CHAPTER IV
CHARACTERIZATION OF TREES WITH EMBEDDING INDEX 1 WITH RESPECT TO ALMOST SELF-CENTERED \((r+1, r)\)-GRAPHS

1. INTRODUCTION:

In the second chapter, we introduced the concept of embedding index \(\theta_{3,2}(G)\) with respect to almost self-centered graph, that is \((3,2)\) - graph. For the sake of completeness, we define \(\theta_{3,2}(G)\) as below:

Let \(G\) be any graph and \(\mathcal{F}\) be a family of \((3,2)\) - graph \(H\) containing \(G\) as an induced subgraph. The embedding index \(\theta_{3,2}(G)\) of \(G\) is defined as:

\[
\theta_{3,2}(G) = \min_{H \in \mathcal{F}} \{O(H) - O(G)\}.
\]

From this definition we overcome the following facts:

**FACT-1:** If \(G\) itself is a \((3,2)\) - graph, then \(\theta_{3,2}(G) = 0\).

**FACT-2:** If \(G\) is not \((3,2)\) - graph, then \(\theta_{3,2}(G) \geq 1\).

**FACT-3:** If \(\theta_{3,2}(G) = 1\), then there exists \((3,2)\) - graph \(H\) containing \(G\) as an induced subgraph such that \(H - u = G\), for some vertex \(u\) in \(H\).

In theorem 3.5 of chapter II, it is proved that every graph can be embedded in some \((3,2)\) - graph. The proof of the theorem 3.5 suggest that, the embedding of any given graph \(G\), it requires at most two vertices.
and hence by the definition of the embedding index $\theta_{3,2}(G)$ one can immediately conclude that $\theta_{3,2}(G) \leq 2$. This fact prompts us to classify the graphs into two categories, namely the graph with $\theta_{3,2}(G) = 1$ and $\theta_{3,2}(G) = 2$. If one is able to characterize one of the class of graphs, the other will be automatically characterized by taking negation. Thus we are in characterization of class of graphs with $\theta_{3,2}(G) = 1$. The characterization of this also seems to be difficult. So we restrict over selves to characterize the class of trees with $\theta_{3,2}(G) = 1$.

2. MAIN RESULTS

**Theorem 2.1:** If $\theta_{3,2}(G) = 1$, then $\text{diam}(G) \leq 5$

**Proof:** Suppose $\theta_{3,2}(G) = 1$, we claim that $\text{diam}(G) \leq 5$. If possible assume that $\text{diam}(G) = k \geq 6$. Let $u_1, u_2, \ldots, u_k, u_{k+1}$ be the diametral path in $G$. Since $\theta_{3,2}(G) = 1$, then there exists $(3,2)$-graph $H$ is formed by introducing a new vertex $x$ and join it to the vertices of $G$ so that all vertices of $H$ have eccentricity two except two vertices. Without loss of generality let $u_1$ and $u_4$ be the vertices in $H$ with eccentricity 3. Clearly $x$ is adjacent to either $u_1$ or $u_4$ (not both) otherwise $d_H(u_1, u_4) = 2$. Let $x$ be adjacent to $u_1$ but not to $u_4$. Partition the vertex set of $H$ as $D_0 = \{u_i\}$, $D_1 = \{v : d(v, u_1) = 1\}$.
$D_2 = \left\{ v/d(v,u_1) = 2 \right\}$ and $D_3 = \{ u_4 \}$. Since $H$ has only two vertices of eccentricity 3, clearly the vertex $x$ is in $D_1$ and all the vertices $v$, $i \in \{ 5, 6, \ldots, k+1 \}$ are in $D_2$. As $x$ is in $D_1$, $x$ must be adjacent to all the vertices $u_i$ for $i \geq 5$ for otherwise $e_H(u_i) \geq 3$, a contradiction. As $k \geq 6$, $d(u_4, u_i) = 3$. This implies that $e_H(u_i) = 3$. Again a contradiction. This proves that $\text{diam}(G) \leq 5$.

Converse of the above theorem is not true. For this consider the graph $G$ as labeled in the Fig. 1.

![Fig.1]

Clearly $\text{diam}(G) = 2$ but $\theta_{3,2}(G) = 2$. This can be seen in the following argument. If possible assume that $\theta_{3,2}(G) = 1$. Then there exists a $(3,2)$-graph $H$ such that $H - x = G$ for some vertex $x$. Clearly $e_H(x) = 3$. Since $e_G(z) \leq 2$ for all vertices $z$ in $G$. As $H$ is a $(3,2)$-graph, there must be exactly one more vertex $z$ in $H$ (other than $x$) whose eccentricity is 3 and rest are of eccentricity 2. This is depending on the vertex $x$ adjacent to the vertices of $G$ in $H$. Let a vertex $x$ be adjacent to a vertex
If a vertex $z$ is a vertex $u$, then $e_H(w) = 3$. A contradiction. If a vertex $z$ is a vertex $w_i$, then $e_H(u) = 3$. A contradiction. We arrive at the same contradiction as above, if a vertex $x$ is adjacent to a vertex $w_i$ or to the vertex $w_j$. Hence in no cases one vertex $x$ is enough to embed $G$ in $(3,2)$-graph. Therefore $\theta_{3,2}(G) \geq 2$. On the other hand the graph $H$ of the figure containing $G$ as an induced subgraph shows that $\theta_{3,2}(G) \leq 2$ and hence $\theta_{3,2}(G) = 2$.

From the above theorem one can conclude that for any graph $G$ with $diam(G) \geq 6$, $\theta_{3,2}(G) = 2$ and in particular for any tree $T$, $diam(T) \geq 6$, $\theta_{3,2}(T) = 2$. Holds. Thus we seek the class of trees $T$ with $diam(T) \leq 5$. Even if $diam(T) \leq 5$, there are many class of trees having embedding index $\theta_{3,2}(G) = 2$. For example the trees $S_{m,n}$ - the double star is one such class with $\theta_{3,2}(T) = 2$.

![Fig.3](image)

**Remark 1:** If given tree $T$ (or a graph $G$), the introduction of a new vertex and making any number of adjacencies will not result into $(3,2)$-graph.
then obviously it requires at least one more vertex to form a (3,2) - graph containing a tree $T$ (or $G$) as an induced subgraph.

From this remark one can at once conclude the following:

1. If $diam(T) = 1$, that is $T = K_2$, then $\theta_{3,2}(T) = 2$.

2. If $diam(T) = 2$, then the tree $T$ is a star, $K_{1,n}$ for $n \geq 2$ one can easily conclude that $\theta_{3,2}(T) = 1$ if and only if $T = K_{1,2}$.

3. If $diam(T) = 3$, then the tree $T$ is a double star, $S_{m,n}$.

In this case consider the double star, $S_{2,3}$ as labeled in the Following Fig.6.

$S_{2,3}$:

Clearly, there are five vertices of eccentricity three and two vertices of eccentricity two. Now we claim that $\theta_{3,2}(T) = 2$. For if $\theta_{3,2}(T) = 1$. Then, there exist a (3,2) - graph $H$ such that $H - x = S_{2,3}$. Clearly $e_H(x) = 2$ and all other vertices in $H$ are of eccentricity two, except two vertices which are having eccentricity three. This is impossible, even if we make an

Fig. 4
adjacencies to all vertices of $S_{2,3}$ accept one vertex. Thus to embed $S_{2,3}$ in $(3,2)$-graph it requires one more vertex.

On the other hand one can easily see that $\theta_{3,2}(S_{m,n}) = 1$, if either $m = 1$ or $n = 1$, from the following Fig. 5

Thus we conclude that $\theta_{3,2}(S_{m,n}) = 1$ if and only if $m$ or $n$ is one.

Now we consider the embedding index of the class of trees with diameter four.

It is interesting to note the class of trees with diameter four can be characterized easily and is seen in the following proposition.

**Proposition 2.2:** For any tree $T$ the following statements are equivalent

1. $\text{diam}(T) = 4$

2. The removal of end vertices results into star.

3. $\text{rad}(T) = 2$ and $|C(T)| = 1$. 
Where $C(T)$ denotes the center of a tree $T$.

Above proposition gives the clear picture of trees with diameter four

Now we characterize the trees with $diam(T) = 4$ having embedding index $\theta_{3,2}(T) = 1$ in terms of two forbidden subtrees.

**Theorem 2.3:** For any tree $T$ with $diam(T) = 4$, $\theta_{3,2}(T) = 1$ if and only if $T$ does not contain $T_1$ and $T_2$ as an induced subgraph, where

\[ T_1: \]
\[ \text{Fig. 6} \]

\[ T_2 \]
\[ \text{Fig. 7} \]

**Proof:** Let $T$ be a tree with $diam(T) = 4$. Assume that $T$ does not contain $T_1$ and $T_2$ as an induced subgraph. Then by Proposition 2.2, the only class of trees $T$ with $T_1$ or $T_2$ as forbidden subgraphs is as in the following Fig.
Now, let us label one of the diameter path $P$ as $u_1, u_2, u_3, u_4, u_5$, and rest of the vertices as $u_6, u_7, \ldots, u_s$ randomly. Let $H$ be a graph obtained from $T$ by taking a new vertex $x$ whose vertex set $V(H) = V(T) \cup \{x\}$ and an edge set $E(H) = E(T) \cup \{xu_i \text{ for all } i, \text{ except } i = 4\}$. Then clearly, $e_H(z) = 2$ for all $z$ except $z = u_1$ and $u_4$, where $e_H(u_1) = e_H(u_4) = 3$. Thus $H$ is $(3,2)$–graph containing $T$ as an induces subgraph and thus $\theta_{3,2}(T) = 1$ holds.

Conversely suppose $\theta_{3,2}(T) = 1$, we claim that neither $T_1$ nor $T_2$ is an induced subgraph. On the contrary assume that $T$ is an induced subgraph of $T$. Let us label the vertices of $T$ in $T$ as in the Fig. 12.

![Fig. 8](image_url)

![Fig. 9](image_url)
As $\theta_{3,2}(T) = 1$, then by FACT 3., there exists a $(3,2)$-graph $H$ such that $H - x = T$ for some vertex $x$. Clearly $e_H(x) = 2$. One can easily see that whatever the adjacencies from $x$ to the vertices $T$, will give at least three vertices of eccentricity three. Since if we take $u, u_1, u_2, u_4$ as a diametral path in $H$, then $e_H(u) = e_H(u_1) = 3$, but then $e_H(u_4) = 3$ holds automatically. In fact, this happens if we choose any of the diametral path (that is the existence of at least three vertices of eccentricity three) a contradiction. Similar argument holds if $T$ contains $T_4$ as an induced subgraph.

As far as the trees $T$ with $\text{diam}(T) = 5$, the same way can be characterized in terms of forbidden subgraphs.

**Theorem 2.4:** For any tree $T$ with $\text{diam}(T) = 5$, $\theta_{3,2}(T) = 1$ if and only if $T$ does not contain $T_1$ or $T_4$ as an induced subgraphs. Where,

![Diagram](Fig. 10)
Proof: Proof is similar to the proof of the above theorem.

From the Theorems 2.3 and 2.4 one can have the following Theorem.

**Theorem 2.5:** For any tree $T$, $\theta_{s_s}(T) = 1$ if and only if $T$ is one of the following Fig. 14.

$T_1$:

$T_2$:

$T_3$:
Fig. 12.

$T_4$:

$T_5$:

Fig. 12.
REFERENCES:


