CHAPTER – I

EXCEPTIONAL VALUES AND UNIQUENESS THEOREMS
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Gopalakrishna H. S. and Subhas S. Bhoosnurmath [5] proved the following theorem on exceptional values.

THEOREM [A]: Let \( f(z) \) be a meromorphic function of order \( p, 0 \leq p \leq \infty \) if there exist distinct elements \( a_1, a_2, \ldots, a_p ; b_1, b_2, \ldots, b_q ; c_1, c_2, \ldots, c_s \) in \( \mathbb{C} \) such that \( a_1, a_2, \ldots, a_p \) are evB for \( f(z) \), for distinct zeros of order \( \leq k \), \( b_1, b_2, \ldots, b_q \) are evB for \( f(z) \), for distinct zeros of order \( \leq l \) and \( c_1, c_2, \ldots, c_s \) are evB for \( f(z) \), for distinct zeros of order \( \leq m \), where \( k, l \) and \( m \) are positive integers. Then

\[
\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2.
\]

CONSEQUENCES OF THEOREM A:

1) There exist atmost two elements in \( \mathbb{C} \) which are evB for \( f(z) \), for distinct zeros of order \( \leq k \), where \( k \geq 3 \).

2) There exist atmost four elements in \( \mathbb{C} \) which are evB for \( f(z) \), for simple zeros.

3) There exist atmost three elements in \( \mathbb{C} \) which are evB for \( f(z) \), for simple and double zeros.
4) If there exist at most three elements in $\overline{C}$ which are evB for $f(z)$, for simple and double zeros, then there is no other element of $\overline{C}$ which is an evB for $f$ for simple zeros.

5) If there exists one element in $\overline{C}$ which is evB for $f(z)$, for simple and double zeros, then there exists at most two elements in $\overline{C}$ which are evB for $f$ for simple zeros.

Gopalakrishna H. S. and Subhas. S. Bhoosnurmath [6] also proved the following theorem.

**THEOREM [B]**: Let $f_1(z)$ and $f_2(z)$ be two distinct meromorphic functions with $N(r, a, f_i) = o(T(r, f_1) + T(r, f_2))$ for $i = 1, 2$ as $r$ tends to $\infty$ outside a set of finite measure for some $a \in \overline{C}$. If there exist distinct elements $a_1, a_2, \ldots, a_m$ in $\overline{C} - \{a\}$ such that $E(a_i, k_i, f_1) = E(a_i, k_i, f_2)$ for $i = 1, 2, \ldots, m$ for some $k_1, k_2, \ldots, k_m$ each of which is a positive integer or $\infty$ with $k_1 \geq k_2 \geq k_3, \ldots, \geq k_m$, and if

$$\sum_{i=1}^{m} \frac{k_i}{k_i + 1} - \frac{k_1}{k_1 + 1} > 1,$$

then, $f_1(z) \equiv f_2(z)$.

**CONSEQUENCES OF THEOREM B**:

1) If $m = 5$, $k_1$ is a positive integer or $k_1 = \infty$ and $k_2 \geq 2$, then $f_1(z) \equiv f_2(z)$.

2) If $m = 4$, $k_3 \geq 2$ and $\frac{k_1}{k_1 + 1} < \frac{k_2}{k_2 + 1} + \frac{1}{6}$, then $f_1(z) \equiv f_2(z)$. 


Subhas. S. Bhoosnurmath [28] proved the following theorem.

**THEOREM [C]**: Let \( f_1(z) \) and \( f_2(z) \) be two distinct meromorphic functions with

\[
N(r, a, f_i) = o(T(r, f_i) + T(r, f_2)) \quad \text{and}
\]

\[
N(r, b, f_i) = o(T(r, f_i) + T(r, f_2)) \quad \text{for} \quad i = 1, 2 \quad \text{as} \quad r \to \infty \quad \text{outside a set of finite measure for some distinct elements 'a' and 'b' in } \overline{C}. \]

If there exist distinct elements \( a_1, a_2, \ldots, a_m \) in \( \overline{C} \setminus \{a, b\} \) such that \( E(a_i, k_i, f_i) = E(a_i, k_i, f_2) \) for \( i = 1, 2, \ldots, m \) for some \( k_1, k_2, \ldots, k_m \) each of which is a positive integer or \( \infty \) with \( k_1 \leq k_2 \leq \ldots \leq k_m \) and

\[
\sum_{i=1}^{m} \frac{k_i}{k_i + i} - \frac{k_i}{k_i + 1} > 0, \quad \text{then} \quad f_1(z) \equiv f_2(z).
\]

**CONSEQUENCE OF THEOREM C**: 

There exist three elements \( a_1, a_2, a_3 \) such that \( E(a_i, k_i, f_i) = E(a_i, k_i, f_2) \) for \( i = 1, 2, 3 \) and \( k_1 \geq k_2 \geq k_3 \) and \( k_2 \geq 2 \), if \( k_1 = \infty \), then \( f_1(z) \equiv f_2(z) \).

In 1999, Hong-Xun-Yi [15] proved the following theorems.

**THEOREM [D]**: Suppose that \( f(z) \) and \( g(z) \) are two meromorphic functions satisfying \( E(a_i, k_i, f) = E(a_i, k_i, g) \) for \( i = 1, 2, \ldots, q \) where \( a_1, a_2, \ldots, a_q \) are distinct elements in \( \overline{C} \) and \( k_1, k_2, \ldots, k_q \) are positive integers or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \geq \ldots, k_q \).
1) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} > 2 \), then \( f(z) \equiv g(z) \).

2) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} = 2 \), then outside a set of \( r \) of finite measure \( \lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1 \).

3) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} > 2 \), then outside a set of \( r \) of finite measure

\[
\limsup_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{i=3}^{q} \frac{k_i}{k_i + 1}}{\sum_{i=3}^{q} \frac{k_i}{k_i + 1} - 2}
\]

**THEOREM [E]:** Let \( f(z) \) and \( g(z) \) be two meromorphic functions satisfying

\( \overline{N}(r, a, f) = S(r, f) \), \( \overline{N}(r, a, g) = S(r, g) \), where \( a \) is an element in \( \overline{C} \). Suppose

\( \overline{E}(a_i, k_i, f) = \overline{E}(a_i, k_i, g) \) for \( i = 1, 2, \ldots, q \), where \( a_1, a_2, \ldots, a_q \) are distinct elements in \( \overline{C} \{-a\} \) and \( k_1, k_2, \ldots, k_q \) are integers or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_q \).

1) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} > 1 \), then \( f(z) \equiv g(z) \).

2) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} = 1 \),

then outside a set of \( r \) of finite measure,

\[
\lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1.
\]

3) If \( \sum_{i=3}^{q} \frac{k_i}{k_i + 1} > 1 \),
then outside a set of \( r \) of finite measure,

\[
\limsup_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{i=1}^a \frac{k_i}{k_i + 1}}{\sum_{i=1}^a \frac{k_i}{k_i + 1} - 1}.
\]

If exceptional values are the shared values for \( f(z) \) and \( g(z) \) then we can extend the theorems B, C, D and E by introducing exceptional values as a shared values for \( f(z) \) and \( g(z) \). We can improve some of above theorems.

In order to prove our theorems following Lemmas are required.

**Lemma 1**: Let \( k \) be positive integer or \( \infty \), then for \( a \in \mathbb{C} \).

\[
\overline{N}(r, a, f) \leq \frac{k}{k+1} \overline{N}_k(r, a, f) + \frac{1}{k+1} N(r, a, f).
\]

--- (1.1)

**PROOF**: If \( k = \infty \) then the lemma is obvious.

If \( k < \infty \), then

\[
(k+1) \overline{n}(r, a, f) = (k + 1) \overline{n}(r, a; f, \leq k) + (k + 1) \overline{n}(r, a; f, > k)
\]

\[
= (k+1) \overline{n}_k (r, a, f) + n(r, a, f) - n_k(r, a, f)
\]

\[
\leq (k+1) \overline{n}_k (r, a, f) + n(r, a, f) - \overline{n}_k (r, a, f)
\]

\[
(k+1) \overline{n}(r, a, f) \leq k \overline{n}_k (r, a, f) + n(r, a, f)
\]

\[
\overline{n}(r, a, f) \leq \frac{k}{k+1} \overline{n}_k (r, a, f) + \frac{1}{k+1} n(r, a, f), \quad \text{from which the Lemma follows.}
\]
Lemma 2 [15]: Let \( f(z) \) be meromorphic function, \( a \in \overline{C} \) and \( k \) be positive integer or \( \infty \). Then

\[
\overline{N}(r, a, f) \leq \frac{k}{k+1} \overline{N}_k(r, a, f) + \frac{1}{k+1} T(r, f) + o(1)
\]  

Lemma 3 [15]: Let \( f(z) \) be meromorphic function \( a_1, a_2, \ldots, a_q \) be distinct elements in \( \overline{C} \) and \( k_1, k_2, \ldots, k_q \) be positive integers or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_q \) then

\[
\left( \sum_{i=1}^{q} \frac{k_i}{k_i + 1} - 2 \right) T(r, f) \leq \sum_{i=1}^{q} \frac{k_i}{k_i + 1} \overline{N}_{k_i}(r, a_i, f) + S(r, f).
\]

We rephrase Theorem A in the following form, to suit the proof of our main theorem.

**Theorem**: Let \( f(z) \) be a meromorphic function of order \( \rho \), \( 0 \leq \rho \leq \infty \). If there exist distinct elements \( a_1, a_2, \ldots, a_p \) that are evB for \( f(z) \), for distinct zeros of order \( \leq l_1, l_2, \ldots, l_p \) respectively with \( l_1 \geq l_2 \geq \ldots \geq l_p \), where \( l_1, l_2, \ldots, l_p \) are positive integers or \( \infty \). Then

\[
\sum_{i=1}^{p} \frac{l_i}{l_i + 1} \leq 2.
\]

**Proof**:

By the second fundamental theorem of Nevanlinna

\[
(p-2) T(r, f) \leq \sum_{i=1}^{p} \overline{N}(r, a_i, f) + S(r, f)
\]
By Lemma 1

\[(p-2)T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r, a_i, f) + \sum_{i=1}^{p} \frac{1}{l_i+1} N(r, a_i, f) + S(r, f)\]

By Lemma 2

\[(p-2)T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r, a_i, f) + \sum_{i=1}^{p} \frac{1}{l_i+1} T(r, f) + S(r, f)\]

\[\left( p - \sum_{i=1}^{p} \frac{1}{l_i+1} - 2 \right) T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r, a_i, f) + S(r, f)\]

\[\left( \sum_{i=1}^{p} \frac{l_i}{l_i+1} - 2 \right) T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r, a_i, f) + S(r, f)\]  \[= (1.4)\]

But, by hypothesis, \( N_i(r, a_i, f) = S(r, f) \) for \( i = 1, 2, \ldots, p \) and \( \frac{l_1}{l_1+1} \geq \frac{l_2}{l_2+1} \geq \ldots \geq \frac{l_p}{l_p+1} \), then (1.4) gives

\[\left( \sum_{i=1}^{p} \frac{l_i}{l_i+1} - 2 \right) T(r,f) \leq \frac{l_i}{l_i+1} S(r, f)\]

Now dividing by \( T(r,f) \) and taking limit as \( r \to \infty \), we get

\[\left( \sum_{i=1}^{p} \frac{l_i}{l_i+1} - 2 \right) \leq \frac{l_i}{l_i+1} \frac{S(r, f)}{T(r,f)}\]

Since \( \frac{S(r, f)}{T(r,f)} \to 0 \) as \( r \to \infty \)

\[\Rightarrow \sum_{i=1}^{p} \frac{l_i}{l_i+1} \leq 2.\]

This completes the proof of the theorem.
MAIN THEOREM: Suppose that $f(z)$ and $g(z)$ are two meromorphic functions satisfying

$$E(a_j, k_j, f) = E(a_j, k_j, g) \text{ for } j = 1, 2, \ldots, q, \text{ where } a_1, a_2, \ldots, a_q \text{ are distinct elements in } \mathbb{C} \text{ and } k_1, k_2, \ldots, k_q \text{ are positive integers or } \infty \text{ with } k_1 \geq k_2 \geq \ldots \geq k_q.$$

Also $b_1, b_2, \ldots, b_p$ are evB for $f(z)$ and $g(z)$ for distinct zeros of order $\leq l_1, l_2, \ldots, l_p$ respectively such that $l_1 \geq l_2 \geq l_3 \geq \ldots \geq l_p$.

PART I: When $\sum_{i=1}^{p} \frac{l_i}{l_i + 1} < 2$ and

1. If $\left[ \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=3}^{q} \frac{k_j}{k_j + 1} \right] > 2$, then $f(z) \equiv g(z)$ \hspace{1cm} (A1)

2. If $\left[ \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=3}^{q} \frac{k_j}{k_j + 1} \right] = 2$, then outside a set of $r$ of finite measure

$$\lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1 \text{ and } \sum_{j=1}^{q} N_{k_j}(r, a_j, f) = 2T(r, f) + S(r, f) \hspace{1cm} (B1)$$

3. If $\left[ \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=3}^{q} \frac{k_j}{k_j + 1} \right] > 2$, then outside a set of $r$ of finite measure

$$\limsup_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{j=3}^{q} \frac{k_j}{k_j + 1}}{\sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=3}^{q} \frac{k_j}{k_j + 1} - 2} \hspace{1cm} (C1)$$
PART II : When $\sum_{i=1}^{q} \frac{l_i}{l_i + 1} = 2$ and

1) If $\sum_{j=1}^{q} \frac{k_j}{k_j + 1} > 0$, then $f(z) \equiv g(z)$  


2) If $\sum_{j=1}^{q} \frac{k_j}{k_j + 1} = 0$, then outside a set of $r$ of finite measure,

$$\lim_{r \to \infty} \frac{T(r,f)}{T(r,g)} = 1 \quad \text{and} \quad \sum_{j=1}^{2} N_{k_j}(r,a_j,f) = 2T(r,f) + S(r,f)$$  


3) If $\sum_{j=1}^{q} \frac{k_j}{k_j + 1} > 0$, then outside a set of $r$ of finite measure,

$$\limsup_{r \to \infty} \frac{T(r,f)}{T(r,g)} \leq 1.$$

Thus, we see that these results lead to some improvements of the results of
Gundersen [12] and Hong Xun Y, I. [15].

PROOF OF PART I OF MAIN THEOREM :

By the second fundamental theorem of Nevanlinna

$$(p+q-2)T(r,f) \leq \sum_{i=1}^{q} N(r,b_i,f) + \sum_{j=1}^{q} N(r,a_j,f) + S(r,f)$$

By Lemma 1

$$(p+q-2)T(r,f) \leq \sum_{i=1}^{q} \frac{l_i}{l_i + 1} N_{b_i}(r,b_i,f) + \sum_{i=1}^{q} \frac{1}{l_i + 1} N(r,b_i,f) + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} N_{k_j}(r,a_j,f) + \sum_{j=1}^{q} \frac{1}{k_j + 1} N(r,a_j,f) + S(r,f)$$  

--- (1.5)
As \( b_1, b_2, \ldots, b_p \) are evB for \( f(z) \), for distinct zeros of order \( l_i : i = 1, 2, \ldots, p \)

\[ \Rightarrow \bar{N}_{k_i}(r, b_i, f) = S(r, f) \text{ for } i = 1, 2, \ldots, p. \]

And by the first fundamental theorem of Nevanlinna

\[ N(r, b_i, f) \leq T(r, f) + O(1) \text{ and } N(r, a_j, f) \leq T(r, f) + O(1) \]

Now by Lemma 2, 3 and (1.5) we have

\[
\left( p + q - \sum_{i=1}^{p} \frac{1}{l_i + 1} - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2 \right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \bar{N}_{k_j}(r, a_j, f) + S(r, f)
\]

\[
\left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2 \right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \bar{N}_{k_j}(r, a_j, f) + S(r, f)
\]

\[ \text{--- (1.6) } \]

We obtain that outside a set of finite measure

\[
\left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2 + o(1) \right) T(r, f) \leq \left( \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \right) T(r, g)
\]

\[ \text{--- (1.7) } \]

Since \( \bar{N}_{k_j}(r, a_j, f) = \bar{N}_{k_j}(r, a_j, g) \leq T(r, g) \) and (1.7) gives

\[
\limsup_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{j=1}^{q} \frac{k_j}{k_j + 1}}{\sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2}
\]

Next assume that \( f(z) \not\equiv g(z) \).
Suppose that $a_1, a_2, \ldots, a_q$ and $b_1, b_2, \ldots, b_p$ are all finite. We obtain (1.6). Noting that $k_1 \geq k_2 \geq \ldots \geq k_q$ and $l_1 \geq l_2 \geq l_3 \geq \ldots \geq l_p$ we have

$$
\frac{k_1}{k_1 + 1} \geq \frac{k_2}{k_2 + 1} \geq \ldots \geq \frac{k_q}{k_q + 1} \quad - - - (1.8)
$$

From (1.6) and (1.8) we have

$$
\left[ \sum_{j=1}^{q} \frac{l_j}{l_j + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2 \right] T(r, f) \leq \frac{k_2}{k_2 + 1} \sum_{j=1}^{q} \overline{N}_{k_j}(r, a_j, f) + \left[ \frac{k_1}{k_1 + 1} - \frac{k_2}{k_2 + 1} \right] \overline{N}_{k_1}(r, a_1, f) + S(r, f) \quad - - - (1.9)
$$

Noting $\overline{N}_{k_j}(r, a_j, f) \leq T(r, f) + o(1)$ from this and (1.9) we obtain

$$
\left[ \sum_{j=1}^{q} \frac{l_j}{l_j + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} + \frac{2k_2}{k_2 + 1} - 2 \right] T(r, f) \leq \frac{k_2}{k_2 + 1} \sum_{j=1}^{q} \overline{N}_{k_j}(r, a_j, f) + S(r, f) \quad - - - (1.10)
$$

From $E(a_j, k_j, f) \equiv E(a_j, k_j, g)$ for $j = 1, 2, \ldots, q$. We get

$$
\sum_{j=1}^{q} \overline{N}_{k_j}(r, a_j, f) \leq N(r, 0, f-g) \leq T(r, f) + T(r, g) + o(1) \quad - - - (1.11)
$$

Substituting (1.11) in to (1.10) we obtain

$$
\left[ \sum_{j=1}^{q} \frac{l_j}{l_j + 1} + \sum_{j=1}^{q} \frac{k_j}{k_j + 1} + \frac{2k_2}{k_2 + 1} - 2 \right] T(r, f) \leq \frac{k_2}{k_2 + 1} T(r, g) + S(r, f) \quad - - - (1.12)
$$
From (1.2) and (1.12) we have that outside a set of r of finite measure

\[(1+o(1)) T(r,f) \leq T(r, g) \quad -\quad (1.13)\]

Similarly we have that outside a set of r of finite measure

\[(1 + o(1)) T(r, g) \leq T(r, f) \quad -\quad (1.14)\]

Combining (1.13) and (1.14), we obtain

\[\lim_{r \to \infty} \frac{T(r,f)}{T(r,g)} = 1 \quad \text{and} \quad \sum_{j=1}^{q} N_{k_j}(r,a_j,f) = 2T(r,f) + S(r,f).\]

Suppose now that some \(a_j\) or \(b_j\) is \(\infty\), then let 'a' be a complex number different from \(a_1, a_2, \ldots, a_q\) and \(b_1, b_2, \ldots, b_p\). Then

\[
\frac{1}{a_1-a}, \frac{1}{a_2-a}, \frac{1}{a_3-a}, \ldots, \frac{1}{a_q-a}
\]

and

\[
\frac{1}{b_1-a}, \frac{1}{b_2-a}, \frac{1}{b_p-a}
\]

are all distinct and finite. Let

\[F = \frac{1}{g-a} \quad \text{and} \quad G = \frac{1}{g-a}.
\]

Then \(F \equiv G\) and also \(E\left(\frac{1}{a_j-a}, k_j, F\right) = E\left(\frac{1}{a_j-a}, k_j, G\right)\) for

\[j = 1,2,\ldots, q \quad \text{and} \quad E\left(\frac{1}{b_i-a}, l_i, F\right) = E\left(\frac{1}{b_i-a}, l_i, G\right)\] for \(j = 1, 2, \ldots, p\).

Hence by what we have proved above [B1] holds.

In order to prove the Case I of the theorem, without loss of generality, we assume that \(a_1, a_2, \ldots, a_q\) and \(b_1, b_2, \ldots, b_p\) are all finite. If \(f(z) \equiv g(z)\) proceeding as above we have (1.12)
\[ \left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \frac{q}{\sum_{j=3}^{q} k_j + 1} + \frac{k_2}{k_2 + 1} - 2 \right) T(r,f) \leq \frac{k_2}{k_2 + 1} T(r,g) + S(r,f) \quad - - (1.15) \]

Similarly we have

\[ \left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \frac{q}{\sum_{j=3}^{q} k_j + 1} + \frac{k_2}{k_2 + 1} - 2 \right) T(r,g) \leq \frac{k_2}{k_2 + 1} T(r,f) + S(r,g) \quad - - (1.16) \]

Adding (1.15) and (1.16) we obtain

\[ \left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \frac{q}{\sum_{j=3}^{q} k_j + 1} - 2 \right) \{ T(r,f) + T(r,g) \} \leq \{ S(r,f) + S(r,g) \} \]

\[ \Rightarrow \left( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \frac{q}{\sum_{j=3}^{q} k_j + 1} - 2 \right) \leq \frac{S(r,f) + S(r,g)}{T(r,f) + T(r,g)} \]

which is a contradiction to our hypotheses. Then \( f(z) \equiv g(z) \). This completes the proof of Part I of main theorem.

**PROOF OF PART II OF MAIN THEOREM:**

Since \( \sum_{i=1}^{p} \frac{l_i}{l_i + 1} = 2 \)

A2 : In Part (1) substituting this condition in (A1), we obtain

\[ \sum_{j=3}^{q} \frac{k_j}{k_j + 1} > 0. \quad \text{Then} \quad f \equiv g. \]

C2 : In Part (1) substituting this condition in (C1), we obtain

\[ \lim_{r \to \infty} \sup \frac{T(r,f)}{T(r,g)} \leq \frac{\sum_{j=3}^{q} \frac{k_j}{k_j + 1}}{\sum_{i=1}^{p} \frac{l_i}{l_i + 1} + \frac{q}{\sum_{j=3}^{q} k_j + 1} - 2} = 1 \]
B2: By the second fundamental theorem of Nevanlinna

\[(p+q-2)T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r,b_i,f) + \sum_{j=1}^{q} \frac{1}{l_j+1} N(r,b_j,f) + \sum_{j=1}^{q} \frac{k_j}{k_j+1} N_k(r,a_j,f) + S(r,f)\]

By Lemma 1

\[(p+q-2)T(r,f) \leq \sum_{i=1}^{p} \frac{l_i}{l_i+1} N_i(r,b_i,f) + \sum_{j=1}^{q} \frac{1}{l_j+1} N(r,b_j,f) + \sum_{j=1}^{q} \frac{k_j}{k_j+1} N_k(r,a_j,f) + S(r,f)\]

\[\frac{1}{1.17} \]

\[= \left(\sum_{j=1}^{q} \frac{k_j}{k_j+1}\right) T(r,f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j+1} N_k(r,a_j,f) + S(r,f)\]

\[\Rightarrow \left(\sum_{j=3}^{q} \frac{k_j}{k_j+1} + \frac{2k_2}{k_2+1}\right) T(r,f) \leq \frac{k_2}{k_2+1} \sum_{j=1}^{q} N_k(r,a_j,f) + S(r,f)\]

\[\text{Since } \sum_{j=3}^{q} \frac{k_j}{k_j+1} = 0 \text{ and } \]

\[= \left(\sum_{j=1}^{q} \frac{k_j}{k_j+1}\right) T(r,f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j+1} N_k(r,a_j,f) + S(r,f)\]

\[\text{--- (1.17)}\]

In view of Lemma 2 and 3, \(\sum_{i=1}^{p} \frac{l_i}{l_i+1} = 2\), and

\[\overline{N}_i(r,b_i,f) = S(r,f) \text{ for } i = 1, 2, ..., p, \text{ and}\]

\[N(r,a_j,f) \leq T(r,f) + o(1) \text{ and } N(r,b_i,f) \leq T(r,f) + o(1)\]

(1.17) now becomes

\[\left(\sum_{j=3}^{q} \frac{k_j}{k_j+1}\right) T(r,f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j+1} N_k(r,a_j,f) + S(r,f)\]

\[\Rightarrow \left(\sum_{j=3}^{q} \frac{k_j}{k_j+1} + \frac{2k_2}{k_2+1}\right) T(r,f) \leq \frac{k_2}{k_2+1} \sum_{j=1}^{q} N_k(r,a_j,f) + S(r,f)\]

\[\text{--- (1.18)}\]

Since \(\sum_{j=3}^{q} \frac{k_j}{k_j+1} = 0\) and
\[
\sum_{j=1}^{s} N_{k_j}(r, a_j, f) \leq N(r, o, f-g) \leq T(r, f-g) \leq T(r, f) + T(r, g),
\]

(1.18) becomes

\[
\frac{2k_2}{k_2 + 1} T(r, f) \leq \frac{k_2}{k_2 + 1} \sum_{j=1}^{s} N_{k_j}(r, a_j, f) + S(r, f),
\]

and hence

\[
2T(r, f) \leq T(r, f) + T(r, g) + o(1) \quad \text{--- (1.19)}
\]

\[
\Rightarrow T(r, f) \leq T(r, g) + o(1) \quad \text{--- (1.20)}
\]

Similarly we prove

\[
T(r, g) \leq T(r, f) + o(1) \quad \text{--- (1.21)}
\]

from (1.19), (1.20) and (1.21), we have

\[
\lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1 \quad \text{and} \quad 2T(r, f) = \sum_{j=1}^{s} N_{k_j}(r, a_j, f) + S(r, f).
\]

This completes the proof of main theorem.

**COROLLARY 1.1** : Let \( f \) and \( g \) be two distinct meromorphic functions satisfying

\[
E(a_j, k_j, f) = E(a_j, k_j, g), \quad \text{for} \quad j = 1, 2, \ldots, q, \quad \text{where} \quad a_1, a_2, \ldots, a_q \quad \text{are distinct elements and} \quad k_1, k_2, \ldots, k_q \quad \text{are positive integers or} \ \infty \ \text{with} \ k_1 \geq k_2, \ldots, k_q.
\]

(1) If \( \sum_{j=1}^{s} \frac{k_j}{k_j + 1} > 2 \), then \( f(z) \equiv g(z) \).
(2) If \( \sum_{j \neq 1} \frac{k_j}{k_j + 1} = 2 \), then outside a set of \( r \) of finite measure, \( \lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1 \).

(3) If \( \sum_{j \neq 1} \frac{k_j}{k_j + 1} > 2 \), then outside a set of \( r \) of finite measure
\[
\lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{j \neq 1} k_j}{\sum_{j \neq 1} k_j + 1}.
\]

PROOF OF COROLLARY 1.1: When \( p = 0 \) in Part 1 of theorem (A1), (B1) and (C1), the above results follow.

This corollary is Theorem D of Hong-Xun-Yi [15].

COROLLARY 1.2: Let \( f \) and \( g \) be two meromorphic functions satisfying
\( \overline{N}(r, a, f) = S(r, f) \) and \( \overline{N}(r, a, g) = S(r, g) \), where \( a \) is an element in \( \overline{C} \).

Suppose that \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \) for \( j = 1, 2, \ldots, q \), where \( a_1, a_2, \ldots, a_q \) are the distinct elements in \( \overline{C} \{-a\} \) and \( k_1, k_2, \ldots, k_q \) are positive integers or \( \infty \) with \( k_1 \geq k_2 \geq \ldots, k_q \).

(1) If \( \sum_{j \neq 1} \frac{k_j}{k_j + 1} > 1 \), then \( f(z) \equiv g(z) \).

(2) If \( \sum_{j \neq 1} \frac{k_j}{k_j + 1} = 1 \), then outside a set of \( r \) of finite measure, \( \lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = 1 \).

(3) If \( \sum_{j \neq 1} \frac{k_j}{k_j + 1} > 1 \), then outside a set of \( r \) of finite measure,
\[
\limsup_{r \to \infty} \frac{T(r, f)}{T(r, g)} \leq \frac{\sum_{j=1}^{q} \frac{k_j}{k_j + 1}}{\sum_{j=1}^{q} \frac{1}{k_j + 1} - 1}.
\]

**PROOF OF COROLLARY 1.2:**

When \( p = 1, l_1 \to \infty \Rightarrow \frac{l_1}{l_1 + 1} \to 1 \), then in Part (1) of theorem (A1), (B1) and (C1), the above results follow.

This corollary is Theorem E of Hong-Xun-Yi [15].

**COROLLARY 1.3:** Let \( f \) and \( g \) be two meromorphic functions satisfying:

\[
N (r, a, f) = S(r, f), \quad N (r, b, f) = S(r, f) \quad \text{and} \quad N (r, a, g) = S(r, g), \quad N (r, b, g) = S(r, g),
\]

where "a" and "b" are elements in \( \overline{C} \). Suppose that \( \overline{E} (a_j, k_j, f) = \overline{E} (a_j, k_j, g) \), for \( j = 1, 2, \ldots, q \), where \( a_1, a_2, \ldots, a_q \) are distinct elements in \( \overline{C} - \{a, b\} \) and \( k_1, k_2, \ldots, k_q \) are positive integers or \( \infty \) with \( k_1 \geq k_2 \geq \ldots, k_q \).

If \( \sum_{j=1}^{q} \frac{k_j}{k_j + 1} > 0 \), then \( f(z) \equiv g(z) \).

This is an improvement of Theorem C of Subhas S. Bhoosnurmath [28].

**PROOF OF COROLLARY 1.3:**

When \( p = 2 \) and \( l_1 \to \infty \) and \( l_2 \to \infty \)

\[
\Rightarrow \sum_{i=1}^{2} \frac{l_i}{l_i + 1} \to 2,
\]

then Part (1) of Theorem (A1), (B1) and (C1), the above results follow.
CONSEQUENCES OF MAIN THEOREM:

(1) If there exist four evB for f and g for distinct zeros of order 1

\[ p = 4, \quad \sum_{i=1}^{4} \frac{l_i}{l_i + 1} = 2 \quad \text{and} \quad \sum_{j=1}^{4} \frac{k_j}{k_j + 1} > 0 \]

Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, 3 \), where each \( k_j \); \( j = 1, 2, 3 \)
is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \), then \( f(z) \equiv g(z) \).

\[ T(r, f) = \overline{N}_{k_j}(r, a_j, f) + S(r, f) \quad \text{for} \quad j = 1, 2, 3 \]

\[ T(r, g) = \overline{N}_{k_j}(r, a_j, g) + S(r, g) \quad \text{for} \quad j = 1, 2, 3. \]

(2) If there exist three evB for f and g for the distinct zeros of order \( \leq 2 \).

Then \( p = 3, \quad \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = 2 \quad \text{and} \quad \sum_{j=1}^{3} \frac{k_j}{k_j + 1} > 0 \)

Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, 3 \), where each \( k_j \); \( j = 1, 2, 3 \)
is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \), then \( f(z) \equiv g(z) \).

\[ T(r, f) = \overline{N}_{k_j}(r, a_j, f) + S(r, f) \quad \text{for} \quad j = 1, 2, 3 \]

\[ T(r, g) = \overline{N}_{k_j}(r, a_j, g) + S(r, g) \quad \text{for} \quad j = 1, 2, 3. \]

(3) If there exist three evB for f and g for the distinct zeros of simple and double zeros, i.e. one evB for simple zeros, two evB for simple and double zeros.

Then \( p = 3, \quad \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = \frac{2}{3} + \frac{2}{3} + \frac{1}{2} = \frac{11}{6} \quad \text{and} \quad \sum_{j=1}^{3} \frac{k_j}{k_j + 1} > \frac{1}{6} \)
Further, if $\overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g)$, for $j = 1, 2, 3$, where each $k_j; j = 1, 2, 3$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq k_3$, then $f(z) \equiv g(z)$.

(4) If there exists three evB for $f$ and $g$ for distinct zeros of simple and double zeros. i.e. Two evB for simple zeros and one evB for simple and double zeros.

Then $p = 3, \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = \frac{2}{3} + \frac{1}{2} + \frac{1}{2} = \frac{5}{3}$ and $\frac{3}{3} \sum_{j=3}^{4} \frac{k_j}{k_j + 1} > \frac{1}{3}$

Further, if $\overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g)$, for $j = 1, 2, 3$, where each $k_j; j = 1, 2, 3$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq k_3$, then $f(z) \equiv g(z)$.

(5) If there exist one evB for $f$ and $g$ for distinct zeros of order 1.

Then $p = 1, \frac{l}{l+1} = \frac{1}{2}$ and $\sum_{j=1}^{4} \frac{k_j}{k_j + 1} > \frac{3}{2}$

$\Rightarrow$

(i) Further, if $\overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g)$, for $j = 1, 2, \ldots, 6$, where each $k_j; j = 1, 2, 3, \ldots, 6$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq \ldots \geq k_6$, then $f(z) \equiv g(z)$.

(ii) Further, if $\overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g)$, for $j = 1, 2, \ldots, 5$, where each $k_j; j = 1, 2, 3, \ldots, 5$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq \ldots \geq k_5$, and $k_3 \geq 2$, then $f(z) \equiv g(z)$.

(iii) Further, if $\overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g)$, for $j = 1, 2, \ldots, 4$ where each $k_j; j = 1, 2, 3, 4$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq \ldots \geq k_4$, and $k_4 \geq 3$, then $f(z) \equiv g(z)$.
(6) If there exist two evB for f and g for distinct zeros of order 1.

Then \( p = 2 \), \( \sum_{i=1}^{2} \frac{l_i}{l_i + 1} = 1 \) and \( \sum_{j=3}^{q} \frac{k_j}{k_j + 1} > 1 \)

\( \Rightarrow \)

(i) Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, \ldots, 5 \), where each \( k_j; j = 1, 2, 3, \ldots, 5 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_5 \), then, \( f(z) = g(z) \).

(ii) Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, 3, \ldots, 4 \), where each \( k_j; j = 1, 2, 3, 4 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_4 \), and \( k_3 \geq 2 \), then \( f(z) = g(z) \).

(7) If there exist three evB for f and g for distinct zeros of order 1.

Then \( p = 3 \), \( \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = \frac{3}{2} \) and \( \sum_{j=4}^{q} \frac{k_j}{k_j + 1} > \frac{1}{2} \)

\( \Rightarrow \)

(i) Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, \ldots, 4 \), where each \( k_j; j = 1, 2, 3, 4 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_4 \), then \( f(z) = g(z) \).

(ii) Further, if \( \overline{E}(a_j, k_j, f) = \overline{E}(a_j, k_j, g) \), for \( j = 1, 2, 3, \) where each \( k_j; j = 1, 2, 3 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \), and \( k_3 \geq 2 \), then \( f(z) = g(z) \).

(8) If there exist one evB for f and g for distinct zeros of order \( \leq 2 \).
Then \( p = 1, \quad \frac{l}{l+1} = \frac{2}{3} \) and \( \sum_{j=3}^{3} \frac{k_j}{k_j+1} > \frac{4}{3} \)

\[ \Rightarrow \]

(i) Further, if \( \overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g) \), for \( j = 1, 2, \ldots, 6 \), where each \( k_j; j = 1, 2, 3 \ldots 6 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_6 \), then \( f(z) \equiv g(z) \).

(ii) Further, if \( \overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g), \) for \( j = 1, 2, \ldots, 4 \), where each \( k_j; j = 1, 2, 3, 4 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_4 \) and \( k_4 \geq 3 \), then \( f(z) \equiv g(z) \).

(9) If there exist two \( \text{evB} \) for \( f \) and \( g \) for distinct zeros of order \( \leq 2 \).

Then \( p = 2, \quad \sum_{i=1}^{2} \frac{l_i}{l_i+1} = \frac{4}{3} \) and \( \sum_{j=3}^{3} \frac{k_j}{k_j+1} > \frac{2}{3} \)

\[ \Rightarrow \]

(i) Further, if \( \overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g), \) for \( j = 1, 2, \ldots, 4 \), where each \( k_j; j = 1, 2, 3, 4 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_4 \), then \( f(z) \equiv g(z) \).

(ii) Further, if \( \overline{E}(a_j,k_j,f) = \overline{E}(a_j,k_j,g), \) for \( j = 1, 2, 3 \), where each \( k_j; j = 1, 2, 3 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3 \), and \( k_3 \geq 3 \), then \( f(z) \equiv g(z) \).

(10) If there exist one \( \text{evB} \) for \( f \) and \( g \) for distinct zeros of order \( l \), \( \to \infty \) and one \( \text{evB} \) for \( f \) and \( g \) of order 1.
Then \( p = 2, \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = \frac{3}{2} \) and \( \sum_{j=1}^{3} \frac{k_j}{k_j + 1} > \frac{1}{2} \)

\( \Rightarrow \)

(i) Further, if \( E(a_j, k_j, f) = E(a_j, k_j, g), \) for \( j = 1, 2, 3, 4, \) where each \( k_j; j = 1, 2, 3, 4 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq \ldots \geq k_4, \)
then, \( f(z) \equiv g(z). \)

(ii) Further, if \( E(a_j, k_j, f) = E(a_j, k_j, g), \) for \( j = 1, 2, 3, \) where each \( k_j; j = 1, 2, 3 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3, \) and \( k_3 \geq 2, \) then \( f(z) \equiv g(z). \)

(11) If there exist one evB for \( f \) and \( f \equiv g \) for the distinct zeros of order \( l_1 \to \infty \) and two evB for the distinct zeros of order \( 1. \)

Then \( p = 3, \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = 2 \) and \( \sum_{j=1}^{3} \frac{k_j}{k_j + 1} > 0 \)

(i) Further, if \( E(a_j, k_j, f) = E(a_j, k_j, g), \) for \( j = 1, 2, 3, \) where each \( k_j, \)
\( j = 1, 2, 3 \) is either positive integer or \( \infty \) with \( k_1 \geq k_2 \geq k_3, \) then \( f(z) \equiv g(z). \)

(12) If there exist one evB for \( f \) and \( g \) for distinct zeros of order \( l_1 \to \infty \) and one evB for the distinct zeros of order \( \leq 2. \)

Then \( p = 2, \sum_{i=1}^{3} \frac{l_i}{l_i + 1} = \frac{5}{3} \) and \( \sum_{j=1}^{3} \frac{k_j}{k_j + 1} > \frac{1}{3} \)
Further, if $E(a_j,k_j,f) = E(a_j,k_j,g)$, for $j = 1, 2, 3$, where each $k_j; j =1, 2, 3$ is either positive integer or $\infty$ with $k_1 \geq k_2 \geq k_3$, then $f(z) = g(z)$.

(13) If there exist four evB for $f$ and $g$ for the distinct zeros of order one and

$E(a_j,k_j,f) = E(a_j,k_j,g)$; for $j = 1, 2$, where each $k_j; j =1, 2$, is either positive integer or $\infty$ with $k_1 \geq k_2$, such that outside a set of $r$ of finite measure,

then, $\lim_{r \to \infty} \frac{T(r,f)}{T(r,g)} = 1$ and $\sum_{j=1}^{2} N_{k_j}(r,a_j,f) = 2T(r,f) + S(r,f)$.

(14) If there exist 3 evB for $f$ and $g$ for distinct zeros of order \leq 2 and

$E(a_j,k_j,f) = E(a_j,k_j,g)$; for $j = 1, 2$, where each $k_j; j =1, 2$ is either positive integer or $\infty$ with $k_1 \geq k_2$, such that outside a set of $r$ of finite measure,

then $\lim_{r \to \infty} \frac{T(r,f)}{T(r,g)} = 1$ and $\sum_{j=1}^{2} N_{k_j}(r,a_j,f) = 2T(r,f) + S(r,f)$.

(15) If there exist 2 evB for $f$ and $g$ for distinct zeros of order $l_i \to \infty; i = 1, 2$ and $E(a_j,k_j,f) = E(a_j,k_j,g)$; for $j = 1, 2$, where each $k_j; j =1, 2$ is either positive integer or $\infty$ with $k_1 \geq k_2$, such that outside a set of $r$ of finite measure,

then $\lim_{r \to \infty} \frac{T(r,f)}{T(r,g)} = 1$ and $\sum_{j=1}^{2} N_{k_j}(r,a_j,f) = 2T(r,f) + S(r,f)$.

This is Theorem of Gunderson [12].
On the other hand

If \( a_1, a_2, \ldots, a_q \) are shared IM and \( b_1, b_2, \ldots, b_p \) are shared values of order \( \leq k_i \),
\( i = 1, 2, \ldots, p \),

and

\[
\left[ \sum_{i=1}^{q} \frac{k_i}{k_i + 1} + (q - 2) \right] > 2, \quad \text{then } f(z) = g(z).
\]

**CONSEQUENCES:**

(1) If \( a_1, a_2, a_3, a_4 \) are shared IM, then there is at least one shared value for simple zeros.

(2) If \( a_1, a_2, a_3 \) are shared IM, then there are at least 2 shared values for simple and double zeros.

(3) If \( a_1, a_2 \) are shared IM, then there are at least 3 shared values for distinct zeros of order \( \leq 3 \).

(4) If \( a_1 \) is shared IM, then there are at least 4 shared values for distinct zeros of order \( \leq 4 \).

**FUNCTIONS SHARING VALUES AND A SET:**

Further we have proved some results on sharing values and a set.

In 1999, Ping Li and C. C. Yang [19] proved the following theorem for two entire functions.
THEOREM G: Let $f$ and $g$ be two non constant and distinct entire functions and let $a_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$ be four distinct finite complex numbers if $f$ and $g$ share $a_1, a_2$ IM and further $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$, then

(i) $T(r, g) = 2T(r, f) + S(r, f)$

(ii) $T(r, f-g) = 3T(r, f) + S(r, f)$

(iii) $T(r, f) = \frac{1}{S(r, f)} + S(r, f) \quad i = 1, 2$

(iv) $T(r, f) = \frac{1}{S(r, f)} + \frac{1}{S(r, f)} + S(r, f)$

(v) $T(r, g) = \frac{1}{S(r, g)} + S(r, g) \quad i = 3, 4$

(vi) $T(r, f') = T(r, f) + S(r, f)$ and $T(r, g') = T(r, g) + S(r, g)$

THEOREM H: Let $f$ and $g$ be two non constant entire functions and let $a_i \in \mathbb{C}, i = 1, 2, 3, 4$ be four distinct finite complex numbers. If $f$ and $g$ share $a_1, a_2$, CM and furthermore $f = a_3 \Rightarrow g = a_3$

and $f = a_4 \Rightarrow g = a_4$, then $f \equiv g$.

We now extend these results for meromorphic functions.
**THEOREM** : Let $f$ and $g$ be two non-constant and distinct meromorphic functions and let $a_i \in \mathbb{C} \text{ (i = 1, 2, 3, 4)}$ be four distinct finite complex numbers. If $f$ and $g$ share $a_1$, $a_2$, $a_3$, and further,

\[
 f = a_3 => g = a_3 \\
 f = a_4 => g = a_4 \text{ and also } \overline{N}(r, a, f) = o(T(r, f)) \\
\]

\[
 \overline{N}(r, a, g) = o(T(r, g)) \text{ for } a \neq a_i ; i = 1, 2, 3, 4, \]

then

(1) $T(r, g) = 2T(r, f) + S(r, f)$

(2) $N(r, f-g) = 3T(r, f) + S(r, f)$

(3) $T(r, f) = \overline{N}(r, a_i, f) + S(r, f)$  $i = 1, 2$

(4) $T(r, f) = \overline{N}(r, a_3, f) + \overline{N}(r, a_4, f) + S(r, f)$

(5) $T(r, g) = \overline{N}(r, a_i, g) + S(r, g); \ i = 3, 4$

(6) $T(r, f') = T(r, f) + S(r, f), \ T(r, g') = T(r, g) + S(r, f)$.

**PROOF THE THEOREM :**

By the second fundamental theorem of Nevanlinna

\[
3T(r, f) \leq \sum_{i=1}^{4} \overline{N}(r, a_i, f) + \overline{N}(r, a, f) + S(r, f)
\]

and by hypothesis \( \overline{N}(r, a, f) = S(r, f) \)
Hence

\[ 3T(r, f) \leq \sum_{i=1}^{4} N(r, a_i, f) + S(r, f), \]

\[ 3T(r, f) \leq N(r, 0, f-g) + S(r, f) \leq T(r, f) + T(r, g) + S(r, f) \]

\[ 2T(r, f) \leq T(r, g) + S(r, f) \quad \cdots (1.23) \]

Now

\[ T(r, g) \leq \sum_{i=1}^{2} N(r, a_i, g) + \overline{N}(r, a, g) + S(r, g) \]

\[ \overline{N}(r, a, g) = S(r, g) \]

\[ T(r, g) \leq \sum_{i=1}^{2} \overline{N}(r, a_i, f) + S(r, g) \]

\[ T(r, g) \leq 2T(r, f) + S(r, g) \quad \cdots (1.24) \]

From (1.23) and (1.24)

\[ T(r, g) = 2T(r, f) + S(r, f) \]

Thus, the result (1) follow.

From (1.22)

\[ 3T(r, f) \leq N(r, 0, f-g) + S(r, f) \leq T(r, f) + T(r, g) \]

\[ \leq T(r, f) + T(r, g) \]
\[ \leq T(r, f) + 2T(r, f) + S(r, f) \]

\[ \leq 3T(r, f) + S(r, f) \]

\[ \Rightarrow N(r, 0, f-g) = 3T(r, f) + S(r, f) \]

Thus, the result (2) follow.

Since \( T(r, g) \leq \sum_{i=1}^{2} N(r, a_i, f) + N(r, a, f) + S(r, g) \)

As \( N(r, a, f) = S(r, f) \)

And \( 2T(r, f) = T(r, g) \)

\[ T(r, g) \leq \sum_{i=1}^{2} N(r, a_i, f) + S(r, f) \]

\[ \leq 2T(r, f) + S(r, f) \]

\[ \Rightarrow 2T(r, f) = \sum_{i=1}^{2} N(r, a_i, f) + S(r, f) \]

\[ \Rightarrow T(r, f) = \overline{N}(r, a_i, f) + S(r, f) \quad i = 1, 2 \]

Thus the result (3) follow.

Next by the second fundamental theorem

\[ 3T(r, f) \leq \sum_{i=1}^{4} \overline{N}(r, a_i, f) + \overline{N}(r, a, f) + S(r, f) \]

Since \( \overline{N}(r, a, f) = S(r, f) \)
\[3T(r, f) \leq \sum_{i=1}^{4} \overline{N}(r, a_i, f) + S(r, f)\]

\[\leq N(r, 0, f - g)\]

\[\leq T(r, 0, f - g)\]

\[\leq T(r, f) + T(r, g) + o(1)\]

\[\leq T(r, f) + 2T(r, f) + o(1)\]

Therefore,

\[3T(r, f) = \sum_{i=1}^{4} \overline{N}(r, a_i, f) + S(r, f)\]

Next \[3T(r, f) = \sum_{i=1}^{3} \overline{N}(r, a_i, f) + \sum_{i=3}^{4} \overline{N}(r, a_i, f) + S(r, f)\]

From (1.25)

\[3T(r, f) = 2T(r, f) + \sum_{i=3}^{4} \overline{N}(r, a_i, f) + S(r, f)\]

\[T(r, f) = \sum_{i=3}^{4} \overline{N}(r, a_i, f) + S(r, f).\]

Thus, the result (4) follow.

Now

\[3T(r, g) \leq \sum_{i=1}^{2} \overline{N}(r, a_i, f) + \sum_{i=3}^{4} \overline{N}(r, a_i, g) + S(r, f)\]
3T(r, g) ≤ 2T(r, f) + \sum_{i=3}^{i} \overline{N}(r, a_i, g) + S(r, g)

3T(r, g) ≤ T(r, g) + \sum_{i=3}^{i} \overline{N}(r, a_i, g) + S(r, g)

2T(r, g) ≤ \sum_{i=3}^{i} \overline{N}(r, a_i, g) + S(r, g)

But \sum_{i=3}^{i} \overline{N}(r, a_i, g) ≤ 2T(r, g) + S(r, g)

⇒ 2T(r, g) = \sum_{i=3}^{i} \overline{N}(r, a_i, g) + S(r, g)

⇒ T(r, g) = \overline{N}(r, a_i, g) + S(r, g) \quad i = 3, 4

Thus, the result (5) follow.

When a = ∞

and \overline{N}(r, a, f) = \overline{N}(r, a, g) = o(T(r, f))

and hence by taking a = ∞

Θ(∞, f) = Θ(∞, g) = 1

Also \sum_{a \in \mathbb{C}} Θ(a_i, f) = \sum_{a \in \mathbb{C}} Θ(a_i, g) = 2

By using the Theorem 1.4 we get,

\lim_{r \to 0} \frac{T(r, f')}{T(r, f)} = 1 \quad \text{and} \quad \lim_{r \to 0} \frac{T(r, g')}{T(r, g)} = 1
\[ \therefore T(r, f) = T(r, f') + S(r, f) \]

\[ T(r, g) = T(r, g') + S(r, g) \]

This completes the proof of the theorem.

**THEOREM** : Let \( f \) and \( g \) be two non constant meromorphic functions and let \( a_i \in \mathbb{C} \) (\( i = 1, 2, 3, 4 \)) be four distinct finite complex numbers. If \( f \) and \( g \) share \( a_i, a_2, a_3, \) and, furthermore,

\[ f = a_3 \Rightarrow g = a_3 \]

\[ f = a_4 \Rightarrow g = a_4 \]

and also \( N(r, a, f) = o(T(r, f)) \) for \( a \neq a_i ; i = 1, 2, 3, 4. \)

If \( N(r, a_3, g) + N(r, a_4, g) < (\lambda + 1) \left\{ (N(r, a_3, f) + N(r, a_4, f)) + S(r, f) \right\} \)

where \( 0 < \lambda < 3, \)

then \( f(z) \equiv g(z). \)

**PROOF** : By the previous theorem.

\[ 2T(r, f) + S(r, f) = T(r, g) \]

\[ \Rightarrow \lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} = \frac{1}{2} \]

Also \( N(r, a_3, g) + N(r, a_4, g) \leq (\lambda + 1) \left\{ (N(r, a_3, f) + N(r, a_4, f)) + S(r, f) \right\} \)

By the previous theorem (4) and (5)

\[ 2T(r, g) \leq (\lambda + 1) T(r, f) + S(r, f) \]
\[ \frac{2}{\lambda + 1} \leq \lim_{r \to \infty} \frac{T(r, f)}{T(r, g)} \]

\[ \frac{2}{\lambda + 1} \leq \frac{1}{2}, \quad \text{which is a contradiction.} \]

Therefore \( f(z) = g(z) \).

**UNIQUENESS THEOREM AND RELATIVE DEFECTS:**

Now we prove some uniqueness theorems for meromorphic functions and its derivatives by using relative defect.

**NOTATIONS:**

\[ \Theta_r^{(k)}(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a, f^{(k)})}{T(r, f)} \]

\[ \delta_r^{(k)}(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a, f^{(k)})}{T(r, f)} \]

If \( \infty \) is an evB for meromorphic functions \( f(z) \), then \( N(r, f) = S(r, f) \) and \( N(r, f^{(k)}) = N(r, f) + kN(r, f) \)

\[ \Rightarrow N(r, f^{(k)}) = S(r, f) \]

Since \( \delta(\infty, f) = \Theta(\infty, f) = 1 \),

\[ 1 - \delta_r^{(k)}(\infty, f) = \limsup_{r \to \infty} \frac{N(r, f^{(k)})}{T(r, f)} \]

\[ = 0, \]

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and hence

\[ \delta_r^{(k)}(\infty, f) = 1. \]

**THEOREM:**

Let \( f(z) \) be a non constant meromorphic function, \( f^{(k)}(z) \) be its derivative such that

1. \( \sum_{\infty \in \mathbb{C}} a_i, p_i, f = \sum_{\infty \in \mathbb{C}} a_i, p_i, f^{(k)} \) for \( i = 1, 2, ..., q. \)
2. \( \theta(a, f) = 1 \) and \( \theta(\infty, f) = 1 \) i.e. \( \infty \) is evB for \( f. \)
3. If \( \sum_{i=1}^{q} \frac{p_i}{p_i + 1} > 1 \), then \( f \equiv f^{(k)} \)

**PROOF:** Since \( \sum_{\infty \in \mathbb{C}} \theta(a, f) = 1 \) and \( \theta(\infty, f) = 1 \)

\[ \Rightarrow \lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)} = 1 \] and \( \theta(\infty, f) = \theta_r^{(k)}(\infty, f) = 1 \)

By the second fundamental theorem of Nevanlinna

\( (q-1)T(r, f^{(k)}) \leq \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N(r, a_i, f^{(k)}) + N(r, f^{(k)}) + S(r, f^{(k)}) \)

\( (q-1)T(r, f^{(k)}) \leq \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N_p (r, a_i, f^{(k)}) + \sum_{i=1}^{q} \frac{1}{p_i} N(r, a_i, f^{(k)}) + N(r, f^{(k)}) + S(r, f^{(k)}) \)

Since \( N(r, a_i, f^{(k)}) \leq T(r, f^{(k)}) + o(1) \),
\(\Rightarrow \left(q - \sum_{i=1}^{q} \frac{1}{p_i} - 1\right) T(r, f^{(k)}) \leq \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N_p(r, a_i, f) + N(r, f^{(k)}) + S(r, f^{(k)})\)

\[\left(\sum_{i=1}^{q} \frac{p_i}{p_i + 1} - 1\right) T(r, f^{(k)}) \leq \left(\frac{p_1}{p_1 + 1} - \frac{p_2}{p_2 + 1}\right) N_p(r, a_i, f^{(k)}) + \frac{p_2}{p_2 + 1} \sum_{i=1}^{q} \bar{N}_p(r, a_i, f^{(k)})\]

\[+ N(r, f^{(k)}) + S(r, f^{(k)})\]

\[\sum_{i=3}^{q} N_p(r, a_i, f^{(k)}) \leq N(r, 0, f - f^{(k)})\]

\[\leq T(r, f - f^{(k)}) + O(1)\]

\[\left(\sum_{i=3}^{q} \frac{p_i}{p_i + 1} - 1\right) T(r, f^{(k)}) \leq \frac{p_2}{p_2 + 1} \left[T(r, f) + T(r, f^{(k)})\right] + N(r, f^{(k)}) + S(r, f^{(k)})\]

Dividing by \(T(r, f)\) and taking limit as \(r \to \infty\)

\[\left(\sum_{i=3}^{q} \frac{p_i}{p_i + 1} - 1\right) \lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)} \leq \frac{p_2}{p_2 + 1} \left[1 + \lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)}\right]\]

\[+ \limsup_{r \to \infty} \frac{N(r, f^{(k)})}{T(r, f)} + \limsup_{r \to \infty} \frac{S(r, f^{(k)})}{T(r, f)}\]

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Since

$$\lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)} = 1 \quad \text{and} \quad S(r, f^{(k)}) = S(r, f), \text{ by Theorem 1.4}$$

which is a contradiction.

Hence \( f \equiv f^{(k)} \).

CONSEQUENCES:

(1) If there are 4 shared values \( a_1, a_2, a_3, a_4 \) such that \( p_i \to \infty ; i = 1, 2, 3, 4 \)

and

$$\sum_{i=1}^{4} \Theta(a_i, f) = 1 \quad \text{and} \quad \Theta(\infty, f) = 1,$$

then \( f(z) = f^{(k)}(z) \)

(2) If there are \( a_1, a_2, a_3, a_4 \) and \( a_5 \) shared values for simple zeros such that

$$\sum_{i=1}^{5} \Theta(a_i, f) = 1 \quad \text{and} \quad \Theta(\infty, f) = 1, \quad \sum_{i=1}^{5} \frac{p_i}{p_i} > 1, \text{ then } f \equiv f^{(k)}$$

(3) If there are \( a_1, a_2, a_3 \) and \( a_4 \) shared values for simple and double zeros, such that

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\[
\sum_{i=1}^{q} \Theta_i(a_i, f) = 1, \quad \Theta(\infty, f) = 1. \text{ Also } \sum_{i=3}^{q} \frac{p_i}{p_i + 1} = \frac{2}{3} + \frac{2}{3} > 1, \text{ then } f = f^{(k)}.
\]

(4) If there are 4 shared values, \(a_1, a_2, a_3\) for simple and double zeros and \(a_4\) for simple zeros such that, \(\sum_{i=1}^{4} \Theta(a_i, f) = 1, \quad \Theta(\infty, f) = 1\)

\[
\sum_{i=3}^{4} \frac{p_i}{p_i + 1} = \frac{2}{3} + \frac{1}{2} > 1, \text{ then } f = f^{(k)}.
\]

**THEOREM**: Let \(f(z)\) be a non constant meromorphic function \(f^{(k)}\) be its derivative such that

1. \(E(a_i, p_i, f) = E(a_i, p_i, f^{(k)})\) for \(i = 1, 2, \ldots, q\).
2. \(\sum_{a \in C} \Theta(a, f) = 1\) and \(\Theta(\infty, f) = 1\) i.e. \(\infty\) is evB for \(f\).
3. For \(b \in C\) and \(b \neq a_i, i = 1, 2, \ldots, q\) such that \(\delta_i^{(k)}(b, f) = 1\).
4. For \(\sum_{i=3}^{4} \frac{p_i}{p_i + 1} > 0\), then \(f = f^{(k)}\)

**PROOF**: Since \(\sum \Theta(a, f) = 1\) and \(\Theta(\infty, f) = 1\)

From Theorem 1.4,

\[
\lim_{r \to \infty} \frac{T(r, f^{(k)})}{T(r, f)} = 1
\]

By the second fundamental theorem of Nevanlinna
\[ qT(r,f^{(k)}(z)) \leq \sum_{i=1}^{q} N(r,a_i,f^{(k)}) + N(r,b,f^{(k)}) + N(r,f^{(k)}) + S(r,f^{(k)}) \]

\[ qT(r,f^{(k)}(z)) \leq \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N_{r_i}(r,a_i,f^{(k)}) + \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N(r,a_i,f^{(k)}) + \sum_{i=1}^{q} \frac{p_i}{p_i + 1} N_{r_i}(r,a_i,f^{(k)}) \]

\[ N(r,b,f^{(k)}) + N(r,f^{(k)}) + S(r,f^{(k)}) \]

Since \( N(r, a_i, f^{(k)}) \leq T(r, f^{(k)}) + O(1) \)

\[ \left( q - \sum_{i=1}^{q} \frac{1}{p_i + 1} \right) T(r,f^{(k)}) \leq \sum_{i=1}^{q} \frac{p_i}{p_i + 1} \overline{N}_{r_i}(r,a_i,f^{(k)}) + N(r,b,f^{(k)}) + N(r,f^{(k)}) + S(r,f^{(k)}) \]

\[ + N(r,b,f^{(k)}) + N(r, f^{(k)}) + S(r, f^{(k)}) \]

\[ \left( \sum_{i=1}^{q} \frac{p_i}{p_i + 1} + \frac{2p_2}{p_2 + 1} \right) T(r,f^{(k)}) \leq \frac{p_2}{p_2 + 1} \left[ \sum_{i=1}^{q} \overline{N}_{r_i}(r,a_i,f^{(k)}) \right] + N(r,b,f^{(k)}) + N(r,f^{(k)}) + S(r,f^{(k)}) \]

\[ \sum_{i=1}^{q} \overline{N}_{r_i}(r,a_i,f^{(k)}) \leq N(r,0,f^{(k)}) \]

\[ \leq T(r,0,f^{(k)}) \]

\[ \leq T(r, f) + T(r, f^{(k)}) \]

Dividing by \( T(r,f) \) and taking limit as \( r \to \infty \), from Theorem 1.4,

\[ \sum_{i=1}^{q} \frac{p_i}{p_i + 1} + \frac{2p_2}{p_2 + 1} \leq \frac{2p_2}{p_2 + 1} + \limsup_{r \to \infty} \frac{N(r,b,f^{(k)})}{T(r,f)} + \limsup_{r \to \infty} \frac{N(r,f^{(k)})}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r,f^{(k)})}{T(r,f)} \]
\[
\sum_{i=1}^{q} \frac{p_i}{p_i + 1} \leq 0,
\]

which is a contradiction to our hypothesis.

This implies \( f \equiv f^{(k)} \)

**CONSEQUENCE:**

As \( \sum_{i=3}^{q} \frac{p_i}{p_i + 1} > 0 \),

If there are at least 3 shared values \( a_1, a_2, a_3 \) for simple zeros of \( f(z) \), then \( f = f^{(k)} \).