CHAPTER VI

THE TRIPLE \((a,b,c)\) OF INTEGERS WITH RESPECT TO AN EDGE

INTRODUCTION:

For each edge \(e = uv\), of a graph \(G\), we can associate a triple of integers \((a,b,c)\) in a well defined fashion, which will be seen soon. The idea of associating triple of integers \((a,b,c)\) is motivated by Herold Wiener's [33] formula for computing the Wiener index of a tree and also one of the paper by I Gutman [23] where he defined two sets of vertices of a graph, associated with an edge, \(e = uv\), while proving some of the results in [4]. Based on this we partition the vertex set of \(G\) into three sets with respect to each edge \(e = uv\) defined as below:

\[
V_L(e = uv) = \{ x \in V(G) / d(u, x) < d(v, x) \}
\]
\[
V_R(e = uv) = \{ x \in V(G) / d(u, x) < d(v, x) \}
\]
\[
V_R(e = uv) = \{ x \in V(G) / d(u, x) > d(v, x) \}
\]

The triple \((a,b,c)\) of integers is said to associated with an edge, \(e = uv\), if

\[
|V_L(e = uv)| = a, \ |V_R(e = uv)| = b \text{ and } |V_R(e = uv)| = c.
\]
Example:

1. For $a = 1$, $b = 1$, $c = 1$

![Fig. 1](image1)

2. For $a = 2$, $b = 2$, $c = 2$

![Fig. 2](image2)

3. $a = 1$, $b = 2$, $c = 1$

![Fig. 3](image3)

In this chapter we characterize bipartite graphs $G$ in terms of triples $(a, b, c)$ associated with each edge, $e = uv$. And also we define a class of graphs called edge triplet regular graph, graph in which every edge of $G$
has associated with same triple \((a,b,c)\). We prove that some well known class of graphs namely complete graphs, \(K_n\), the star \(K_{1,n}\) and the cycle \(C_n\), the Petersen graph are edge triplet regular graphs.

2. MAIN RESULTS:

Theorem 2.1: The triple \((a,b,c)\) of integers with each edge \(e = uv\) of a graph \(G\) is \((a,0,c)\) if and only if \(G\) is a bipartite graph.

Proof: Suppose each edge of \(G\) is associated with the triple of the form \((a,0,c)\). We claim that \(G\) is a bipartite graph. Suppose on contrary \(G\) is not a bipartite graph. Then \(G\) contains a odd cycle \(C\) (say). Let \(e = uv\) be an edge of \(G\) contained in \(C\). Then there exists a vertex \(w\) on \(C\), which is equidistant from \(u\) and \(v\). Therefore \(b \geq 1\), a contradiction to the assumption.

Conversely, assume that \(G\) is a bipartite graph. If possible assume that there is some edge \(e = uv\) with triple \((a,b,c)\) with \(b \neq 0\). Then there are \(b\) number of vertices which are equidistant from \(u\) and \(v\). That means, there are \(b\) odd cycles containing an edge \(e = uv\) and these \(b\) number of vertices, a contradiction to the assumption.
Proposition 2.2: If \( G = C_n, \) \( n = 2k \) for any integer \( k \geq 2, \) then

\[
|V_L(e = uv)| = \frac{n}{2}, \quad |V_R(e = uv)| = 0 \quad \text{and} \quad |V_R(e = uv)| = \frac{n}{2}.
\]

And if \( G = C_n, \) \( n = 2k + 1 \) for any integer \( k \geq 1, \) then

\[
|V_L(e = uv)| = \left\lfloor \frac{n}{2} \right\rfloor, \quad |V_R(e = uv)| = 1 \quad \text{and} \quad |V_R(e = uv)| = \left\lfloor \frac{n}{2} \right\rfloor.
\]

Proof: Obvious.

Proposition 2.3: The complete bipartite graph \( G = K_{r,r} \) is edge triple regular for each edge \( e = uv. \) That is, for \( G = K_{r,r}, (a,b,c) = \left( \frac{n}{2}, \frac{n}{2}, \frac{n}{2} \right) \)

Proof: Let \( G = K_{r,s} \) for \( r = s. \) Let \( V_1 \) and \( V_2 \) be the partition of the vertex set \( V(G) \) such that, \( |V_1| = |V_2| = r = \frac{n}{2} \) and \( |V_1| + |V_2| = n. \) Since \( G \) is a complete bipartite graph, every vertex of \( V_1 \) is adjacent to all the vertices of \( V_2. \)

\[
\begin{array}{c}
\text{Fig. 4}
\end{array}
\]
Without loss of generality, let \( \nu_i, u_i \in E(G) \). Clearly the vertices which are adjacent to \( u_i \) except \( v_i \) are near to \( u_i \) (since \( d(u_i, v_i) > d(u_i, u_i) \)). Hence for a vertex \( u_i \), \( s-1 \) vertices are near. But a vertex \( u_i \) is always close to \( u_i \).

Hence totally \( s \) number of vertices are in \( |V_{R}(e = uv)| \). Since \( G \) is a complete bipartite graph, it must be true for every edge. Therefore \( |V_{R}(e = uv)| = s = \frac{n}{2} \) and by Theorem 3, \( |V_{R}(e = uv)| = 0 \). Which clearly implies that \( |V_{L}(e = uv)| = s = \frac{n}{2} \). Hence the proof.

**Proposition 2.4:** All \((a, 0, a)\) graphs are of the form \( \left( \frac{n}{2}, 0, \frac{n}{2} \right) \) graphs.

**Proof:** We know that the vertex set with respect to an edge is partitioned into three sets. That is, \( |V_{L}(e = uv)| = a \), \( |V_{R}(e = uv)| = b \) and \( |V_{R}(e = uv)| = c \). By hypothesis, \( |V_{R}(e = uv)| = b = 0 \) and \( |V_{L}(e = uv)| = a = c \). Therefore,

\[
|V_{L}(e = uv)| + |V_{R}(e = uv)| = n = 2a = n = \frac{n}{2}.
\]

Hence the proof.

**Theorem 2.5:** The triple of integers \((a, b, c) = (1, n-2, 1)\) if and only if \( G \) is a complete graph.

**Proof:** Let \( G \) be a complete graph on \( n \) vertices. We claim that the triple of integers \((a, b, c) = (1, n-2, 1)\). Let \( e = uv \) be any arbitrary edge of \( G \). Since
\[ \text{deg}(v) = n-1, \] every vertex \( w \in V(G) \) is adjacent to both \( u \) and \( v \). That is \( n-2 \) vertices are adjacent to both \( u \) and \( v \). Clearly \( n-2 \) vertices are equidistant from an edge \( e = uv \). And moreover \( |V_L(e = uv)| = \phi \), and \( |V_R(e = uv)| = \phi \). Hence \( |V_L(e = uv)| = 1 \), \( |V_R(e = uv)| = n-2 \) and \( |V_R(e = uv)| = 1 \).

Hence \( (a, b, c) = (1, n-2, 1) \).

Converse is obvious.

**Theorem 2.6:** The triple of integers \( (a, b, c) = (1, 0, n-1) \) if and only if \( G \) is a star.

**Proof:** Let \( G \) be a star. We claim that the triple of integers \( (a, b, c) = (1, 0, n-1) \). Let \( v_1, v_2, \cdots, v_n \) be its labeling. Consider a partition \( V_1 \) and \( V_2 \) of a vertex set \( V(G) \) such that, \( |V_1(G)| = 1 \) and \( |V_2(G)| = n-1 \). Without loss of generality, let \( v_1 \in V_1 \) and \( v_i \in V_2 \) for \( i = 2, 3, \cdots n \). Let \( v_iv_2 \in E(G) \). Clearly the vertices which are adjacent to the vertex \( v_1 \) are near to the vertex \( v_1 \). But \( d(v_i, v_1) < d(v_i, v_2) \). That is, \( n-2 \) vertices with a vertex \( v_1 \) are near to the vertex \( v_1 \). Hence for an edge \( e = v_1v_2, v_2 \) is the only vertex which is near to a vertex \( v_2 \). Therefore, \( |V_L(e = uv)| = 1 \) and \( |V_R(e = uv)| = n-1 \). Since star is a bipartite, \( |V_R(e = uv)| = 0 \).
Converse is obvious.

**Proposition 2.7:** For Petersen graph the triple \((a, b, c)\) of integers is \((3, 4, 3)\).

**Proof:** Let \(v_1, v_2, v_3, v_4, v_5\) be the labeling of the outer cycle of the Petersen graph and \(u_1, u_2, u_3, u_4, u_5\) be the labeling of the inner cycle of the Petersen graph such that \(v_i\) is adjacent to \(u_i\) as shown in the Fig. Let \(v_iv_2 \in E(G)\).

The vertices \(v_2, v_3\) and \(u_2\) are near to the vertex \(v_2\). The vertices \(v_1, u_1\) and \(u_3\) are near to the vertex \(v_1\). The vertices \(v_2, v_3\) and \(u_2\) are in \(|V_2(e = v_1v_2)|\) and the vertices \(v_1, u_1\) and \(u_3\) are in \(|V_r(e = v_1v_2)|\). Now the vertex \(u_3\) is adjacent to the vertices \(v_2\) and \(v_1\). But these vertices are adjacent to \(v_2\) and \(v_1\). Hence a vertex \(u_3\) is at a distance 2 from the vertices \(v_2\) and \(v_1\).

Similarly vertices \(u_4, v_4\) and \(u_5\) are at a distance 2 from the vertices \(v_2\) and \(v_1\). Since \(v_1v_2 \in E(G)\) is an edge of an outer cycle it is true for all the edges of the outer cycle.
Let $u,u_4 \in E(G)$. Now the vertices $u_i$, $v$, and $u_3$ are near to the vertex $u_i$ and the vertices $u_4$, $v_4$ and $u_2$ are near to the vertex $u_4$. The vertices $u_i$, $v$, and $u_3$ are in $|V_e(e = u_iu_4)|$ and the vertices $u_4$, $v_4$ and $u_2$ are in $|V_n(e = u_iu_4)|$. Again as above, remaining four vertices are at a distance 2 from both $u_iu_4$ vertices. Hence $|V_e(e = u_iu_4)| = 4$. Since $u_iu_4 \in E(G)$ is an edge of an inner cycle, it must be true for all the edges of an inner cycle of Petersen graph. By the similar argument it can be proved for $u_iv_i \in E(G)$.

Therefore for a Petersen graph the triple $(a, b, c)$ of integers is $(3, 4, 3)$. 
REFERENCES:


