CHAPTER III
ALMOST SELF-CENTERED \((r, r + 1)\)-GRAPHS

1 INTRODUCTION:

In chapter II and IV we have considered a class of graphs in which \(|C(G)| = n - 2\) and we have called them as Almost Self-centered \((r + 1, r)\)-graphs. The detail study of these class of graphs are considered in chapters II and IV. In this chapter we consider other extreme class of graphs that is – the class of graphs in which there is exactly one vertex of eccentricity \(r\) and rest are of eccentricity \(r + 1\). We know that, every graph contains atleast one central vertex and so if \(G\) is not self-centered graph then, it is obvious that, \(|P(G)| \leq n - 1\). The above said graphs are exactly the graphs with \(|P(G)| = n - 1\) and hence we call them as Almost Self-centered \((r, r + 1)\)-graphs. Where \(r\) is the radius of \(G\)

Following are some of the examples of \((r, r + 1)\)-graph:

EXAMPLES:

\(r = 1:\)

![Diagram](image-url)
In this chapter we study the existence and some properties of $(r,r+1)$-graphs and also we prove that every graph can be embedded in some $(r,r+1)$-graph and there by ruled out the possibility of characterization of $(r,r+1)$-graphs in terms of forbidden subgraphs.

Fig. 1

In this chapter we study the existence and some properties of $(r,r+1)$-graphs and also we prove that every graph can be embedded in some $(r,r+1)$-graph and there by ruled out the possibility of characterization of $(r,r+1)$-graphs in terms of forbidden subgraphs.
In section 2 we consider the size of \((r, r+1)\) – graph with given order. In section 3, we introduce the concept of embedding index \(\theta_{r,r+1}(G)\) as has been introduced in case of \((r+1,r)\) – graphs.

2 EXISTANCE AND GENERAL PROPERTIES OF \((r,r+1)\) – GRAPHS:

The existence of \((1,2)\) – graphs is not a problem. The smallest \((1,2)\) – graph (in terms of edges) is the only graph that is \(K_{1,n-1}\). In fact we can characterize \((1,2)\) – graph as in the following proposition.

**Proposition 2.1:** A graph \(G\) of order \(n\) is \((1,2)\) – graph if and only if \(G\) has exactly one vertex of degree \(n-1\) and all other vertices are of degree less than or equal to \(n-2\).

From the above Proposition one can immediately infer that

\[
1 \leq m \leq \frac{n^2 - n + 1}{2}
\]

Where \(m\) denotes the number of edges in \(G\).

But the same is not the case when \(r \geq 2\). Even it is difficult for \(r = 2\). In the following theorem we show the existence of \((r,r+1)\) – graphs for \(r \geq 2\).
**Theorem 2.2:** For any integer \( r \geq 2 \) there exist \((r,r+1)\)-graphs of order \( n \) for every \( n \geq 4r + 1 \).

**Proof:** Consider a graph \( G \) of order \( n \geq 4r + 1 \) as labeled in the Fig. 1

![Graph diagram](image)

To show that the graph \( G \) is \((r,r+1)\)-graph of order \( n \), it is sufficient to show that all the vertices in \( G \) are of eccentricity \( r+1 \) except the vertex \( u_{i0} \); that is the eccentricity of \( u_{i0} \) is \( r \). For this we show that every vertex \( u_{ij} \) \((1 \leq i \leq r, 1 \leq j \leq 4)\) has an eccentric vertex \( u_{si} \) \((1 \leq s \leq r, 1 \leq t \leq 4)\) with \( d(u_{i},u_{s}) = r+1 \) one can easily verify that the two lists of the following pairs of vertices are eccentric vertices with \( d(u_{i},u_{s}) = r+1 \) depending on whether \( r \) is even or odd.

**Case 1:** \( r \) is even.

The pairs of eccentric vertices with distance \( r+1 \) are listed below:

- \( u_{11} \rightarrow u_{r3} \)
- \( u_{12} \rightarrow u_{r4} \)
- \( u_{13} \rightarrow u_{r1} \)
- \( u_{14} \rightarrow u_{r2} \)
- \( u_{21} \rightarrow u_{r-13} \)
- \( u_{22} \rightarrow u_{r-14} \)
- \( u_{23} \rightarrow u_{r-11} \)
- \( u_{24} \rightarrow u_{r-12} \)
And so on ....

\[ u_{2j} \rightarrow u_{\frac{r+1}{2}}^{(\frac{r+1}{2})^3} \]

\[ u_{2j} \rightarrow u_{\frac{r+1}{2}}^{(\frac{r+1}{2})^4} \]

\[ u_{2j} \rightarrow u_{\frac{r+1}{2}}^{(\frac{r+1}{2})^1} \]

\[ u_{2j} \rightarrow u_{\frac{r+1}{2}}^{(\frac{r+1}{2})^2} \]

Further for each \( v \), any of the vertices \( u_{rj} \), \( 1 \leq j \leq 4 \) is an eccentric vertex and eccentricity of \( u_{60} \) is \( r \). Thus \( G \) is \((r, r+1)\)-graph of order \( n \).

**Case 2:** \( r \) is odd.

In this case the pattern of eccentric vertices holds as in case 1. for all vertices in \( G \) except four vertices namely,

\[ u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^1, u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^2, u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^3, u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^4 \].

For these vertices one can easily verify that the eccentric vertex of \( u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^1 \) is \( u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^3 \) and eccentric vertex of \( u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^2 \) is \( u_{\left\lfloor \frac{r+1}{2} \right\rfloor}^4 \) and their distance \( r+1 \). This completes the proof. The following two graphs classifies both the cases as in the proof of above theorem. The examples for the cases \( r = 4 \) and \( r = 5 \) is given below.
The pairs of eccentric vertices are given below:

\[ u_{11} \rightarrow u_{43} \quad u_{21} \rightarrow u_{33} \]

\[ u_{12} \rightarrow u_{44} \quad u_{22} \rightarrow u_{34} \]

\[ u_{13} \rightarrow u_{41} \quad u_{23} \rightarrow u_{31} \]

\[ u_{14} \rightarrow u_{42} \quad u_{24} \rightarrow u_{32} \]

and \( v_i \rightarrow u_{4i} \), for \( i = 1, 2, \ldots, n - 17 \)
Now the question is what is the smallest order of the 

$$(r, r+1)$$-graph. Towards the solution of this question, we look at the 

following graphs as in the Fig. 5 and Fig. 6.

Consider the graph $G_1$ as in the Fig. 5

![Graph G_1](image.png)

Clearly, $G_1$ is $(2,3)$-graph of order 7 which is of the form $4r-1$ with 

$e(v_0) = 2$, $e(v_i) = 3$ for $i \geq 1$. In the following Proposition we prove that $G_1$
is the smallest \((2,3)\)-graph by proving that there is no \((2,3)\)-graph of order \(\leq 6\).

**Proposition 2.3:** There exists no \((2,3)\)-graph of order \(\leq 6\).

**Proof:** This can be easily verified from the table of graphs listed in F. Harary [9].

For \(r = 3\) we have another graph \(G_2\) which is \((3,4)\)-graph containing 11 vertices as shown in Fig. 6.

![Fig. 6](image-url)

In this graph \(e(v_0) = 3\) and \(e(v_i) = 4\), for \(i \geq 1\). This is also \((r, r + 1)\)-graph of order \(4r - 1\). In view of the above two examples we have the following open problem.
**Open Problem:** Prove or disprove: There exists no \((r, r+1)\)-graph of order less than \(4r - 1\).

Next two Propositions gives the structure of \((r, r+1)\)-graph.

**Proposition 2.4:** If \(G\) is \((r, r+1)\)-graph then \(G\) contains at most one cut vertex.

**Proof:** Suppose \(G\) is \((r, r+1)\)-graph. We prove that \(G\) contains at most one cut vertex. First we prove that no cut vertex is a peripheral vertex. If possible assume that a cut vertex \(u\) in \(G\) is a peripheral vertex. Then \(e_{\partial}(u) = r + 1\). Let \(x\) be an eccentric vertex of \(u\). Then \(d(u, x) = r + 1\). Let \(y\) be any other vertex in the block of \(G\) not containing \(x\). Then \(d(x, y) > d(u, x) = r + 1\), a contradiction, since \(\text{diam}(G) = r + 1\). Now it is easy to derive that \(G\) contain at most one cut vertex; for if \(G\) has two cut vertices say \(u\) and \(v\), Then \(e_{\partial}(u) = e_{\partial}(v) = r\). A contradiction to the fact that \(G\) contains exactly one vertex of eccentricity \(r\).

**Corollary 2.5:** If \(G\) is \((r, r+1)\)-graph containing a cut vertex \(u\), then \(u\) must be a center of \(G\).

**Proof:** The proof follows from the above proposition.
Proposition 2.6: Let $G$ be $(r,r+1)$-graph with a cutvertex $u$ and $B_1, B_2, \ldots, B_k$ be the blocks of $G$ containing a cutvertex $u$. If $v \in B_i$ not adjacent to $u$, then all the vertices in other blocks are adjacent to $u$.

Proof: Without loss of generality assume that the vertex $v \in B_i$ is a vertex not adjacent to $u$. If possible assume that $w \in B_2$ is the farthest vertex not adjacent to $u$.

Therefore, $d(u, w) = r$. As $v \in B_i$, not adjacent to $u$ so that $d(u, v) \geq 2$. But every shortest path $v - w$ must contain a vertex $u$ and hence $d(v, w) = d(w, u) + d(u, v) \geq r + 2$. But $d(v, w) \leq r + 1$, which implies $r + 2 \leq r + 1$. This is an absurd. A contradiction to the fact that $diam(G) = r + 1$. Which completes the proof.
The \((r, r+1)\)-graph is said to be minimal if \(G - u\) is not
\((r, r+1)\)-graph for every vertex \(u\) in \(G\).

**Proposition 2.7:** Every minimal \((r, r+1)\)-graph is a block.

**Proof:** Let \(G\) be minimal \((r, r+1)\)-graph. We prove that \(G\) is a block.

Then by the proposition \(G\) has a cut vertex \(x\) (say). \(B_1, B_2, \ldots B_k\) be the
blocks of \(G\). By corollary, \(x\) is a center of \(G\) and by the proposition there
is exactly one block in which some vertex is not adjacent to \(x\) and all the
vertices in the remaining blocks are adjacent to \(x\). Without loss of
generality assume that the \(B_x\) is a block in which one vertex is not adjacent
to \(x\). Then clearly for every vertex \(u \neq x\) in \(B_i, i \geq 2\). \(G - u\) is
\((r, r+1)\)-graph. A contradiction to the minimality ; which proves the
result.

Next Theorem rules out the possibility of characterization of
\((r, r+1)\)-graph in terms of an forbidden subgraph.

**Theorem 2.8:** Every graph can be embedded in some \((r, r+1)\)-graph.

**Proof:** Let \(G\) be any graph of order \(n\) with the labeling \(u_1, u_2, \ldots u_n\). Let \(F\)
be \((r, r+1)\)-graph of order \(n + 4r + 1\) as labeled in the Fig. 8
Let $H$ be the graph obtained from $G$ and $F$ by identifying each vertex $u_i$ for $G$ with $v_i$ of $F$ for every $i = 1, 2, \ldots, n$. Clearly one can verify that $H$ is $(r, r + 1)$-graph of order $n + 4r + 1$ containing $G$ as an induced subgraph.

3 THE SIZE OF $(r, r + 1)$-GRAPH:

In this section we consider the size of $(r, r + 1)$-graph of order $n$ in general and the size of (1,2) – graph and (2,3) – graph in particular.

**Theorem 3.1:** If $G$ is $(r, r + 1)$-graph of order $n \geq 7$ then,

$$m \geq n + 2,$$

for every $r \geq 2$.

**Proof:** $G$ is $(r, r + 1)$-graph of order $n \geq 7$ and $r \geq 2$. To prove $m \geq n + 2$, it is sufficient to show that there is no $(r, r + 1)$-graph of order $n$ with size
\[ m \leq n + 1. \text{ As } G \text{ is } (r,r+1)\text{-graph, so } m \geq n - 1. \text{ If } m = n - 1, \text{ then } G \text{ is a tree.} \]

The only \((r,r+1)\)-graph with \(m = n - 1\) is star. Then \(r = 1\). A contradiction to the fact that \(r \geq 2\). If \(m = n\), then \(G\) is either a cycle or \(G\) is a unicyclic graph with exactly one cut vertex (by the Proposition 2.6). If \(G\) is a cycle, then \(G\) is a self-centered graph, which is redundant. Hence \(G\) must be a unicyclic graph with exactly one cut vertex. But only \((r,r+1)\)-graph containing one cycle is \(K_{1,n-1} + e\), which is \((1,2)\)-graph. Again a contradiction.

Now assume that \(m = n + 1\). In this case we consider two cases.

**Case 1:** \(G\) has cut vertex.

Then by Proposition 2.6 and the assumption that \(m = n + 1\), \(G\) is either of the following forms:

\[
\begin{align*}
\text{Chord} & \\
\text{Fig. 9}
\end{align*}
\]

In both the cases \(G\) is not \((r,r+1)\)-graph.
Case 2: G has no cut vertex.

In this case G is a cycle with the chord and the vertex at the end of the chord whose eccentricity is less than r, a contradiction. Thus there exists no \((r, r+1)\)-graph of order \(n\) with size \(m \leq n+1\) for \(r \geq 2\).

To complete the proof, it is sufficient to show that there exist \((r, r+1)\)-graph of order \(n\) with \(m = n+2\).

The following two graphs \(G_1\) and \(G_2\) which are \((2,3)\)-graph and \((3,4)\)-graph respectively, contains \(m = n+2\) edges.

\[G_1: \]

\[G_2: \]

Fig. 10

Fig. 11
Two graphs $G_1$ and $G_2$ which are $(2,3)$ - graph and $(3,4)$ - graph contains $m = n + 2$ edges.

The next two theorems deals with the size of $(1,2)$ - graph and $(2,3)$ - graph.

**Theorem 3.2**: If $G$ is $(1,2)$ - graph of order $n$ and size $m$, then,

$$n - 1 \leq m \leq \begin{cases} \frac{(n-1)^2}{2}, & \text{if } n \text{ is odd} \\ \frac{n(n-2)}{2}, & \text{if } n \text{ is even} \end{cases}$$

Further the lower bound is attainable if and only if $G = K_{1,n-1}$ and the upper bound is attainable if and only if $G = (K_{n-1} - Q) + K_1$. Where $Q = \left(\binom{n-2}{2} K_2 \cup K_{1,2}\right)$ if $n$ is even and $Q = \left(\binom{n-1}{2} K_2\right)$ if $n$ is odd.

**Proof**: Let $G$ be $(1,2)$ - graph of order $n$. As $G$ is connected, $m \geq n - 1$ always holds. Further if $m = n - 1$, then $G$ is a tree and as $diam(G) = 2$, it must be a star $K_{1,n-1}$. To prove the upper bound, one can see the properties of $(1,2)$ - graph. As $G$ is $(1,2)$ - graph, $G$ must contain exactly one vertex of degree $n - 1$ and for the rest of $n - 1$ vertices, degree of each vertex is $\leq n - 2$. If $n$ is odd, then $n - 1$ is even, so that degree of each vertex is $\leq n - 2$. In this case,

$$2m = \sum_{u \in V} \deg u \leq (n - 1) + (n - 1)(n - 2)$$

$$= (n - 1)(n - 1)$$
\[ m = \frac{(n-1)^2}{2}. \]

On the other hand if \( n \) is even, so that \( n-1 \) is odd, the degree of \( n-2 \) is \( < n-2 \) and one vertex is \( n-3 \). Therefore,

\[ m \leq \frac{(n-1) + (n-2)(n-2) + n - 3}{2} \]
\[ = \frac{n(n-2)}{2} \]

The attainment of the upper bound in both the cases is to derive.

**Theorem 3.3:** If \( G \) is \((2,3)\) - graph of order \( n \), ad size \( m \), then

\[ n + 2 \leq m \leq \frac{n^2 - 7n + 18}{2} \]

Further the bounds are attainable.

**Proof:** The lower bound is the particular case of the general result proved in Theorem 3.1.

To prove the upper bound we employ the induction method. As \( G \) is \((2,3)\) - graph, so that \( n \geq 7 \). For the case \( n = 7 \), the graph \( G \) of the Fig. 10 is the smallest one and hence \( m = 9 = \frac{7^2 - 7.7 + 18}{2} \) holds good. Assume that the result is true for \((2,3)\) - graph of order \( n \geq 8 \). Then there exists a vertex \( u \) in \( G \) such that \( G - u \) is a \((2,3)\) - graph. Thus while removing a vertex \( u \), we have removed \( 4 + n - 8 = n - 4 \) edges. Thus,
\[
m \leq \frac{(n-1)^2 - 7(n-1) + 18}{2} + n - 4
\]

\[
= \frac{n^2 - 2n + 1 - 7n + 7 + 18 + 2n - 8}{2}
\]

\[
= \frac{n^2 - 7n + 18}{2}.
\]

Hence the result holds for \( n \geq 7 \).

The following graph \( H_1 \) and \( H_2 \) show that both lower and upper bounds are attainable (respectively).

\[H_1: \]

\[H_2: \]

\[K_{n-7} \]

Fig. 12

Fig. 13
4 EMBEDDING INDEX OF (2,3)-GRAPH

As in case of \((r+1,r)\)-graph, we introduce the concept of embedding index with respect to \((r,r+1)\)-graph as below:

The embedding index \(\theta_{r,r+1}(G)\) of a graph \(G\) is defined as

\[
\theta_{r,r+1}(G) = \min_{H \in \mathcal{A}_{r,r+1}} \{O(H) - O(G)\}
\]

where, \(\mathcal{A}_{r,r+1}\) denotes the set of all \((r,r+1)\)-graphs.

By the construction of \((r,r+1)\)-graphs, in the theorem one can conclude that

\[
\theta_{r,r+1} \leq 4r + 1
\]

In this section we restrict ourselves to the value, \(r = 2\), that is \(\theta_{2,3}(G)\).

To carry out the study of \(\theta_{2,3}(G)\), we have the following observations:

1. By Theorem 2.8 we have, \(0 \leq \theta_{2,3}(G) \leq 9\).

2. If \(G\) is not \((2,3)\)-graph, then \(\theta_{2,3}(G) \geq 1\).

3. If \(\theta_{2,3}(G) = 1\), then there exists \((2,3)\)-graph \(H\) containing \(G\) as an induced subgraph such that \(H - u = G\), for some vertex \(u\) in \(H\).

In this section we establish the exact values of \(\theta_{2,3}(G)\) for \(G = K_n, P_n\) and \(C_n\) respectively.
Proposition 4.1: \( \theta_{2,3}(C_n) = \begin{cases} 
5 & \text{for } n = 3 \\
3 & \text{for } n = 4,5 \\
1 & \text{for } n = 6,7 \\
3 & \text{for } n \geq 8 
\end{cases} \)

Proof: To prove the result, we use the fact that the graph \( G_i \) of the Fig. 5 as the smallest \((2,3)\)-graph.

It is obvious to observe that any graph other than the subgraph in \( G_i \) contained in \((2,3)\)-graph must have an order at least eight. Since the smallest \((2,3)\)-graph is of order seven, with this observation, we have the following cases for \( C_n \) and the graph \( G_i \), we mean the graph of Fig. 5 only.

Case 1: \( n = 3 \)

As \( G_i \) does not contain \( C_3 \) by the above remark, the order of \((2,3)\)-graph, containing \( C_3 \) must be at least eight. Thus \( \theta_{2,3}(C_3) \geq 8 - 3 = 5 \).

But, by the graph \( H \) of the Fig. 14 shows that \( \theta_{2,3}(C_3) \leq 5 \).

Fig. 14

Thus, \( \theta_{2,3}(C_3) = 5 \) holds.
Case 2: \( n = 4, 5 \)

As \( G \) contains \( C_4 \) as an induced subgraph, thus \( \theta_{2,3}(C_4) = 3 \) holds. And also \( G \) contains \( P_4 \) as an induced subgraph, taking a new vertex and joining it to both the end vertices of the induced subgraph \( P_4 \) as shown in Fig. 16 and as in case of \( C_3 \) one can argue in similar fashion. We conclude that \( \theta_{2,3}(C_3) = 3 \). Both the instances can be observed from the Fig. 15 and in Fig. 16.

![Fig. 15](image1)

![Fig. 16](image2)

And also one can show that \( \theta_{2,3}(C_5) = 0 \) and \( \theta_{2,3}(C_7) = 1 \) by embedding of \( C_6 \) and \( C_7 \) in the following \((2,3)\)-graphs given in the Fig. 17 and Fig. 18.

![Fig. 17](image3)

![Fig. 18](image4)
For $n \geq 8$, the cycle $C_n$ can be embedded in $(2,3)$-graph $H$ of the following Fig. 19.

\[ H : \]

Fig. 19

The graph $H$ is of order $n+3$ and hence $\theta_{2,3}(C_n) = 3$. This completes the proof.

\[ \text{Proposition 4.2: } \theta_{2,3}(P_n) = \begin{cases} 
4 & \text{for } n = 3 \\
2 & \text{for } n = 5 \\
3 & \text{for } n = 4, \ n \geq 6 
\end{cases} \]

\[ \text{Proof: } \] As the graph $G$ of the Fig. 20 contains $P_3$, $P_4$ and $P_5$ as an induced sub graphs and hence the result holds for $n = 3, 4, 5$.

$\theta_{2,3}(P_n) = 4$ for $n = 3$

Fig. 20
For $w > 6$, consider a $(2,3)$-graph $H$ of order $n + 3$ as labeled in the Fig. 23.

The vertices on the dark line induces a path $P_n$ in $H$ of Fig 23 and hence

\[ \theta_{2,3}(P_n) = 3. \]
Proposition 4.2: $\theta_{2,3}(K_n) = 5$

Proof:

Let $H$ be a minimal $(2,3)$-graph. We know that every contains $K_2$ as an induced subgraph. $K_n$ can be embedded in some $(2,3)$-graph, by concatenating the edges $w_iw_e$ of a minimal $(2,3)$-graph with the edges $u_1u_2$ of $K_n$. The resulting graph $F$ contains $K_n$ as an induced subgraph. Hence $\theta_{2,3}(K_n) = \min_{H\in\mathcal{G}_{2,3}} \{ O(H) - O(G) \} \leq n + 7 - n - 2 \leq 5$. On the other hand removal of any 4 vertices results into either $K_1 \cup K_n$ or $K_1 + K_n$. Hence $\theta_{2,3}(K_n) = 5$. 

Fig. 1
Open Problems:

2. Prove or disprove: There exists no \((r, r + 1)\) - graph of order less than 
   \(4r - 1\).

3. If \(G\) is \((r, r + 1)\) - graph of order \(n \geq 7\) then, what is the maximum 
   number of edges, for every \(r \geq 2\)?
REFERENCES:


