Chapter 4

Generalization of safe 1 - secure sets

The number of guards required to defend an attack is much less in the case of eternal $m$ - security compared to eternal 1 - security in graphs. Same is true for safe eternal version of domination. This motivates to define safe eternal $m$ - security in graphs. In this chapter we initiate a study on safe eternal $m$ - security in graphs and give bounds for safe eternal $m$ - security number. We also find the number for some classes of graphs and give characterizations in certain cases. The safe eternal $m$ - secure set is a safe eternal secure set such that more than one guard can move in response to an attack. The number of vertices in a smallest safe eternal $m$ - secure set of a graph $G$ is the safe eternal $m$ - security number and denoted by $\sigma_{sm}(G)$. It is a simple observation that, $\sigma_{sm}(K_n) = 1$. Next result is an immediate consequence of the observation that every safe eternal $m$ - secure set is an eternal $m$ - secure set.
Theorem 4.0.1. For any graph $G$,

$$\sigma_m(G) \leq \sigma_{sm}(G).$$

Replacing the safe eternal $1$ - secure sets in the definition of $\alpha_1$ - graph and $\beta_1$ - graph, by safe eternal $m$ - secure sets we define their counterparts $\alpha_m$ - graph and $\beta_m$ - graph in safe eternal $m$ - domination.

A graph $G$ is an $\alpha_m$ - graph if it has no safe eternal $m$ - secure set. Every safe eternal $1$ - secure set of a graph is a safe eternal $m$ - secure set of the graph. If a graph has no safe eternal $m$ - secure set, then it has no safe eternal $1$ - secure set. A graph is a $\beta_m$ - graph if it has at least one safe eternal $m$ - secure set. Let $S$ be a safe eternal $1$ - secure set of the graph $G$. This set is a safe eternal $m$ - secure set with $m = 1$.

The above ideas together give the following result.

Theorem 4.0.2. 1. All $\alpha_m$ - graphs are $\alpha_1$ - graphs.

2. All $\beta_1$ - graphs are $\beta_m$ - graphs.

Proof. (1) : Let $G$ be an $\alpha_m$ - graph. It has no safe eternal $m$ - secure set. So $G$ cannot have any safe eternal $1$ - secure set. Thus $G$ is an $\alpha_1$ - graph.

(2) : Let $G$ be a $\beta_1$ - graph. The graph has at least one safe eternal $1$ - secure set. The same set is a safe eternal $m$ - secure set (with $m = 1$). So $G$ is a $\beta_m$ - graph. \qed

Next we proceed to find $\sigma_{sm}(G)$ for some familiar classes of graphs.

Theorem 4.0.3. 1. $\sigma_{sm}(P_n) = \theta_s(P_n) = \lceil \frac{n}{2} \rceil$.

2. $\sigma_{sm}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$. 
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Proof. 1. Let the vertices of the path be labeled $v_1, v_2, \ldots, v_n$ from one end. Place the guards at $v_2, v_4, \ldots, v_{(n-1)}$ if $n$ is odd and $v_2, v_4, \ldots, v_{(n-2)}$ and at $v_{(n-1)}$ if $n$ is even. This arrangement gives a safe eternal $m$-secure set and it requires $\lceil \frac{n}{2} \rceil$ guards.

2. If $n = 3$, one guard standing at any one vertex is a safe eternal $m$-security set. Next when $n > 3$, guards can be arranged at the minimum dominating set. This arrangement is an eternal $m$-secure set. Always we can have an arrangement in which the guards stand at $v_1$ and $v_4$ leaving $v_2$ and $v_3$ vacant. Hence any minimum eternal $m$-secure set is a safe eternal $m$-secure set. 

\[ \square \]

Theorem 4.0.4. If the graph $G$ is a complete multipartite graph, then $\sigma_{sm}(G) = 2$.

Proof. Let $G = K_{r_1,r_2,\ldots,r_n}$ with partition of vertex set as $V_1, V_2, \ldots, V_n$. Place a guard at $u \in V_i$ and another guard at $v \in V_j$. All vertices in $V_i \cup V_j$ are safe vertices. If the enemy attacks at $x \in V_i$ where $x \neq u$, the guard standing at $v$ moves to defend it. Simultaneously, the guard at $u$ can move to any other subset $V_k$. Clearly this is the minimum possible case, because single guard cannot keep all vertices. 

\[ \square \]

Let $\Gamma$ be a group and $C \subseteq \Gamma - \{e\}$, where $e$ is the identity element. The Cayley graph $G = (\Gamma, C)$ is a graph with the vertex set represented by the elements of $\Gamma$ and the elements $f$ and $g$ of the vertex set are connected if and only if there exists $h \in C$, such that $f = gh$. If we take $\Gamma = \mathbb{Z}_n$, we get a special subclass of cayley graphs called circulant graphs. Wayne Goddard et al. [13] have proved that if $G$ is a Cayley graph, then $\sigma_m(G) = \gamma(G)$. The
following theorem can be proved in a similar way for circulant graphs and it is expected that it is true for the general class of Cayley graphs.

**Theorem 4.0.5.** If the circulant graph $G$ has an initial safe $m$-secure set $S$, then it is a $\beta_m$-graph and $\sigma_{sm}(G) = |S|$.

**Proof.** Let $Z_n$ be the underlying group and $C$ be the subset of $Z_n$ used to define the Circulant graph $G$. Suppose that $S$ is an initial safe $m$-secure set. If there is an attack at the safe vertex $u$, then a guard from $v \in S \cap N(u)$ defends it. It is possible if and only if there exists $d \in C$ such that $u = dv$. Then $dS$ is another safe eternal $m$-secure set and $|S| = |dS|$. □

The set $D \subseteq V$ is a total dominating set of a graph $G$, if and only if every vertex in the graph is adjacent to at least one vertex in $D$. Let $D$ be a total dominating set of the graph $G$ and $\langle D \rangle$ be the induced subgraph of $D$. We use $C(D)$ to denote the number of components of $\langle D \rangle$. Also let $C'(D)$ denote the total number of components of $\langle D \rangle$ such that there exists at least one vertex say $x \notin D$ and $|N(x) \cap D| = 1$. We shall call the vertex $x$, a safe leaf. Clearly $C'(D) \leq C(D)$ for each $D$.

**Example 4.0.6.** Consider the graph given below. The set $D = \{v_3, v_4, v_7, v_8\}$ is a total dominating set. The induced subgraph $\langle D \rangle$ contains two paths $(v_3, v_4)$ and $(v_7, v_8)$. But $\langle D' \rangle$ contains only one path $(v_3, v_4)$. So $|D'| \leq |D|$. 
Lemma 4.0.7. If the graph $G$ has a total dominating set $D$ and there exist two vertices $u_1, u_2 \in (V(G) - D)$ such that $|N[u_i] \cap D| = 1$ for $i = 1$ and 2, then $\sigma_{sm}(G) \leq |D| + C'(D) \leq |D| + C(D)$.

Proof. Consider a total dominating set $D$ having the given properties. Let the guards be placed at each of the vertices in $D$. Consider a component $K$ of $\langle D \rangle$. If none of the vertices adjacent to $V(K)$ is a safe leaf, the enemy will not attack these vertices. So suppose that some vertices adjacent to $V(K)$ are safe leaves. Then provide one additional guard at a vertex $y \notin V(K)$, but adjacent to a vertex in $V(K)$. In response to an attack at a safe leaf a guard standing near the attacked vertex can move. Simultaneously all remaining vertices in the component can rearrange their places to accommodate the guard standing at $y$. This can be successfully repeated to defend any number of attacks at the safe leaves of the component. Thus for the component $K$ having safe leaves, exactly $|V(K)| + 1$ soldiers are required. Next take the whole components together. The arrangements of guards at the vertices in $D$ is a safe $m$ - secure set. Hence we get the result. \hfill $\Box$

Now we define $C'(G) = \min\{|D| + |C'(D)|\}$ : such that $D$ is a total
dominating set of $G$ having at least two safe leaves} and $C(G) = \min\{|D| + |C(D)| : \text{such that } D \text{ is a total dominating set of } G \text{ having at least two safe leaves}|$.

**Lemma 4.0.8.** For the graph $G$, $C'(G) \leq C(G)$.

**Proof.** Consider the total dominating set $D$ such that $(|D| + |C(D)|) = C(G)$. Then $C'(G) \leq (|D| + |C'(D)|) \leq (|D| + |C(D)|) = C(G)$.

**Theorem 4.0.9.** For any graph $G$, $\sigma_{sm}(G) \leq C'(G) \leq C(G)$.

**Proof.** From the given graph, select a subset $D$ of $V(G)$ such that there exist two vertices $x_1, x_2 \notin D$ and $|N(x_i) \cap D| = 1$ for $i = 1$ and $2$. Then following the steps of Lemma 4.0.7, we can show that, $D$ is a safe eternal $m$-secure set. Using all the possible total dominating sets and the Lemma 4.0.8, we get the result.

### 4.1 Conclusion and future directions

In this chapter, we have seen a general version of safe eternal 1 - secure sets in graphs, named safe eternal $m$ - secure set. Minimum cardinality of all existing safe eternal $m$ - secure sets of $G$ is also defined ($\sigma_{sm}(G)$) and this number is found for paths, cycles and multi-partite graphs etc. Safe eternal $m$ - security number of many other classes of graphs has to be determined. Spider graphs, Petersen family of graphs, $n$ - dimensional cubes etc. are some examples. Whether all trees have a safe eternal $m$ - secure set is unknown. If it exists, determining $\sigma_{sm}(T)$, where $T$ is a tree is another possible area.