Chapter 3

Safe Eternal Domination

3.1 Introduction

In this chapter we define a new version of eternal domination. We introduce this idea in the form of a game between two players. One player is the defender and the other player is the attacker. The defender will select a dominating set $D_i$ in the $i^{th}$ step of the play, such that there exists at least one vertex $v$ satisfying $N[v] \cap D_i = 1$. Let $A_{D_i} = \{v \in V | N[v] \cap D_i = 1\}$. In each step an attacking player selects a vertex $r_i \in A_{D_i}$. Then the first player modifies his set $D_i$ to obtain $D_{i+1} = (D_i - v_i) \cup r_i$, where $v_i \in N[r_i]$. While playing the game, if the set $A_{D_i}$ becomes empty, then the game stops. We ask the question whether for a given graph $G$, the game terminates after some time or continue indefinitely? We try to find answer to these questions in the following section. The vertices in the set $A_{D_i}$ are called \textit{safe vertices}. If a vertex $x \in A_{D_i}$ is in $D_i$ also, then $x$ is a \textit{safe dominating vertex}. The dominating set $D_i$ is called a \textit{safe 1-secure set} if $A_{D_i} \neq \emptyset$.

If the second player selects a safe dominating vertex, then we get $D_i = D_{i+1}$ and we call it a \textit{trivial case}. An initial dominating set $D_0$ is a \textit{failed
dominating set, if in the \(i^{th}\) step there exists a vertex in the graph, which is not adjacent to any element in \(D_i\).

In the next section we prove that every graph has at least one safe 1-secure set and give a characterization of the graphs in which all vertices excluding one vertex is a safe vertex. We also classify the whole family of graphs into three classes based on the existence of safe 1-secure set.

In the third section, we define a new kind of directed graphs which represent the transformation from one safe 1-secure set to another safe 1-secure set of a given graph. Most of the results that appear in this chapter are included in [35].

### 3.2 Safe eternal 1-security in graphs

We recall the definition of the above concepts and subsequently define safe eternal 1-secure set. A vertex \(v \in V\) in a graph \(G = (V,E)\) is called safe with respect to a 1-secure set (or a dominating set) \(D \subseteq V\) if and only if \(|N(v) \cap D| = 1\), that is \(v\) is adjacent to exactly one vertex in \(D\). A 1-secure set \(D\) is a safe 1-secure set if and only if there exists a vertex \(v \in V\) such that \(v\) is safe with respect to \(D\). A safe 1-secure set is a safe eternal 1-secure set if for any finite sequence of safe vertices \(r_1, r_2, \ldots, r_k\), there exists a sequence of vertices \(v_1, v_2, \ldots, v_k\) and a sequence \(D = D_0, D_1, D_2, \ldots, D_k\) of safe 1-secure sets of \(G\), such that for every \(i, 1 \leq i \leq k\), \(D_i = (D_{i-1} - \{v_i\}) \cup \{r_i\}\), where \(v_i \in D_{i-1}, r_i \in N(v_i)\) and \(r_i\) is safe with respect to the set, \(D_i\).

**Observation 3.2.1.** If for any finite sequence of safe vertices \(r_1, r_2, \ldots, r_k\), there exists a sequence of vertices \(v_1, v_2, \ldots, v_k\) and a sequence \(D = D_0, D_1, D_2, \ldots, D_k\) of safe 1-secure sets of \(G\), which satisfy the conditions given above, then the same is true for any infinite sequence of safe vertices \(r_1, r_2, \ldots\), and
We proceed to prove that every graph has at least one safe 1-secure set.

**Theorem 3.2.2.** Every graph has at least one safe 1-secure set.

**Proof.** Let \( x \in V \) be any leaf in a graph \( G \). Then the set \( D = V - \{ x \} \) is a safe 1-secure set. If the graph \( G \) does not have any leaf, then we have the following cases.

Case 1: If there is a vertex \( v \in V \) such that all vertices in \( N[v] - \{ v \} \) are adjacent to a vertex in \( V - N[v] \). Make \( D \) using exactly one vertex, which is in \( N[v] - \{ v \} \) and all the vertices in \( V - N[v] \).

Case 2: There is no vertex \( v \in V \) such that all vertices in \( N[v] - \{ v \} \) are adjacent to a vertex in \( V - N[v] \). Since there is no leaf in \( G \), for any \( x \in V \), there exist some \( y \in N(x) \) such that \( N[y] \subseteq N[x] \). If there exists only one such vertex in \( N(x) \) for some \( x \), say \( w \), then the set \( D = (V - N[w]) \cup \{ x \} \) is a safe 1-secure set.

Next assume that \( G \) is a graph such that for any \( v \in V \), there exists two vertices \( x_{v1}, x_{v2} \in N(v) \) and \( N[x_{v1}], N[x_{v2}] \subseteq N[v] \). If \( x_{v1} \) and \( x_{v2} \) are adjacent, then both \( (V - N[x_{v1}]) \cup \{ x_{v1} \} \) and \( (V - N[x_{v2}]) \cup \{ x_{v2} \} \) are safe 1-secure sets.

Finally assume that \( G \) is a graph such that for any \( v \in V \), there exists two vertices \( x_{v1}, x_{v2} \in N(v) \), \( N[x_{v1}], N[x_{v2}] \subseteq N[v] \) and \( x_{v1} \) and \( x_{v2} \) are not adjacent. We claim that this case is impossible. Otherwise the inclusion described above is proper and this implies the existence of a proper chain of subsets of \( V \); \( V \supset N[x] \supset N[y] \supset N[z] \supset \ldots \), where \( y \in N(x), z \in N(y) \) etc. This contradicts the fact that the graph is finite. \( \square \)

Next theorem is a characterization of the graphs such that all safe 1-secure sets are safe eternal 1-secure sets and all vertices in \( V - D \) are safe
with respect to $D$, for any safe eternal $1$-secure set $D$.

**Theorem 3.2.3.** Let $D_0$ be an eternal $1$-secure set of a graph $G$ and the subsequent safe $1$-secure sets are $D_1, D_2, \ldots$ All vertices in $V - D_i$ are safe with respect to $D_i$, where $i = 0, 1, 2, \ldots$ if and only if $G$ is a disjoint union of complete graphs.

**Proof.** Let $G$ be a disjoint union of complete graphs. We claim that any safe eternal $1$-secure set of $G$ contains exactly one vertex from each component. Suppose, two vertices in a component are present in a safe eternal $1$-secure set. None of the vertices in that component is a safe vertex. So it is clear that any safe eternal $1$-secure set of $G$ contains exactly one vertex from each component and all other vertices in each component are safe vertices.

To prove the converse, first we shall prove that if $u, v$ and $w$ are any three vertices in the graph such that the edges $(u, v)$ and $(v, w)$ are present in it, then $(u, w)$ is also present. Suppose that $(u, w)$ is not present in $G$. Now we have to consider the following two cases.

Case 1: The vertex $v$ is a member of the safe eternal $1$-secure set $D_0$. Then $u$ and $w$ cannot be in $D_0$. Suppose that the vertex $u$ is selected by the second player. The first player modifies $D_0$ by replacing $v$ by $u$ to get the eternal $1$-secure set $D_1$. If the second player selects the vertex $w$ subsequently, then $D_2 = (D_1 - \{u\}) \cup \{w\}$ must be an eternal $1$-secure set. Otherwise $|D_0 \cap N[w]| \geq 2$, which is a contradiction.

Case 2: The vertex $v \notin D_0$. Then there exists exactly one vertex $x \in N(v)$, such that $x \in D_0$. The possibility that $x$ is either $u$ or $w$ is not ruled out. All vertices in $N(v)$ must be adjacent to $x$. If $x = u$ or $x = w$, we are done. Otherwise suppose that the second player selects the vertex $v$ and consequently the first player selects a new dominating set which contains $v$. Since $u$ and $w$ are not in the new dominating set, by the steps of case 1, we
can show that $u$ and $w$ are adjacent.

Thus whenever there are three vertices $u, v$ and $w$, such that the edges $(u, v)$ and $(v, w)$ are present in the graph, then the edge $(u, w)$ is also present in $G$. Thus if $G$ is connected, then it is a complete graph. Otherwise it is a disjoint union of complete graphs.

Now we define three different types of graphs, which form the basis of the classification of the family of all graphs. A graph $G$ is an $\alpha_1$ - graph if it has no safe eternal 1 - secure set. A graph is a $\beta_1$ - graph if it has at least one safe eternal 1 - secure set. A graph is a $\gamma_1$ - graph if every safe 1 - secure set of the graph is a safe eternal 1 - secure set. The following is an example of $\alpha_1$ - graph. Examples of other two types of graphs are given in the results, which follow. Let $G$ be a path having length two in which the vertex $v$ is adjacent to $u$ and $w$. The set $\{u, v, w\}$ is not a safe 1 - secure set. The set $D = \{u, v\}$ is a safe 1 - secure set. But after defending an attack at $w$, the new set obtained, that is $\{u, w\}$ is not a safe 1 - secure set. The set $D = \{v, w\}$ gives a similar result. Next we have to consider all one vertex cases. The sets $\{u\}$ and $\{v\}$ are not safe 1 - secure sets. Even though $D = \{v\}$ is a safe 1 - secure set, after one attack, the resulting set is not a safe 1 - secure set.

Theorem 3.2.4. Complete multipartite graphs $K_{r_1, r_2, \ldots, r_n}$ where $r_i \geq 2$ for some $i$ are in $\alpha_1$ category.

Proof. Let $G = K_{r_1, r_2, \ldots, r_n}$ be a complete multipartite graph whose vertex set is partitioned in to $V_1, V_2, \ldots, V_n$. Let $D_0$ be a safe 1 - secure set chosen by the first player and let $u \in V(G)$ be a safe vertex. Then $|N(u) \cap D_0| = 1$. We can relabel the partition so that $u \in V_1$ and $v \in V_2$. Since $u$ is a safe vertex, $D_0 \cap (V - V_1 - \{v\}) = \phi$. 
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Now we have to consider two cases. Case 1: \(|V_1| > 1\). We claim that, \(x \in D_0\) for all \(x \in V_1 - \{u\}\). Suppose not. Let \(x \in V_1\) and \(x \notin D_0\). The second player chooses the vertex \(u\) and subsequently the first player selects the set \(D_1 = D_0 - \{v\} \cup \{u\}\). But this set is not a dominating set as the vertex \(x\) is not defended by any guard in \(D_1\). Hence the claim is proved. Next we proceed by assuming that \(|D_0 \cap (V_1 - \{u\})| = 1\). After \(u\) being chosen by the second player, \(D_1 \cap V_1 = V_1\) and \(D_1 \cap (V - V_1) = \phi\). There are only safe dominating vertices left in the graph. So the graph is in the \(\alpha_1\) category.

Case 2: \(|V_1| = 1\). We claim that \(|V_i| = 1\) for all \(i\). Suppose \(|V_i| > 1\) for some \(i > 1\). After the vertex \(u\) being selected by the second player, \(D_1 \cap V_i = \phi\) for all \(i > 1\). Subsequently if the second player chooses a vertex \(x \in V_i\), from \(V_i\) for which \(|V_i| > 1\). Then \(D_2 = D_1 - \{u\} \cup \{x\}\) is not a dominating set. So the graph is in \(\alpha_1\) category.

For a complete graph any dominating set containing exactly one vertex is a safe eternal 1-secure set and no other dominating set is a safe 1-secure set. So all complete graphs are in \(\gamma_1\) category.

Next we proceed to find a class of graphs which fall in \(\beta_1\) - category. A member of this class is constructed using the graphs \(G_1, G_2, \ldots, G_r\) such that \(G_i = K_n\) where \(n \geq 3\). Let \(\mathcal{G}\) be the class of all graphs obtained by using \(G_i\)s, either fusing some vertices in \(V(G_i)\) and \(V(G_j)\) or adding edges between a vertex \(V(G_i)\) and a vertex in \(V(G_j)\) or applying both operations, such that there exists at least one vertex \(v_i \in V(G_i)\) satisfying \(N[v_i] \subseteq V(G_i)\), for each \(i\).

**Theorem 3.2.5.** If \(G \in \mathcal{G}\), then \(G\) is a \(\beta_1\) - graph. Also, \(\sigma_{s1}(G) = r\) where \(r\) is the number of complete graphs used to construct \(G\).

**Proof.** First player can choose \(D_0 = \{v_1, v_2, \ldots, v_r\}\), where \(v_i \in V(G_j)\) if \(i = j\)
and \( v_i \notin V(G_i) \) if \( i \neq j \). Second player can select any vertex which is not in \( D_0 \), say \( r \in G_i \). Then \( D_1 = (D_0 - \{v_i\}) \cup \{r\} \). Renaming \( r \) by \( v_i \), we can repeat the game any number of times. Next suppose that there exists a safe eternal dominating set with \( |D_0| = r - 1 \). Let \( D_0 = \{v_1, v_2, ..., v_{r-1}\} \). Then at least one vertex is common to \( G_i \) and \( G_j \), where \( i \neq j \).

We can place one guard exactly at one vertex in each \( G_i \). This guard can move to defend any attack with in \( G_i \). The presence of at least one vertex in every complete graph \( v_i \in V(G_i) \) with the property \( N[v_i] \subseteq V(G_i) \) ensures minimum one safe vertex in each \( G_i \). This arrangement is the smallest possible. So we get \( \sigma_{s1}(G) = r \).

Another class of graphs \( \mathcal{G} \) is obtained by joining to some vertices in \( G \in \mathcal{T} \), the vertex having degree \( n \) in \( K_{1,n} \), where \( n \geq 2 \) using an edge.

**Theorem 3.2.6.** Let \( G \in \mathcal{G} \) be constructed using \( r \) complete graphs and the stars \( K_{1,n_1}, K_{1,n_2}, ..., K_{1,n_t} \). Then \( G \) is a \( \beta_1 \)-graph and \( \sigma_{s1}(G) = r + \sum_i n_i \).

**Proof.** Arrangement of guards in the clique subgraphs in \( G \) is done as in the pervious proof. In addition, we have to arrange guards at leaves in \( K_{1,n_i} \), for all \( i \). This arrangement is a safe eternal 1 security set. Thus the graph is in \( \beta_1 \) category. The number of guards required is \( r + \sum_i n_i \). Next suppose that the graph has an arrangement of guards in its safe eternal 1 security set which contains less vertices than \( r + \sum_i n_i \). If the guard absent is in a clique, then two connected cliques \( C_1 \) and \( C_2 \) must be guarded by a guard. Let \( v_1 \) and \( v_2 \) be the vertices in \( C_1 \) and \( C_2 \) respectively, where no other clique is joined. An attack either at \( v_1 \) or \( v_2 \) results into a non secure set. If a guard is absent in a star subgraph (say in \( K_{1,n_i} \)) of \( G \), then one guard must be present at the vertex with degree \( n_i \) and two leaves must be vacant. If the enemy attacks one of the leaves, the guard at the center vertex must defend
it. This makes the second vacant leaf un-dominated. So at least $r + \sum_i n_i$ guards are required.

We have proved that one can easily find a safe one secure set in any graph. But for a safe one secure set to be a safe eternal one secure set is extremely difficult. This difficulty naturally reduces the possibility of existence of graphs having the property that all safe one secure sets are safe eternal one secure sets. So we conjecture that the only class of graphs which is in the $\gamma_1$ category is $K_n$. Next we proceed to describe the three categories of graphs in an elegant way. The following figure well explains the interrelationships of the three classes of graphs.

![Diagram](image)

**Figure 3.1**

We denote the set of all subsets of $V$, excluding the empty set, by $\mathcal{P}(G)$. The set of all safe one secure sets of $G$ is denoted by $\mathcal{S}(G)$. Clearly $\mathcal{S}(G) \subseteq \mathcal{P}(G)$. We can construct a directed graph with vertices equivalent to the elements in $\mathcal{P}(G)$ and a directed edge from $S \in \mathcal{S}(G)$ to $P \in \mathcal{P}(G)$ if and only if it is possible to get $P$ as the new arrangement of guards when a safe vertex in the set $S$ is attacked. The graph thus obtained is named the *safe graph*. We denote the safe graph of a graph $G$ by $G_s$. In the following section
we study the general nature of the safe graph of a graph $G$.

## 3.3 Safe graph of a graph

The following properties of safe graphs are very clear.

- There is no directed edge from a vertex in $\mathcal{P}(G) - \mathcal{S}(G)$ to a vertex in $\mathcal{S}(G)$.
- It is possible to exist directed edges in both directions between two vertices.
- For a given graph $G$, $G_s$ is unique.
- The graph $G_s$ is not in general connected because some graphs have safe secure sets of different cardinality. Since rearrangements of guards do not alter their number, there cannot be a directed path connection between two vertices representing safe secure sets of different cardinality. So, if a graph has safe secure sets of different cardinality, then $G_s$ is disconnected.

**Lemma 3.3.1.** If there is a leaf $S$ in $G_s$ with in-degree, then $S \in \mathcal{P}(G) - \mathcal{S}(G)$.

*Proof.* Since $S$ a leaf in $G_s$ with in-degree, it is clear that $S$ is not a safe secure set. Hence the lemma. \qed

**Lemma 3.3.2.** If $G$ has two safe secure sets of different cardinality, then $G_s$ is disconnected.

*Proof.* In $G_s$, if $S_1$ and $S_2$ represent two safe secure sets of different cardinality, then it is not possible to arrange guards in $S_2$ starting from $S_1$ or
vice versa, defending a sequence of attacks at the vertices in $G$. Hence the result. 

**Theorem 3.3.3.** A graph $G$ is an $\alpha_1$ graph if and only if there is no vertex $s$ in $G_s$ such that the induced subgraph of all the vertices reachable from $s$ has no vertex, which is an element of $\Psi(G) - \mathcal{S}(G)$.

*Proof.* Suppose that the graph $G_s$ has a vertex $s$ such that the induced subgraph of all the vertices reachable from $s$ has no vertex, which is an element of $\Psi(G) - \mathcal{S}(G)$. Consider all the vertices which are reachable through directed paths from the vertex $s$. Name the set of all vertices on such paths by $R$. By our assumption, $R \cap \Psi(G) - \mathcal{S}(G) = \emptyset$. Since $\mathcal{S}(G)$ is a finite set, vertices in each directed path must repeat an infinite number of times. Thus some vertices in $R$ make directed cycles in $G_s$. If two directed cycles have some vertices in common, then take the union of the cycles. Consider the biggest such subgraph $C$ of $G_s$, which is the join of directed cycles. Let $S$ be a safe one secure set corresponding to a vertex $v$ in $C$. We can defend any sequence of attacks at the vertices in $G$ arranging guards at the vertices in $S$. So $S$ is a safe eternal one secure set and $G$ is not an $\alpha_1$ graph.

Conversely, if $G$ is not an $\alpha_1$ graph, then it is a $\beta_1$ graph. By definition there exists a safe eternal one secure set $S$. Then reversing the arguments in the first part of the proof, we get a vertex $s$ which corresponds to $S$, such that such that the induced subgraph of all the vertices reachable from $s$ has no vertex, which is an element of $\Psi(G) - \mathcal{S}(G)$. 

**Theorem 3.3.4.** A graph $G$ is a $\beta_1$ graph if and only if there is a connected induced subgraph of $G_s$, which has all vertices in $\mathcal{S}(G)$.

*Proof.* Suppose that the graph $G_s$ has a safe 1 secure set. vertex $s$ such that
the induced subgraph of all the vertices reachable from \( s \) has no vertex, which is an element of \( \mathcal{P}(G) - \mathcal{S}(G) \). Consider all the vertices which are reachable through directed paths from the vertex \( s \). Name the set of all vertices on such paths by \( R \). By our assumption, \( R \cap \mathcal{P}(G) - \mathcal{S}(G) = \emptyset \). Since \( \mathcal{S}(G) \) is a finite set, vertices in each directed path must repeat infinite times. Thus some vertices in \( R \) make directed cycles in \( G_s \). If two directed cycles have some vertices in common, then take the union of the cycles. Consider the biggest such subgraph \( C \) of \( G_s \), which is the join of directed cycles. Let \( S \) be a safe one secure set corresponding to a vertex \( v \) in \( C \). We can defend any sequence of attacks at the vertices in \( G \) arranging guards at the vertices in \( S \). So \( S \) is a safe eternal one secure set and hence \( G \) is not an \( \alpha_1 \) graph.

Conversely, if \( G \) is not an \( \alpha_1 \) graph, then it is a \( \beta_1 \) graph. By definition, there exists a safe eternal one secure set \( S \). Then reversing the arguments in the first part of the proof, we get a vertex \( s \), which corresponds to \( S \), such that the induced subgraph of all the vertices reachable from \( s \) has no vertex, which is an element of \( \mathcal{P}(G) - \mathcal{S}(G) \).

**Theorem 3.3.5.** A graph \( G \) is a \( \gamma_1 \) graph if and only if there is no edge directed from a vertex in \( \mathcal{S}(G) \) to a vertex in \( \mathcal{P}(G) - \mathcal{S}(G) \).

**Proof.** Suppose that the graph \( G_s \) has an edge directed from a vertex \( S \in \mathcal{S}(G) \) to a vertex \( P \in \mathcal{P}(G) - \mathcal{S}(G) \). Clearly \( S \) is not a safe eternal 1 secure set. Hence \( G \) is not a \( \gamma_1 \) graph.

Conversely, suppose that \( G \) is not a \( \gamma_1 \) graph. Then there exists at least one safe 1 secure set which is not safe eternal. So after defending a finite sequence of attacks we get an arrangement of guards, which does not correspond to a safe 1 secure set.
3.4 Conclusion and future directions

In this paper the idea of safe eternal 1-secure security is introduced as a game between two players. It is proved that every graph has a safe 1-secure set. A characterization of the graphs such that all safe 1-secure sets are safe eternal 1-secure sets is given. Actual values of the eternal security number is determined for only a few classes of graphs. Many classes of graphs are yet to be explored. Three different types of graphs are defined and some classes of graphs are proved to belong to the classes. The three type of graphs are compared and the relation among the classes is given in the form of a Venn diagram. A directed graph is defined, named safe graph, on the set of all subsets of $V$, excluding the empty set and some of its properties are discussed. This study can be extended to find more families of graphs which belong to each type of the three classes. Another interesting area of study is that of safe graphs. Safe graph of the graphs such as Complete graphs, Bipartite graphs, Paths, Cycles, $n$-dimensional cubes etc. can be found.