Chapter 8

Some Applications and

Conclusion

A theory becomes meaningful if it has applications in real life. Almost all branches in graph theory have developed mainly due to variety of applications and motivation derived from the applications. Application of a mathematical method or technique to model or represent a real problem is called mathematical modeling. We make models either for representing an object or phenomenon or to solve an underlying problem to obtain a better solution. In this chapter we discuss a few modeling situations of some variations of Roman domination and the simplification or result that is offered by models.

The concept of domination in graph theory originated from a chess board problem. It goes back to 1850’s. Chess players were interested in the minimum number of queens such that every square on the chess board either contains a queen or is attacked by a queen. On a chess board a queen can move either vertically or horizontally or diagonally. All other moves of queen are invalid. This problem was first mathematically described by C. F. de Jaenisch [21] in the year 1862.
Domination is widely applied in many other fields. It is being used in facility location problems (see [12]), where the number of facilities (e.g., hospitals, fire stations etc.) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying (e.g., minimizing the number of places a surveyor must stand in order to take height measurements for an entire region). In the following sections we discuss various applications of Roman domination and its variations.

8.1 Applications in Military and Defence Strategic Planning

Motivation behind Roman domination is the “defence in depth” strategy of emperor Constantine, adopted in the beginning of 4th century AD. Details of his strategy and the situation which forced him to adopt a minimum defence strategy is explained in the first chapter itself. We know discuss the importance of the strategy to defence planning. Due to the huge expense of setting up military camps at a strategically important place, some places are selected to set up camps. Selection of such places are made after considering the total cost of maintaining the troops at the stations and the cost of defending the places which could be targeted by enemies.

Defence centers should be selected so that troops could defend all other places effectively from the places where they are based at. We can minimize
the cost of defence by minimizing the number of centers. Thus the problem
is equivalent the problem to find the minimal Roman dominating function
of the underlying graph. It is based on the oversimplified assumption that
total cost of maintaining defense center at a vertex is same.

We can restate the problem in a more realistic way. Let the cost of setting
up a defense station at the vertex \( v_i \) be \( s_i \) and the running cost be \( r_i \). The
total cost of \( v_i \) is \( t_i = s_i + r_i \). It is assumed that cost of a military operation
from a vertex to another vertex is same, between any pairs of vertices. The
total cost of a Roman dominating set \( R = \{ v_1, v_2, ..., v_s \} \) is \( C_R = \sum_{i=1}^{s} t_i \).
The best strategy of the problem is the Roman dominating function for
which \( C_R \) is minimum. This minimum cost corresponds to a minimal Roman
dominating function.

The problem can be more generalized assuming that the cost of main-
taining two units of army at a place is double the cost of maintaining one
unit. Thus we have two choices of cost at each vertex. At \( v_i \) if there is only
one unit, \( t_i = s_i + r_i \) and \( t_i = s_i + 2r_i \) otherwise. Minimum cost of this prob-
lem may not correspond to a minimal Roman dominating function. Thus we
have to consider all Roman dominating function in this case, to obtain the
solution.

8.2 Applications in the Analysis of Social Re-
lations

Social network analysis is a well established area of intensive research in
Social Science. Graph Theoretic methods are proved to be very effective in
modeling problems in social science. For detailed study, the book written by
Ted G. Lewis \cite{25} is very useful.
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Edge vertex Roman domination can be used to analyze the social relations among individuals and to select representatives of a group subject to some constrains. Members of a group usually have different opinions and they divide among themselves based on their opinion. Thus two or more subgroups are formed. Good relations among two members can be represented by the presence of an edge and the absence of edge indicates that the members hate with each other. We want to select some representative of the group subject to the following conditions:

- At least one member should represent each opinion group.
- Every member of the group must have good relation with at least one representative.
- All representative should have good relations mutually.
- Number of representative should be minimum.

In almost all situations people divide into two based on their opinion. So here we particularly analyze this situation leaving the more general case for another occasion.

With reference to the discussions in the previous chapter on a special case of edge vertex Roman domination we give a graph theoretic model to represent the problem. If the graph modeling the problem has a partition of vertex set into $V_1$ and $V_2$ and there exists an edge $e = uv$ such that

- $u \in V_1$, $v \in V_2$
- $u$ is connected to all vertices in $V_2$.
- $v$ is connected to all vertices in $V_1$. 
That is, the associated graph has an edge vertex Roman dominating function $f$ such that $f(uv) = 2$ and $f(a) = 0$ for all remaining edges in $G$. If there does not exist an edge having the properties stated, answer could be obtained seeking a bigger set of edges, which all together constitute a complete subgraph of $G$.

8.3 Conclusion

In this thesis our research mainly focused on the variations of Roman dominating functions. Roman domination has many applications in the areas such as facility location problems, planning of defence strategies, surveillance related problems etc. In the eight chapters of the thesis first chapter is exclusively for presenting basic ideas and definitions.

Chapter two is based on the concept of connected eternal domination, which is a restriction of eternal domination, a concept introduced by A. P. Burger and his coauthors in the year 2004. The main result of the chapter helps us to categorize the whole family of graphs into two. One, the set of all graphs for which $\sigma_{cm}(G) = \gamma_c(G)$ and the other is the set of all graphs for which $\sigma_{cm}(G) = \gamma_c(G) + 1$.

In the third chapter we saw safe eternal 1-secure sets and discussed the properties. It is a conditional restriction of eternal 1-secure sets. We characterized the graphs having safe eternal 1-secure sets, which has the property that all vertices - excluding those in the safe 1-secure set - are safe. We also introduced a new kind of directed graphs, which represent the transformation from one safe 1-secure set to another safe 1-secure set of a given graph. In the fourth chapter, we further generalized safe eternal 1-secure sets, allowing $m$ guards to move in response to an attack. In the fifth
chapter, we dealt with the directed graph version of Roman domination. A vertex based orientation of a digraph is suggested, which would simplify the difficulties associated with the calculation of Roman domination number of digraphs, to some extent.

Sixth chapter is on the linear relaxation of Roman dominating function of a graph $G$. It is followed by a preliminary study on convexity of fractional Roman dominating functions. In the seventh chapter we explored in detail the vertex - edge and edge - vertex versions of Roman dominating functions and presented many related results.

The last chapter contains discussions on some important applications of the theory developed in the previous chapters and few final words of conclusion.