Chapter 5

Roman Domination in Digraphs

Kamaraj and Jakkamma [22] extended the idea of Roman domination to Digraphs, which is followed by the study of Sheikholeslami and Volkmann [46].

A Roman dominating function (RDF) on a digraph \( D = (V, A) \) is a function \( f : V \to \{0, 1, 2\} \), which satisfies the condition that every vertex \( v \) for which \( f(v) = 0 \) has an in-neighbour \( u \), such that \( f(u) = 2 \). The weight of an RDF \( f \) is the value \( \omega(f) = \sum_{v \in V} f(v) \). The minimum weight of an RDF on \( D \) (denoted by \( \gamma_R(D) \)) is the Roman domination number of a digraph \( D \). A \( \gamma_R(D) \) - function (or \( \gamma_R \) function) is a Roman dominating function of \( D \) with weight \( \gamma_R(D) \). A Roman dominating function \( f : V \to \{0, 1, 2\} \) can be represented by the ordered partition \( (V_0, V_1, V_2) \) (or \( (V^f_0, V^f_1, V^f_2) \) to refer \( f \)) of \( V \), where \( V_i = \{v \in V | f(v) = i\} \). In this representation, its weight is \( \omega(f) = |V_1| + 2|V_2| \). Since \( V^f_1 \cup V^f_2 \) is a dominating set when \( f \) is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we have.

\[
\gamma(D) \leq \gamma_R(D) \leq 2\gamma(D)
\]  

(5.0.1)
Here $\gamma(D)$ is the domination number of a digraph.

The following result is given by Lee [24].

**Proposition 5.0.1.** [24]

Let $D$ be a digraph with order $n$ and minimum in-degree $\delta^-(D) \geq 1$. Then,

$$\gamma(D) \leq \frac{2n}{3}.$$  

Kamaraj et al. [22] proved the following results which are straightforward extensions of the properties of Roman dominating functions in ordinary graphs.

**Proposition 5.0.2.** [22]

Let $f = (V_0, V_1, V_2)$ be any $R(D)$ - function of a digraph $D$. Then

1. $\Delta^+(D[V_1]) \leq 1$.

2. If $w \in V_1$, then $N^-_D(w) \cap V_2 = \emptyset$.

3. If $u \in V_0$, then $N^+_D(u) \cap V_1 \leq 2$.

4. $V_2$ is a $\gamma(D)$ - set of the induced sub-digraph $D[V_0 \cup V_2]$.

5. Let $H = D[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N^-(v) \cap V_2 \neq \emptyset$ has at least two private neighbors relative to $V_2$ in the sub-digraph $H$.

They also gave an upper bound for Roman domination number of digraphs.

**Proposition 5.0.3.** [22]

Let $D$ be a digraph with order $n$. Then

$$\gamma_R(D) \leq n - \Delta^+(D) + 1.$$
According to the characterization given by Sheikholeslami and Volkmann [46], the domination number and the Roman domination number of a digraph are equal if and only if $\Delta^+(D) = 0$. They proved the existence of a lower bound in terms of the order or domination number of the same graph.

**Proposition 5.0.4.** [46] If $D$ is a digraph on $n$ vertices, then

$$\gamma_R(D) \geq \min\{n, \gamma(D) + 1\}$$

Further the characterized the digraphs such that their Roman domination number equals $\gamma(D) + 1$ and $\gamma(D) + 2$.

Another result gives the condition for the Roman domination number to be less than the order of the graph. It indicates that there are many graphs having equal Roman domination number and order.

**Proposition 5.0.5.** [46] Let $D$ be a digraph of order $n$. Then $\gamma_R(D) < n$ if and only if $\Delta^+(D) \geq 2$.

The above result has an immediate corollary.

**Corollary 5.0.6.** [46] If $D$ is a directed path or directed cycle of order $n$, then $\gamma_R(D) = n$.

In [46], the authors have successfully characterized the digraphs having Roman domination number 2, 3, 4 and 5.

### 5.1 More on Roman domination on Digraphs

In the rest of the chapter we discuss some techniques, which would simplify the task of determining Roman domination number of digraphs. The main results are included in the paper [36] communicated.
Observation 5.1.1. If a cycle, which contains directed edges, has \( r \) vertices of in-degree 2, then it has at least \( r \) vertices having out-degree 2. This fact can be used to reduce the upper bound for many orientated cycles.

Proposition 5.1.2. If a directed cycle \( C \) of order \( n \) has \( r \) vertices of in-degree 2, then \( \gamma_R(C) \leq n - r \).

Proof. All vertices having out degree 2 can be included in \( V_2 \) and other vertices which are not in the out neighbourhood of the vertices in \( V_2 \) are included in \( V_1 \) to obtain a Roman dominating set.

This result can be used to reset the upper bound for digraphs which contains a Hamiltonian cycle in the underlying graph and orientation of the edges on the Hamiltonian circuit contains \( r \) vertices of in degree 2. Other arcs present in the graph, except those in the Hamiltonian circuit, can only decrease the Roman domination in \( D \). So the following result is evident.

Proposition 5.1.3. If a directed graph \( D \) of order \( n \) contains a Hamiltonian cycle \( C \), which has \( r \) vertices of in-degree 2 together with its orientation, then \( \gamma_R(C) \leq n - r \).

Proposition 5.1.4. Let \( V_1 \) and \( V_2 \) be a partition of \( V(D) \) into two subsets, such that \( D \) doesn’t contain any arc from \( \langle V_1 \rangle \) to \( \langle V_2 \rangle \). Then

\[
\gamma_R(D) \leq \gamma_R(\langle V_1 \rangle) + \gamma_R(\langle V_2 \rangle)
\]

Proof. Let \( f_1 \) and \( f_2 \) be the Roman dominating functions of the subgraphs \( \langle V_1 \rangle \) and \( \langle V_2 \rangle \) respectively and \( V_{ij} \) where \( i = 1, 2 \) and \( j = 0, 1, 2 \) the partitions of the vertex sets of the subgraphs with respect to the functions. Since all arcs between \( \langle V_1 \rangle \) and \( \langle V_2 \rangle \) are directed from \( \langle V_1 \rangle \) to \( \langle V_2 \rangle \), union of the minimal
Roman dominating sets of subgraphs acts as a Roman dominating set of the whole graph.

A digraph contains a set of vertices $Q = \{u_1, u_2, ..., u_q\}$ such that $N^+(u_i)$ doesn’t contain $v_j$, where $i \neq j$ and $|N^+(u_i)| \geq 2$. But $N^+(u_i) \cap N^+(u_j)$ can be nonempty. Such a digraph is called a $D_Q$ digraph. Note that in a digraph another set (say $Q'$) may exist with different cardinality and same property. Then we can call the graph a $D_{Q'}$ graph. Let $n(Q) = |N(u_1) \cup N(u_2) \cup ... \cup N(u_q)|$. Let $Q^1, Q^2, ..., Q^s$ be all sub-sets $V(D)$, having the above property and let $M = \max_i n(Q^i)$. It is convenient to index the number $M$ with a set $Q$ such that $M_Q = n(Q)$. Next result is a direct consequence of the preceding discussion.

**Proposition 5.1.5.** If the directed graph $D$ is a $D_Q$ graph for some subset $Q$ and $Q'$ is a set such that $M_{Q'} = n(Q')$, then $\gamma_R(D) \leq n - M + |Q'|$.

**Proof.** We can define a Roman dominating function $f$ such that $V_0^f = N^+(Q')$, $V_2^f = Q'$ and $V_1^f = V(D) - N^+[Q']$.

We define an orientation of a connected digraph based on a point. An orientation of a digraph is a *vertex based orientation* if there exists a vertex $v$ such that all edges incident with $v$ are oriented outward from $v$. All remaining edges are oriented conditionally step by step. We call $v$ the vertex at level 0. Also we define, $V_0 = \{v\}$. All vertices in $N(v)$ are vertices at level 2. Correspondingly we define $V_1 = N[v] - V_0$. $V_2 = \cup_{v_i \in V_1} N[v_i] - V_0$ and in general we define $V_r = \cup_{v_i \in V_{r-1}} N[v_i] - (V_0 \cup V_1 \cup ... \cup V_{r-1})$. If there is an edge between $V_{r-1}$ and $V_r$, then it is oriented from $V_{r-1}$ to $V_r$. An edge connecting two vertices in $V_r$ is arbitrarily oriented. All the vertices in $V_r$ are called vertices at $r^{th}$ level.
Lemma 5.1.6. The point orientation of a digraph with respect to a vertex is unique excluding the directed edges at the same level.

Proof. The procedure of orienting the edges starting from the given vertex is exact excluding the edges at a particular level.

Corollary 5.1.7. The point orientation of a tree with respect to a vertex is unique.

Proof. Tree contains no cycle. So there is no edge connecting two vertices at a particular level. So the result follows from Theorem 5.1.6.

An orientation of the graph is given in the Figure 5.1 with respect to the vertex $v_4$. The level sets related to the vertex $v_4$ are $V_0 = \{v_4\}$, $V_1 = \{v_3, v_5, v_6\}$, $V_2 = \{v_1, v_2, v_7, v_8\}$ and $V_3 = \{v_9\}$. We can reverse the orientations of the edges $(v_1, v_2)$, $(v_5, v_6)$ and $(v_8, v_7)$ to obtain seven other different orientations of the graph. Thus total number of orientations is $2^3$.

Figure 5.1

Lemma 5.1.8. If the point orientation of a digraph with respect to a vertex has $s$ edges at the same level, then the total number of different orientations is $2^s$. 
Proof. Each edge at the same level can contribute two different orientations. Thus all together we have $2^s$ orientations.

Given a vertex based orientation of a graph and let $n_i$ be the number of vertices in $V_i$ which are adjacent to at least one vertex in $V_{i+1}$ and $m_i$ be the number of remaining vertices in the $i^{th}$ level set. Then $\gamma_R(D) \leq 2(n_0 + n_1 + \ldots + n_s) + m_0 + m_2 + \ldots + m_s$. Here $s$ is the highest possible even index.

**Theorem 5.1.9.** Let $D$ be a digraph and its underlying undirected graph be a complete bipartite graph. Also let $(V_1, V_2)$ be a bipartition of $G$ and $|V_1| = n_1$ and $|V_2| = n_2$. Then the following are true.

1. If all edges are directed from $V_1$ to $V_2$, then $\gamma_R(D) = n_1 + 1$.

2. If $G$ is oriented starting with a vertex in $V_1$, then $\gamma_R(D) = 4$ if $|V_1| \geq 3$.

   If $|V_1| = 2$, then $\gamma_R(D) = 3$ and if $|V_1| = 1$, then $\gamma_R(D) = 2$.

Proof. Proof of 1: We set $f(v) = 2$ for one vertex in $V_1$ and $f(x) = 1$ for all other vertices in $V_1$. $f(y) = 0$ for all $y \in V_2$. The value of the function $f(V) = n_1 + 1$. There is no function having value less than $n_1 + 1$. So $\gamma_R(D) = n_1 + 1$.

Proof of 2:

Case 1: $|V_1| \geq 3$. Let $v$ be the vertex that we take as the base of orientation. We assign $f(v) = 2$ and $f(x) = 2$ for exactly one vertex $x \in V_2$. All other vertices get zero as function value. This function is a $\gamma_R(D)$ function. So $\gamma_R(D) = 2$.

Case 2: If $|V_1| = 2$, then we set $f(v) = 2$, $f(x) = 1$ where $x$ is the other vertex in $V_1$ and $f(y) = 0$ for all remaining vertices. So $\gamma_R(D) = 3$.

Case 3: This case is same as part 1, when $n_1 = 1$. □
In the following discussion, we deal with deletion of arcs from a directed graph and the change it causes on the value of $\gamma_R(D)$.

Let $f$ be a $\gamma_R(D)$ function of a digraph with the corresponding vertex partition $V^f_0$, $V^f_1$ and $V^f_2$. If an arc is deleted from $V^f_2$ to $V^f_0$, then the Roman domination number of the graph $D'$ may be affected. On the other hand, If we remove any other arc, the Roman domination number of the new graph is same as that of the initial graph. In this discussion graphs include both connected and disconnected graphs because removal of arcs makes the graph disconnected.

**Proposition 5.1.10.** If $D'$ is a new digraph obtained by deleting an arc other than one from $V^f_2$ to $V^f_0$, then $\gamma_R(D) = \gamma_R(D')$.

**Proof.** Let $f$ be a $\gamma_R$ function of $D$. It remains as a Roman dominating function in the new graph $D'$. Suppose that $D'$ has a Roman dominating function $g$ with $g(V) < f(V)$, the function $g$ can become an RDF in $D$ as addition of arc doesn’t undermine the state of a function as an RDF. This contradicts the minimality of the $\gamma_R$ function $f$. \hfill $\Box$

As a direct consequence of the above result, we get a set of graphs having same order and same Roman domination number but varying number of arcs. We denote the set of all digraphs having order $n$ and and Roman domination number $r$ by $G_{n,r}$ and the set of all digraphs on $n$ vertices by $\mathcal{U}_n$. Then $G_{n,r} \subseteq \mathcal{U}_n$.

For a given orientation of a digraph, how many new digraphs can be obtained by deleting arcs and still have $\gamma_R(D) = \gamma_R(D')$? Can we establish any relationship between the number of digraphs and the structure of the digraph? Can we extend research in the same line, deleting vertices? These are few questions that may motivate us for further research. Next we proceed
5.2 Generalization of Roman Domination

A more generalized version of Roman domination is defined here. Its motivation comes directly from the definition of Roman dominating function and indirectly from the facility location problem. A Roman dominating function, associates function values either 0, 1 or 2 to the vertices in a graph. This is equivalent to the assignment of same number of persons at the vertex. If two is assigned to a vertex, it is assumed that the first person can take care of the home vertex and the second person keeps all vertices adjacent to the home vertex. An assignment is a Roman domination assignment if all vertices are either directly guarded by a person at the vertex or indirectly guarded by a second person at a neighbouring vertex, provided that the vertex is empty.

Ordinary domination is an assignment of single person to some vertices in a graph so that each person can guard the vertex where he is assigned and all the adjacent vertices.

We can view it in a very general setting. We define neighbourhoods of a vertex of increasing order. Let \( v \in V \). \( N^0[v] = \{v\}, \ N^1[v] = N[v], \ N^2[v] = \{x \in V | d(v, x) \leq 2\}, \) etc. In general, \( N^r[v] = \{x \in V | d(v, x) \leq r\}. \ N^0(v) = \{v\}, \ N^1(v) = N(v), \ N^r(v) = N^r[v] - N^{r-1}[v]. \) A \([0, 1, 2, \ldots, r]\) - Roman domination (or an \( N - \) Roman domination, where \( N \) stands for \([0, 1, 2, \ldots, r]\)) of a graph \( G \) is the assignment of the integer \( f(v_i) \), where \( 0 \leq f(v_i) \leq r \) at the vertex \( v_i \) of \( G \). Equivalently, we can say that \( f(v_i) \) persons are assigned to the vertex \( v_i \). The first person will guard the home vertex. The second person will guard all vertices in \( N^1(v_i) \). In general the \( i^{th} \) person will guard all vertices in \( N^{i-1}(v) \), where \( i = 1, 2, \ldots, r. \) An
arrangement of a set $r$ or lesser number of people at the vertices of a graph, such that all vertices are guarded by at least one person is a $[0, 1, 2, \ldots, r]$-Roman domination arrangement. A vertex $x$ is $i$-Roman dominated by a vertex $v$, if $x \in N^i(v)$ for some $v$ and $f(v) \geq i + 1$. The $N$-Roman domination number of a graph $G$ is the minimum of the sum of the function values, taken over all $N$-Roman dominations of $G$.

We can allow a person to guard all the vertices in many adjacent levels. For example, suppose in an $N$-Roman domination, first person guards all the vertices in $N^0(v) \cup N^1(v) \cup N^2(v)$ and the second person guards only those vertices in $N^3(v)$, we call the Roman dominating function a $[[0, 1, 2], {3}]$-Roman dominating function. In this sense, the ordinary dominating function is a $[[0, 1]]$-Roman dominating function. The Roman dominating function is a $[[0], {1}]$-Roman dominating function.

Eccentricity, center and radius of a graph might have an important role in the determination of $N$ Roman domination number. We discuss some examples of $N$-Roman dominating function of very simple family of graphs.

**Proposition 5.2.1.** Let $G$ contains a vertex $v$ such that, $d(v) = n$, where $n$ is the number of vertices in $G$. Then $\gamma_{NR}(G) \leq 2$.

**Proof.** Since $G$ contains a vertex $v$ such that, $d(v) = n$, where $n$ is the number of vertices. If a person assigned to guard only his home vertex, A second person at the same vertex can take care of all other vertices. So $\gamma_{NR}(G) \leq 2$. Otherwise a single person at $v$ can guard all vertices in $G$. □

The following corollaries are the immediate consequence of the above result.

**Corollary 5.2.2.** Let $G = K_n$. Then $\gamma_{NR}(G) \leq 2$.

**Corollary 5.2.3.** Let $G = K_{1,n}$, where $n \geq 1$. Then $\gamma_{NR}(G) \leq 2$. 
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Proposition 5.2.4. Let $G$ be a graph having radius $r$ and the center contains $s$ vertices. Then $\gamma_{NR}(G) \leq s(r + 1)$.

Proof. Since radius of $G$ is $r$, $r+1$ people must be present to Roman dominate all the vertices which are reachable step by step from each vertex in the center. Also if the center contains $s$ vertices, then the total number of guards required is at most $s(r + 1)$. Hence we get the result.

5.3 Conclusion and future directions

Roman domination in digraphs is not well studied by graph theorists. In this chapter first we found $\gamma_R$ of some classes of graphs. Then we define an orientation of the edges of a graph that is based on a specific starting. It is followed by many results giving $\gamma_R(D)$ for some classes of graphs. Major difficulty that we face while we study properties and parameters of digraphs is the variety generated by its orientations. In many cases a general study is either restricted or limited. We can overcome this problem to some extent, if we specify some orientation of the edges and try to derive results based on this. The second half of the chapter contains a general definition of Roman domination and some related results. These ideas open a vast area of future research.