5.1, schematically shows the algorithm of a standard multigrid V-cycle. The left arm of the Fig. 5.1., indicating the first part of the cycle, where one goes from finer grid to the coarser grids and the right arm where one comes back to the finest grid. The *essential multigrid principle* is to approximate the smooth part of the error on coarser grids. The non-smooth or rough part is reduced with small number of iterations on the finer grid. Unlike other known methods, multigrid offers the possibility of solving problems with $N$ unknowns with $O(N)$ operations and storage, resulting in a substantial computational saving and multigrid solution approaches exact solution as grid size tends to zero, overcoming limitations of classical numerical schemes. This method has greatest advantage of minimizing the errors using correction schemes in obtaining accurate solution, as grid size tends to zero. Another noteworthy property of this scheme is that there is no deterioration in the convergence rate when mesh points are increased. Recently, Bujurke and Kudenatti (2005, 2006) have successfully used multigrid method with half weighting restriction and bilinear interpolation operators to solve linear elliptic type equations arising in fluid dynamics.

**Similarities between Multiresolution Analysis and Multigrid Scheme:**

This section focuses on the suggestive similarities between the MRA of wavelet theory and multigrid algorithm, noted by Briggs and Henson (1993). Our main purpose in drawing comparison between the two is to see how each benefits from the other. They are as follows.
1) The space $V_0$ of the highest level of resolution in the MRA corresponds to $\Omega^h$, the space of fine grid vectors in the multigrid method.

2) The wavelet analysis (decomposition) step is identical with a restriction operation. Equivalently, the FWT (forward wavelet transforms) corresponds to $I_{2h}^h$.

3) The wavelet synthesis (reconstruction) step is identical with a prolongation operation. Equivalently, the IWT (inverse wavelet transform) is identical with $I_{2h}^h$.

4) $WW^T = I$, $W$ being the matrix of the wavelet transform (This is precisely the condition for “perfect reconstruction” in image compression) and

$$I_{2h}^h I_{2h}^h = I.$$

5) $V_1 = \text{span}\{\phi_{1,k}\} \oplus \text{span}\{\psi_{1,k}\} = V_0 \oplus W_0$, in the MRA is identical with

$$\Omega^h = \text{Range}\{I_{2h}^h\} \oplus \text{Nullspace}\{I_{2h}^h\}$$

of the multigrid algorithm.

We can thus draw a parallel between multiscale and multiresolution properties of wavelet transforms and error smoothing /correcting properties of intergrid operators.

5.3. Daubechies Wavelet Series-Multigrid Method

We have seen in Section 5.2, how a wavelet decomposition of a function leads to a multiresolution representation of the function. If we combine these inherent multiresolution properties with multigrid ideas, then it may enable us to reduce the computation time and simplify the implementation procedure.
(Geodecker & Chauvin, (2003)). Here, in fact, wavelets are not only the bases of the approximation spaces but they also play the fundamental role of *prolongation* and *restriction* between the consecutive levels. We employ, in this direction, a modified V-cycle, the underlying principle of which is illustrated in Fig. 5.2.

Consider a general operation equation of the form (5.2.3) representing, for example, an elliptic boundary value problem. Let the finite difference discretization of this equation be:

\[ Ax = y \]  

(5.3.1)

Let \( \hat{x} = W_s x, \hat{y} = W_s y \) and \( \hat{A} = W_s A W_s^T \),

(5.3.2)

Then we obtain,

\[ \hat{A} \hat{x} = \hat{y} \]  

(5.3.3)

Equation (5.3.3) is in the wavelet domain. From this one can solve for \( x \), using
This gives a single grid solution.

Let $J$ be the level of wavelet decomposition. The modified V-cycle consists of the following steps:

1) Perform FWT on $y$ and $A$ to obtain, $\hat{y}_i$ and $\hat{A}_i$, $i = -1, -2, -3, ..., J$, Set $l = -J$.

2) Solve $\hat{A}_l \hat{x}_l = \hat{y}_l$, to obtain $\hat{x}_l$ at the coarsest level.

3) Perform IWT on $\hat{x}_l$, $l = -J, ..., -2, -1$ to yield the required solution.

The method gives the solution at resolution level $\leq \log_2 2^J$, where $J$ represents the level of wavelet decomposition (Zhu and Wang, 1997), i.e., the convergence to the correct solution is guaranteed with number of iterations not exceeding $J$.

5.4. Formulation of the Problem and Solution procedure

Fluid film region:

On the basis of Stokes microcontinuum theory, the equations of continuity and momentum for an incompressible couple-stress fluid in the absence of body forces and body couples are

$$\nabla \cdot \mathbf{v} = 0, \quad (5.4.1)$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} - \eta \nabla^4 \mathbf{v}, \quad (5.4.2)$$

where $\mathbf{v}$ is the fluid velocity vector, $\rho$ is the density, $p$ is the pressure, $\mu$ is the viscosity and $\eta$ is the material constant with dimension of momentum giving couple-stress property of the lubricant. The ratio $\eta/\mu$ has the dimension of length
squared and characterizes the material length of the fluid. The flow problems discussed by Stokes present the significant effects of couple stresses and give hints for measuring various material constants and describe the influence of size effects in couple-stress fluid that is not present in the non-polar cases.

The physical configuration of the problem is shown in Fig. 5.3(b), which is the simplified form of synovial knee joint (shown in Fig. 5.3(a)). The upper poroelastic cartilage surface is approaching the lower poroelastic matrix normally with a constant velocity \( \frac{dH}{dt} \). Couple-stress fluid is taken to be a lubricant of the joint cavity. Following Walker and Erkman (1972), as the load bearing area of the synovial knee joint is small, the two surfaces may be considered to be parallel under high loading conditions and the average of three layers of the cartilage is modeled as single poroelastic layer. So, the problem considered is that of three-dimensional squeeze film lubrication between two rectangular surfaces. The moving film thickness is characterized by

\[
H = h(t).
\]  

(5.4.3)

Under fluid film lubrication, all articulations of knee joints involve cartilage-couple-stress fluid-cartilage interactions. With usual assumptions of fluid film lubrication, the governing equations in cartesian coordinates reduce to

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, 
\]  

(5.4.4)

\[
\mu \frac{\partial^2 u}{\partial y^2} - \eta \frac{\partial^4 u}{\partial y^4} = \frac{\partial p}{\partial x},
\]  

(5.4.5)
Fig. 5.3(a). A schematic diagram of a synovial knee joint.

Fig. 5.3(b). A simplified model for a synovial knee joint.
\[ 0 = \frac{\partial \mathcal{P}}{\partial y}, \quad (5.4.6) \]

\[ \mu \frac{\partial^2 w}{\partial y^2} - \eta \frac{\partial^4 w}{\partial y^4} = \frac{\partial \mathcal{P}}{\partial z}, \quad (5.4.7) \]

where \( u, v \) and \( w \) are the velocity components in \( x, y \) and \( z \) directions respectively.

**Poroelastic Region:**

Following Torzilli and Mow (1978), we write coupled equations of motion for the deformable cartilage matrix and the flow of fluid contained in its pores in the form (Collins, 1982)).

Matrix:
\[ \rho_m \frac{\partial^2 \mathbf{U}}{\partial t^2} = \text{div} \sigma_m - \frac{\mu}{k} \left( \frac{\partial \mathbf{U}}{\partial t} - \mathbf{V} \right), \quad (5.4.8) \]

Fluid:
\[ \rho_f \frac{D \mathbf{V}}{D t} = \text{div} \sigma_f + \frac{\mu}{k} \left( \frac{\partial \mathbf{U}}{\partial t} - \mathbf{V} \right), \quad (5.4.9) \]

where \( \rho_m \) and \( \rho_f \) denote the densities of solid matrix and fluid respectively, \( \mathbf{U} \) is the corresponding displacement vector, \( k \) is the permeability of the cartilaginous matrix to fluid. The left hand terms denote the local forces (mass times acceleration), which are counterbalanced by the right hand terms namely the surface forces, \( \text{div} \sigma \), and the porous medium driving forces (Darcy's law) respectively. These two component equations may be simply viewed as generalized forms of Darcy's law for unsteady flow in a deformable porous medium in terms of relative velocity between the moving cartilage and the fluid contained in its pores. The classical stress tensor \( \sigma \) for a continuous homogeneous
medium may be expressed for the matrix (cartilage) and fluid (synovial) respectively, in the forms

\[ \sigma_m = p^* I + 2Ne + AeI, \quad (5.4.10) \]

\[ \sigma_f = -p^* I + EeI, \quad (5.4.11) \]

in terms of the elastic parameters \( N, E \) and \( A \) of the cartilage, the hydrostatic pressure \( p^* \), \( I \) the identity tensor and \( e \) the cartilage dilation. The inertial terms in (5.4.8) and (5.4.9) are neglected because in the balance of momentum equation the fluid-fluid viscous stress is negligible compared with the drag between the fluid and solid matrix (Barry and Holms, 2001)). After neglecting inertia terms, addition of equations (5.4.8) and (5.4.9) eliminates the pressure and fluid velocity and then after taking divergence of the result, gives

\[ \nabla^2 e = 0, \quad (5.4.12) \]

The cartilage dilatation is characterized by a simple linear equation in terms of the corresponding average bulk modulus \( K \) (Hori and Mockers, 1976)).

\[ e = e_0 + \frac{p^*}{K}. \quad (5.4.13) \]

The equation describing pressure in the poroelastic region obtained using (5.4.13) in (5.4.12) is

\[ \nabla^2 p^* = 0. \quad (5.4.14) \]
Boundary Conditions

The relevant boundary conditions for the velocity field \((0 < y < H)\) are

\[
\begin{align*}
u(x,0,z) &= u(x,H,z) = w(x,0,z) = w(x,H,z) = 0, \quad (5.4.15) \\
v(x,0,z) &= -v_n, \quad v(x,H,z) = v_n - \frac{dH}{dt}, \quad (5.4.16) \\
\end{align*}
\]

\[
\begin{align*}
\left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0} &= \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=H} = \left. \frac{\partial^2 w}{\partial y^2} \right|_{y=0} = \left. \frac{\partial^2 w}{\partial y^2} \right|_{y=H} = 0, \quad (5.4.17)
\end{align*}
\]

where \(v_n\) represents the normal component of the relative velocity of the fluid at the cartilage surface. Conditions (5.4.15) are no-slip conditions and (5.4.17) are due to vanishing of couple stresses.

Solution procedure

Integrating equation (5.4.14) with respect to \(y\) over the porous layer thickness \((-\delta < y < 0)\) and using the solid backing boundary condition \(\frac{\partial p^*}{\partial y} = 0\) at \(y = -\delta\), we get,

\[
\left. \frac{\partial p^*}{\partial y} \right|_{y=0} = -\int_{-\delta}^{0} \left( \frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial z^2} \right) \, dy, \quad (5.4.18)
\]

where \(\delta\) is the thickness of the poroelastic layer. Using the Cameron-Morgan approximation (Morgan and Cameron, (1957)) which is valid for the poroelastic layer thickness \(\delta\) to be very small and using pressure continuity condition \((p = p^*)\), at the porous interface \((y = 0)\), we get

\[
\left. \frac{\partial p^*}{\partial y} \right|_{y=0} = -\delta \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} \right), \quad (5.4.19)
\]
After neglecting inertia terms, equation (5.4.9) may be arranged in terms of relative velocity in the form

\[
\left( \mathbf{V} - \frac{d \mathbf{U}}{dt} \right) = -\frac{k}{\mu} \left( \nabla p^* - E \nabla e \right). \tag{5.4.20}
\]

Elimination of \( e \) through (5.4.13) and (5.4.20) gives

\[
\left( \mathbf{V} - \frac{d \mathbf{U}}{dt} \right) = -\nabla p^* \cdot \left( \frac{k}{\mu} \left( 1 - \frac{E}{K} \right) \right). \tag{5.4.21}
\]

The normal component of the relative fluid velocity at the cartilage surface is

\[
v_n = -\left( \mathbf{V} - \frac{d \mathbf{U}}{dt} \right)_n = -\frac{k}{\mu} \left( \frac{E}{K} - 1 \right) \frac{\partial p^*}{\partial y} \bigg|_{y=0}. \tag{5.4.22}
\]

Using equation (5.4.19) in equation (5.4.22), we get

\[
v_n = \frac{k \delta}{\mu} \left( \frac{E}{K} - 1 \right) \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} \right). \tag{5.4.23}
\]

Equations (5.4.5) and (5.4.7) can be integrated for \( u \) and \( w \) with respect to \( y \) using boundary conditions (5.4.16) and (5.4.18). Substituting \( u \) and \( w \) in the continuity equation (5.4.4) and integrating across the film thickness with respect to \( y \) using boundary conditions (5.4.17), we obtain modified Reynolds equation

\[
\frac{\partial}{\partial x} \left[ F(H,l) + 24 \delta k \left( \frac{E}{K} - 1 \right) \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial z} \left[ F(H,l) + 24 \delta k \left( \frac{E}{K} - 1 \right) \frac{\partial p}{\partial z} \right] = 12 \mu \frac{dH}{dt} \tag{5.4.24}
\]

Since \( H \) is independent of \( x \) and \( z \), the modified Reynolds equation (5.4.24) becomes

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{12 \mu dH/dt}{F(H,l) + 24 \delta k \left( \frac{E}{K} - 1 \right)}, \tag{5.4.25}
\]
where
\[ F(H, l) = -H^3 + \frac{12}{l^2} \left( H - \frac{2}{l} \tanh(\frac{\pi H}{2}) \right), \quad l = \sqrt{\frac{\eta}{\mu}}. \]

The relevant boundary conditions for the pressure are

\[ p = 0 \quad \text{at} \quad x = 0, a \quad \text{and} \quad z = 0, b, \quad (5.4.26) \]

where \( a \) and \( b \) are dimensions of plate in \( x \) and \( z \) directions respectively.

Introduce non-dimensional parameters and variables as follows

\[ \bar{x} = \frac{x}{a}, \bar{z} = \frac{z}{b}, \lambda = \frac{a}{b}, \bar{H} = \frac{H}{h_0}, \bar{h} = \frac{h}{h_0}, \tau = \frac{h_0}{h_0}, \bar{k} = \frac{k\delta}{h_0^2}, \text{and} \quad \bar{p} = \frac{p h_0^2}{\mu a^2 dh/dt}, \]

where \( h_0 \) is the initial film thickness, \( \lambda \) is the aspect ratio, \( \tau \) is couple-stress parameter, \( \bar{k} \) is the permeability parameter and \( \bar{p} \) is the non-dimensional fluid film pressure. With these quantities equation (5.4.25) becomes

\[ \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} + \frac{1}{\lambda^2} \frac{\partial^2 \bar{p}}{\partial \bar{z}^2} = \frac{12}{F(\bar{H}, \tau)} \left( \frac{E}{K} - 1 \right), \quad (5.4.27) \]

where

\[ F(\bar{H}, \tau) = -\bar{H}^3 + \frac{12}{\tau^2} \left( \bar{H} - \frac{2}{\tau} \tanh(\frac{\pi H}{2}) \right). \]

and the boundary conditions for the pressure field are

\[ \bar{p} = 0 \quad \text{at} \quad \bar{x} = 0,1 \quad \text{and} \quad \bar{z} = 0,1. \quad (5.4.28) \]

**Numerical Solution**

The modified Reynolds equation (5.4.27) is of elliptic type, which can be solved numerically using Daubechies Wavelet series multigrid method. So, using
second order finite difference scheme, derivative terms in equation (5.4.27) are approximated by

\[
\frac{\partial^2 p}{\partial x^2} = \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{(\Delta x)^2}, \quad \frac{\partial^2 p}{\partial z^2} = \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{(\Delta z)^2}.
\]

After substituting above finite difference schemes into equation (5.4.27), we get following discretised equation

\[
\begin{align*}
 p_{m+1,j} + p_{i-1,j} + B_0 p_{i,j+1} + B_0 p_{i,j-1} - B_1 p_{i,j} &= (\Delta x)^2 D_{i,j}, \\
\end{align*}
\]

(5.4.29)

where coefficients are given by

\[
\begin{align*}
h_i &= \frac{\Delta x}{\Delta x}, \quad B_0 = \frac{1}{\lambda^2} h_i^2, \quad B_1 = 2 + 2B_0, \quad D_{i,j} = \frac{12}{F(H_{i,j}, \tau) + 24k \left( E \frac{K}{K} - 1 \right)}. \\
\end{align*}
\]

To enforce the boundary conditions, we set

\[
\begin{align*}
 p_{0,j} &= p_{N,j} = p_{i,0} = p_{i,N} = 0. \\
\end{align*}
\]

Let the matrix formulation of (5.4.29) be

\[
Ax = y.
\]

(5.4.30)

Now FWT is performed, according to the procedure explained in Sec.5.3 on \( A \) and \( y \) of equation (5.4.30) recursively till the coarsest level is reached at level \(-J\). Then \( \tilde{A}_i \hat{x}_i = \tilde{y}_i \) is solved to obtain \( \hat{x}_i \), at the coarsest level using Gauss-Jordan method. Finally, IWT performed on \( \hat{x}_i \) (\( l = -J, \ldots, -2, -1 \)), which gives the fluid film pressure \( \bar{p} \) of required accuracy.
Once, fluid film pressure is obtained by using Daubechies wavelet series multigrid method, the load carrying capacity $\bar{W}$ per unit area of the joint surface in non-dimensional form is

$$\bar{W} = \int_0^1 \int_0^1 p(x, z) dxz .$$  \hspace{1cm} (5.4.31)

5.5. Results and Discussion

In this Chapter, for all numerical experiments D-4 wavelets are employed. However, we have the freedom and flexibility to choose other wavelet bases also. To test the accuracy, we have solved the problem at resolutions $2^4$ and $2^5$. It is observed that, there is a marginal increase in the accuracy of the solution. Better accuracy can be achieved by increasing the resolution and / or the order of the wavelet family. It is also observed that, the amount of computational effort is considerably less than that of classical multigrid method. In fact, It is found that, 6-7 cycles are required to obtain a reasonably accurate solution in the multigrid scheme, whereas, only one cycle is required to obtain the solution in the wavelet-multigrid method. Also, if $A$ denotes the matrix obtained from the above scheme, with $N = 4$ for example, the condition number of $A \approx 9.6$. The condition number of $W.A \approx 0.99$. Hence, this preconditioning technique applied to $A$ recursively produces a system, having condition number $O(1)$. Furthermore, this ensures the fast convergence of the Daubechies wavelet series multigrid method (Hachbush. W. (1989), Bujurke et al. (2006a)).
A simplified mathematical model has been developed for analyzing the effects of couple-stress fluid and poroelasticity on lubrication characteristics of synovial joints. All the bearing characteristics are functions of non-dimensional parameters $\tau(= l h_o)$, $\bar{k}(= k \delta / h^o)$ and $\lambda(= a/b)$. The introduction of the new material constant $\eta$ is due to polar additives in the non-polar lubricant. The ratio $\eta/\mu$ has dimension of length squared and this may be regarded as the chain length of the polar additives in the non-polar lubricants. It is expected that the polar effects should be more prominent either when the minimum film thickness is small or molecular size of the additives is large i.e. when $\tau$ is small. On the other hand for large value of $\tau$, the couple-stress effects are not significant. The values for elastic parameters $E,K$ and for $\bar{k}$ are taken from Torzilli & Mow (1976), which are associated with healthy human cartilage during normal articulation. In the graphs, dotted lines correspond to either Newtonian case ($\tau \to \infty$) or non-elastic case ($E/K = 0$). In the limiting case, $\tau \to \infty$, $k = 0$, the Reynolds equation (5.4.25) corresponds to classical case (Pinkus and Stremlicht, (1961)).

The fluid film pressure distribution for different couple-stress parameters are plotted in Fig. 5.4(a-c). The effect of couple stresses is to increase the pressure maximum compared to Newtonian case ($\tau \to \infty$). The two material constants $\mu$ and $\eta$ which are present in couple-stress fluid as polar additives are responsible for long chain hyaluronic acid (HA) molecules. The water and low molecular weight substances present in the lubricant are forced into poroelastic cartilage by
the squeeze film action of the impinging cartilaginous surfaces. This leads to an increase of the concentration of the polymer additives on the cartilage surfaces, producing larger pressure due to increased viscosity of the lubricant. In certain pathological changes which occur in synovial joints due to abnormal joint mechanics or the process of aging, the fluid becomes Newtonian \((\tau \rightarrow \infty)\) which is responsible for degenerative joint disease (Sokoloff, (1969)).

Fig. 5.5 shows variation of non-dimensional load capacity \(\bar{W}\) with film thickness \(\bar{h}\) for different values of elasticity parameter \(E/K\). The load \(\bar{W}\) for elastic parameter \(E/K = 2.0\) is quite large compared to that of non-elastic case \((E/K = 0.0)\). Further, load carrying capacity \(\bar{W}\) decreases as intra articular gap between two articular surfaces decreases for all values of elastic parameter \(E/K\).

The effect of permeability parameter \(\bar{k}\) on the variation in \(\bar{W}\) with \(\log_{10}(\lambda)\) is shown in Fig. 5.6. It is observed that load carrying capacity \(\bar{W}\) decreases with increase in permeability parameter \(\bar{k}\). This is because, the large permeability value means there are more voids available in the poroelastic surface, which permits the quick escape of the fluid. The interstitial fluid which is present in the lubricant region is free to escape through poroelastic surfaces. Fluid motion occurs due to large permeability of poroelastic matrix which produces decrease in pressure resulting in decrease of load carrying capacity. Further, for large \(\lambda\), the length of the plate to be longer in \(z\) direction than that of \(x\) direction to achieve the maximum load capacity that can be sustained by joint surface.
Fig. 5.4(a). Fluid film Pressure distribution for $\tau = 5$ and $\lambda = 1.0$.

Fig. 5.4(b). Fluid film Pressure distribution for $\tau = 10$ and $\lambda = 1.0$. 
Fig. 5.4(c). Fluid film Pressure distribution for \( \tau = \infty \) and \( \lambda = 1.0 \)
Fig. 5.5. Variation of load capacity $\bar{W}$ with $\bar{h}$ for different elasticity Parameter $E'/K$ with $\bar{\tau} = 5$ and $\bar{k} = 7.65 \times 10^{-5}$. 
Fig. 5.6. Variation of load capacity $\bar{W}$ with $\log_{10}(\lambda)$ for different cartilage permeability parameter $\bar{k}$ and $\bar{r} = 5$. 