Chapter 6

Fractional Roman Domination

It is important to discuss minimality of Roman domination functions before we get into the details of fractional version of Roman domination. Minimality of domination set is clearly defined. A dominating set $S$ is minimal if no proper subset of $S$ is a dominating set. Equivalently, the dominating function $f$ representing $S$ is minimal if we cannot obtain another dominating function $g$ by reducing the function values of $f$. In this sense $f$ is irreducible. Minimality and convexity of dominating functions is a well studied area. More about minimality of dominating functions can be obtained from the following publications [6, 7, 8, 14, 34, 37, 38, 39, 40, 41, 43].

An RDF $f = (V_0, V_1, V_2)$ is reducible if it is possible to reduce the value of $f(v)$ for some vertex $v$ and still the resulting function be an RDF. An RDF is irreducible if it is not reducible. But minimality of Roman dominating function is more than irreducibility. Minimality has four versions, which are described in [12]. An RDF is Type I minimal if it is irreducible.

A second type of minimality has been suggested by Renu Laskar [19]. An RDF $f = (V_0, V_1, V_2)$ is Type II minimal if it is irreducible and $V_0$ is not empty.
The third definition of minimality was developed by Steve Hedetniemi [19], which uses the notion of sliding tokens between vertices in the graph. For a given RDF \( f = (V_0, V_1, V_2) \), \( f(v) = i \) means that there are \( i \) tokens on \( v \). A slide of a token from \( u \in V_1 \cup V_2 \) to \( v \in V_0 \cup V_1 \) is called a valid slide if the new function \( f' \), defined by \( f'(u) = f(u) - 1, f'(v) = f(v) + 1, f'(w) = f(w), w \neq u, v \) is also an RDF. If \( f \) is an irreducible RDF, there are only two types of possible valid slides: sliding a token from a vertex in \( V_1 \) to another vertex in \( V_1 \), or sliding a token from a vertex in \( v \in V_2 \) to vertex in \( w \in V_0 \), provided that \( w \) is the only external \( V_2 - pn \) of \( v \). An RDF \( f \) is Type III minimal if (1) \( f \) is irreducible and (2) no sequence of valid slides changes \( f \) into a reducible RDF \( f' \).

The last version of minimality, due to Alice McRae [19]. An RDF \( f = (V_0, V_1, V_2) \) is Type IV minimal if (1) \( f \) is irreducible and (2) \( V_1 \) is independent.

In Graph Theory many concepts are dealt with treating it as presence or absence of elements in a specified set. Presence of a vertex is denoted by the number 1 and absence by 0. For example, a dominating set \( D \) of \( G \) is the whole vertex set \( V \) of the graph together with a function \( f \) defined on it such that \( f(v) = 1 \) if \( v \in D \) and \( f(v) = 0 \) if \( v \notin D \). This function is an integer valued function.

Fractional graph theory deals with the generalization of integer-valued graph theoretic concepts such that they take fractional values. One of the standard methods for converting a graph theoretic concept from integer version to fractional version is to formulate the concept as an integer program and then to consider the linear programming relaxation. We first present some basic definitions and results on fractional domination.
6.1 Different types of fractional Roman domination

We can define different varieties of fractional Roman domination taking into account different aspects of Roman domination. First we consider the fact that a Roman dominating function \( f = (V_0, V_1, V_2) \) has the property that every vertex in \( V_0 \) is adjacent to at least one vertex in \( V_2 \). We can fractionalize the Roman dominating function by allowing the function values to vary freely in the closed interval \([0, 2]\). The vertices having function values 2 can supply one item to one of the vertices having zero as function value and adjacent to it. The vertex can do this only if it has one item for own use. We can fractionalize the situation as follows. A function \( f : V \to [0, 2] \) such that every vertex having function value zero is adjacent to at least one vertex having function value greater than one is a kind of fractional Roman dominating function.

To make the discussion more clear we need the following definitions. For the function \( f : V \to [0, 2] \), \( V_0^f = \{ v \in V \mid f(v) = 0 \} \), \( V_1^f = \{ v \in V \mid f(v) = 1 \} \), \( V_2^f = \{ v \in V \mid f(v) = 2 \} \), \( V_{[0,1]}^f = \{ v \in V \mid 1 \geq f(v) > 0 \} \) and \( V_{[2,1)}^f = \{ v \in V \mid 2 \geq f(v) > 1 \} \). If there is no chance of confusion, we write \( V_0 \) for \( V_0^f \), \( V_1 \) for \( V_1^f \) and so on. A function \( f : V \to [0, 2] \) is a fractional Roman dominating function if \( V_{[2,1)}^f \) dominates \( V_0^f \) (i.e. \( V_{[2,1)}^f \to V_0^f \)). A fractional Roman dominating function \( f \) is reducible if it is possible to reduce the value of \( f(v) \) for some vertex \( v \) and obtain a new fractional Roman dominating function of the same graph. Otherwise it is irreducible. We can reduce the function value of a fractional Roman dominating function \( f \) freely, if the function value \( f(v) > 1 \). This reduction will not change the vertex sets \( V_{[2,1)}^f \), \( V_0^f \) and \( V_1^f \). But the function value can be reduced to one (then the vertex is removed from \( V_{[2,1)} \) and included in \( V_1 \)) only if all vertices in \( V_0 \), which are
adjacent to \( v \in V_{(2,1)} \) are adjacent to another vertex in \( V_{(2,1)} \). Thus if the new function obtained is \( g \), then \( V_{(2,1)}^g \subset V_{(2,1)}^f \), \( V_1^g \supset V_1^f \) and \( V_0^g = V_0^f \). In addition to the above, if a vertex \( u \in V_1^g \) is adjacent to some vertices in \( V_{(2,1)}^g \), we can decrease the function values at the vertices to zero and obtain a new fractional Roman domination function, say \( h \). Then \( V_{(2,1)}^g = V_{(2,1)}^h \), \( V_0^h \supset V_0^g \) and \( V_1^h \subset V_1^g \). Now we define a special type of fractional Roman domination functions called \( \alpha \) type. A fractional Roman domination function \( f \) is \( \alpha \) type if it satisfies the following conditions:

- There exists no other fractional Roman domination function \( g \) such that \( V_{(2,1)}^g \subset V_{(2,1)}^f \), \( V_1^g \supset V_1^f \) and \( V_0^g = V_0^f \)
- There is no edge in \( G \) connecting \( V_1^f \) and \( V_{(2,1)}^f \).

**Theorem 6.1.1.** If the fractional Roman dominating function \( f \) is \( \alpha \) type, then there exists another minimal Roman dominating function \( g \) such that \( V_{(2,1)}^g = V_{(2,1)}^f \), \( V_0^g = V_0^f \) and \( V_1^g \supset V_1^f \). There exist infinitely many fractional Roman dominating functions of \( \alpha \) type.

**Proof.** Let \( f \) be a \( \alpha \) type fractional Roman dominating function and \( v \in V_{(2,1)}^f \). Since \( f(v) > 1 \), we can define another fractional Roman dominating function \( g \) such that \( g(v) = f(v) - \Delta \), where \( \Delta \to 0 \) and \( g(v) = f(v) \) for all other vertices. So \( V_{(2,1)}^g = V_{(2,1)}^f \), \( V_0^g = V_0^f \) and \( V_1^g \supset V_1^f \). Since \( \Delta \) is arbitrarily chosen and \( \Delta \to 0 \), the second part of the result follows.

The above Theorem makes it clear that an \( \alpha \) type fractional Roman dominating function can still be reducible. Now we proceed to prove the condition for an \( \alpha \) type fractional Roman dominating function to be irreducible.

**Theorem 6.1.2.** An \( \alpha \) type fractional Roman dominating function \( f \) is irreducible if and only if \( V_{(2,1)}^f = \emptyset \).
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Proof. If \( V_{[2,1]}^f = \emptyset \) for a minimal fractional Roman dominating function \( f \), then \( f(v) = 0 \) for all \( v \in V \) and hence \( V^f_0 = \emptyset \). So \( f \) is irreducible. Conversely suppose \( f \) is irreducible and \( V_{[2,1]}^f \neq \emptyset \). Let \( v \in V_{[2,1]}^f \). We can reduce the function value at \( v \) to obtain another minimal fractional Roman dominating function and thus get a contradiction.

A more generalized version of fractional Roman dominating function is possible to define by letting the vertices in \( V_1^f \) to take a value from the interval \((0,1]\). Then the vertex set equivalent to \( V_1^f \) becomes \( V_{[0,1]}^f \). Now we apply the condition that there is no edges from a vertex in \( V_{[0,1]}^f \) to a vertex in \( V_{[2,1]}^f \) to obtain a fractional Roman dominating function of the \( \beta \) type.

Example 6.1.3. Consider the graph \( G = K_3 \) given in Figure 6.1. \( V(G) = \{x, y, z\} \). The function \( f : V \to [0,2] \) such that \( f(x) = \delta_x^f \), \( f(y) = \delta_y^f \) and \( f(z) = \delta_z^f \). We can allow all the function values tend to zero and get a fractional Roman dominating function. \( \gamma_{fr}^f(G) = \inf \{ \delta_x^f + \delta_y^f + \delta_z^f \mid f \text{ is a fractional Roman dominating function of } G \} = 0 \).

If \( G \) has a fractional Roman dominating function \( f(x) = 1 + \delta \) then the other two function values must be zero. Thus \( \inf \{ f(V) \mid f \text{ is a fractional Roman dominating function of } G \text{ where one of the function values greater than one} \} = 1 \). A different type of fractional Roman dominating function \( f' \) is obtained by assigning non-zero real numbers which are less than one (say \( \delta_x, \delta_y, \delta_z \)). Since \( f' \) is a fractional Roman dominating function of fourth type, \( \delta_x + \delta_y + \delta_z = 1 \). So \( \inf \{ f'(V) \mid f' \text{ is a fractional Roman dominating function of } G \} = 1 \). Hence \( \gamma_{fr}^f(G) = 1 \).
A fractional Roman dominating function $f$ is \textit{minimal} (irreducible) if there does not exist a minimal Roman dominating function $g \neq f$ for which $g(v) \leq f(v)$ for all $v \in V$ and $g(v) < f(v)$ for some $v \in V$. Now we proceed to define a fractional Roman dominating function of $\gamma$ type. A fractional Roman dominating function $f$ is of $\gamma$ type, if it satisfies the following conditions.

- For all $v \in V_0^f$, $f(N[v]) \geq 2$.
- For all $v \in V_{(0,1)}^f$, $f(N[v]) \geq 1$.
- For all $v \in V_{(1,2)}^f$, $f(N[v]) \geq 2$.

Now we can define minimal fractional Roman dominating function in the same way the different types of minimal Roman dominating functions are defined. The above defined minimal fractional Roman dominating function is named as \textit{Type I minimal fractional Roman dominating function}.

Thus $f$ is minimal if either for each $v \in P'_1$, there exists $w \in P_0 \cap N(v)$ such that $|N(w) \cap P'_1| = 1$ or for each $v \in P'_0$, $N(v) \cap P'_1 = \emptyset$. The \textit{minimal fractional Roman dominating number} of a graph is $\gamma^f_R = \inf \{ f(V) | \text{ where } f \text{ is a fractional Roman dominating function of } G \}$. As illustrated in the following
example, $\gamma^f_R$ of a graph may not have a fixed real number. However we can
determine the value as a limit. The above description makes it clear that the
function stated here is a fractional version of Type I minimality. So we can
call it the \textit{Type I fractional Roman domination}. Note that if the function $f$
is Type I fractional Roman dominating function, then $P_0 = \emptyset$, $P'_1 = \emptyset$ and
only $P'_0 \neq \emptyset$.

Second type of fractional minimality is obtained by assuming that for the
function $f$, $V^f_0$ is not empty. Consequently $V^f_{(1,2)}$: This fractional Roman
domination is \textit{Type II fractional minimal Roman domination}.

As the third version of minimality is more involved and less important, fractional version of third type minimality is not discussed here.

The last version of fractional Roman domination is obtained by extending
conditions of Type IV minimality of Roman dominating functions. Let $f$
be an irreducible fractional Roman dominating function. If it satisfies the
condition that for all $v \in V^f_{(0,1)}$, $f(N[v]) = 1$ then the function a \textit{Type IV
minimal Roman dominating function}.

In the following example all the three types of minimal fractional Roman
dominating function are given. The graph $G$ contains an edge $e = uv$ and
there are $r$ leaves $u_1, u_2, ..., u_r$ and $s$ other leaves $v_1, v_2, ..., v_s$ attached to
$u$ and $v$ respectively, where $r, s \geq 3$. The function $f$ defined by $f(u) = f(v) = 2$ and $f(u_i) = f(v_i) = 0$ for all values of $i$. This function is a Type I
minimal fractional Roman dominating function. Since $V^f_0 \neq \emptyset$, $f$ is a Type
II minimal fractional Roman dominating function as well. The function $g$
defined by $g(u) = 2$, $g(v) = g(u_i) = 0$ for $i = 1, 2, ..., r$ and $g(v_i) = v$ for
$i = 1, 2, ..., s$ is a Type IV minimal fractional Roman dominating function,
because $V^g_1 = \{v_1, v_2, ..., v_s\}$ is independent.
6.2 Convex Combination of Fractional Roman Dominating Functions

Convexity of dominating functions and minimal dominating function is a well studied area. It is the focus of a number of papers [6, 7, 8, 32, 33, 41, 34, 38]. Various aspects of convexity of dominating functions is given in the first chapter. In this section we initiate a study of convex combination of fractional Roman dominating functions. As we have defined various type of fractional Roman dominating functions, it is necessary to consider convexity properties of each of them separately. We just initiate an introductory study and leave all the details for another occasion.

Convex combination of the two fractional Roman dominating functions \( f \) and \( g \) of a graph \( G \), is the function \( h_\lambda = \lambda f + (1 - \lambda)g \) where \( 0 < \lambda < 1 \).

Since any convex combination of DFs is again a DF, it follows that the set of all DFs forms a convex set. However it is evident from the Theorem 1.4.2 that the convex combination of two MDFs need not be an MDF always. So the set of all MDFs is not a convex set. Is a convex combination of two fractional Roman dominating functions again a fractional Roman dominating function? To find the answer, consider two fractional Roman dominating functions, say \( f \) and \( g \) of a graph \( G \). Then the following results can be proved.

**Lemma 6.2.1.** Let \( f \) and \( g \) be two fractional Roman dominating functions of \( G \). Then,

\[
V_{(2,1)}^{h_\lambda} \subseteq V_{(2,1)}^f \cup V_{(2,1)}^g
\]

**Proof.** Let \( v \in V_{(2,1)}^{h_\lambda} \). Then \( h_\lambda(v) = [\lambda f + (1 - \lambda)g](v) \). Since \( h_\lambda(v) \geq 1 \), either \( f(v) \geq 1 \) or \( g(v) \geq 1 \). So \( v \in V_{(2,1)}^f \cup V_{(2,1)}^g \). \( \square \)
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But the reverse inclusion need not be true. For example, let \( f(v) = 1.3 \) and \( g(v) = 0.3 \) and \( \lambda = 0.5 \). Then \( v \) is not a member of \( V_{[2,1]}^{h\lambda} \).

**Lemma 6.2.2.** Let \( f \) and \( g \) be two fractional Roman dominating functions of \( G \). Then,

\[
V_f \cap V_g \subseteq V_{[2,1]}^{h\lambda}
\]

**Proof.** Let \( v \in V_f \cap V_g \). Then \( f(v), g(v) \geq 1 \). So \( v \in V_{[2,1]}^{h\lambda} \). \qed

**Theorem 6.2.3.** Convex combination of two fractional Roman dominating functions \( f \) and \( g \) is a fractional Roman dominating function if \( V_f \cup V_g \subseteq V_{[2,1]}^{h\lambda} \).

**Proof.** It is easy to prove that \( V_0^{h\lambda} = V_0^f \cap V_0^g \). From the previous results,

\[
V_{[2,1]}^{h\lambda} = V_0^f \cup V_0^g \rightarrow V_0^{h\lambda} = V_0^f \cap V_0^g.
\]

The above results sheds light on the fact that the convex combination of two fractional Roman dominating functions need not be a fractional Roman dominating function in general. Convexity of the set of all minimal fractional Roman dominating functions of the three types of minimal fractional Roman dominating functions are to be explored further.

6.3 Conclusion and future directions

Study on fractional version of various domination parameters is a very active area of research. In many situations, fractional version allow us to express the values of the parameters as fractions, which give us more insight into the characteristics of the parameter specifically and that of the underlying graph generally. In fractional versions, we choose function values for the vertices or edges from intervals, instead of set of integers. This allows continuity
of functions and gives us opportunity to incorporate limit process in the study. This has been done in the earlier papers \cite{32, 33, 41, 34}. Fractional version of Roman domination is not yet defined anywhere. As such research in this direction should get more attention. Minimality of Roman domination function is defined four different ways. We have initiated study on fractional version of them, excluding the third type. Is it possible to define fractional version of the third type? Which of the three types of fractional Roman dominating functions possess convex combinations? What are the properties of fractional Roman dominating functions? These questions could guide us further in the area.