CHAPTER-III

SOME FIXED POINT THEOREMS IN MENGER SPACES
Chapter 3

Some Fixed Point Theorems in Menger Spaces

3.1 Introduction

Menger [86] introduced the notion of probabilistic metric space (or statistical space or Menger space) which is a generalization of metric space and the study of this space was expanded rapidly with the pioneering work of Schweizer and Sklar [123] & Stevens [140]. Bharucha Reid [9] set out the tradition of proving fixed point theorems in Menger space. Since that time a substantial literature has been developed on this topic. In recent years, some interesting fixed point theorems for four self maps or a collection of maps satisfying contractive type condition in Menger space have been reported in the literature e.g., Singh et al [130], Cho [21, 22], Mishra [91], Singh et al [132]-[135], Kutukcu [73, 74]. These theorems invariably require a commutative or compatibility condition and a contractive condition besides assuming continuity of at least one of the mappings and each theorems aims at weakening one or more of these conditions.

In our first result we employ the notion of reciprocal continuity to obtain common fixed point theorem in Menger space in which the fixed point may be a point of discontinuity. We also investigate the relationship between continuity of mappings and reciprocal continuity in the setting of Menger spaces. Our result improves recent
result of Singh and Chauhan [132] in Menger spaces and extends many known results in metric spaces. Using the notion of reciprocal continuity of mappings we can widen the scope of many interesting fixed point theorems on Menger spaces as well as fuzzy metric spaces (eg. Kutukcu [73, 74], Singh et al [132]-[135], Chug [23], Khan et al [68]).

Some basic definitions for this setting have been given in the first chapter of this thesis, in this section we recall some other useful definitions and known results in Menger space. We begin with;

**Definition 3.1.1** [132]: A sequence \( \{p_n\} \) in a Menger space \( X \) is said to converge to a point \( p \in X \) (written as \( p_n \rightarrow p \)) if for \( \epsilon > 0 \) and \( \lambda > 0 \), there is an integer \( M(\epsilon, \lambda) \) such that \( F_{p_n, p}(\epsilon) > 1 - \lambda \) for all \( n \geq M(\epsilon, \lambda) \).

The sequence is said to be Cauchy sequence if for each \( \epsilon > 0 \) and \( \lambda > 0 \) there exists an integer \( M(\epsilon, \lambda) \) such that \( F_{p_n, p_m}(\epsilon) > 1 - \lambda \) for all \( n, m \geq M(\epsilon, \lambda) \).

A Menger space is said to be complete if every Cauchy sequence converges to a point in it.

**Definition 3.1.2** [92]: Two self-maps \( A \) and \( S \) of a Menger space \( (X, F, t) \) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence point, i.e. if \( Ap = Sp \) for some \( p \in X \) then \( ASp = SAp \).

**Definition 3.1.3** [132]: Two self-maps \( A \) and \( S \) of a Menger space \( (X, F, t) \) are called compatible if \( F_{ASp_n, SAp_n}(x) \rightarrow 1 \) for all \( x > 0 \), whenever \( \{p_n\} \) is a sequence in \( X \) such that \( \{Ap_n\} \rightarrow u \), \( \{Sp_n\} \rightarrow u \), for some \( u \in X \) as \( n \rightarrow \infty \).

**Proposition 3.1.1**: Two self-maps \( A \) and \( S \) of a Menger space \( (X, F, t) \) are compatible then they are weakly compatible.
[However, the converse of the above proposition need not be true, will be shown by means of an example 3.2.1 after Theorem 3.2.1, below]

**Lemma 3.1.1**  [130] Let \( \{p_n\} \) be a sequence in a Menger space \((X, F, t)\) with continuous \( t \)-norm and \( t(x, x) \geq x \). Suppose, for all \( x \in [0,1] \) there exists \( k \in (0,1) \) such that

\[
F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x), \text{ where } n \in \mathbb{N}.
\]

Then \( \{p_n\} \) is a Cauchy sequence in \( X \).

**Lemma 3.1.2**  [92] Let \((X, F, t)\) be a Menger space if there exists \( k \in (0,1) \) such that for \( p, q \in X \), \( F_{pq}(kx) \geq F_{pq}(x) \), then \( p = q \).

**Proof:** As \( F_{pq}(kx) \geq F_{pq}(x) \), we have, \( F_{pq}(x) \geq F_{pq}(k^{-1}x) \). By repeated application of above inequality, we get,

\[
F_{pq}(x) \geq F_{pq}(k^{-1}x) \geq F_{pq}(k^{-2}x) \geq \cdots \geq F_{pq}(k^{-m}x) \ldots, \text{ } m \in \mathbb{N}
\]

which \( \rightarrow 1 \) as \( m \rightarrow \infty \). Hence, \( F_{pq}(x) = 1 \), for all \( x > 0 \) and we get \( p = q \). \( \square \)

Now, in the following we show that the continuity of one of the mappings in compatible pair \((L, AB)\) implies their reciprocal continuity under some contractive condition.

**Theorem 3.1.1:** Let \( A, B, S, T, L \) and \( M \) be self maps on a complete Menger Space \((X, F, t)\) where \( t \) is any continuous \( t \)-norm such that \( t(a,a) \geq a \) for all \( a \in [0,1] \) satisfying:

1. \( L(X) \subseteq ST(X), \quad M(X) \subseteq AB(X), \)
2. \( AB = BA, \quad ST = TS, \quad LB = BL, \quad MT = TM \)
3. the pair of maps \((M, ST)\) is weak compatible,
(4) there exists \( k \in (0, 1) \) such that

\[
F_{Lp, Mg}(kx) \geq \min \{F_{ABp, Lp}(x), F_{STq, Mg}(x), F_{STq, Lp}(\beta x), F_{ABp, Mg}((2 - \beta)x), F_{ABp, STq}(x)\},
\]

for all \( p, q \in X, \beta \in (0, 2) \) and \( x > 0 \).

Then the continuity of one of the mappings in compatible pair \((L, AB)\) implies their reciprocal continuity.

Proof: Part-I: Let us assume that \( AB \) is continuous in the compatible pair of mappings \( L \) and \( AB \) and then we will show that the mapping pair \((L, AB)\) is reciprocal continuous.

For this let \( \{x_n\} \) be any sequence in \( X \) such that \( \lim_{n \to \infty} Lx_n = z \) and \( \lim_{n \to \infty} ABx_n = z \) for some \( z \in X \). To prove our assertion we shall show that \( LABx_n \to Lz \) and \( ABLx_n \to ABz \) as \( n \to \infty \).

Since \( AB \) is continuous we get, \( ABABx_n \to ABz \) and \( ABLx_n \to ABz \) as \( n \to \infty \). Now compatibility of \( L \) and \( AB \) implies that \( \lim_{n \to \infty} F_{LABx_n, ABLx_n}(x) = 1 \), i.e. \( LABx_n \to ABz \) as \( n \to \infty \). Also since \( L(X) \subseteq ST(X) \), for each \( n \), there exists \( \{y_n\} \) in \( X \) such that \( LABx_n = STy_n \) and since \( ABABx_n \to ABz \), \( LABx_n \to ABz \), \( ABLx_n \to ABz \) therefore \( STy_n \to ABz \) as \( n \to \infty \). Now we shall show that \( My_n \to ABz \) as \( n \to \infty \). For this, by contractive condition, we have,

\[
F_{ABz, My_n}(kx) = F_{LABx_n, My_n}(kx) \\
\geq \min \{F_{ABABx_n, LABx_n}(x), F_{STy_n, My_n}(x), F_{STy_n, LABx_n}(\beta x), F_{ABABx_n, STy_n}(x)\} \\
\geq \min \{F_{ABz, ABz}(x), F_{x, z}(x), F_{ABz, ABz}(\beta x), F_{ABz, ABz}((2 - \beta)x), F_{ABz, ABz}(x)\},
\]

which implies that \( My_n \to ABz \) as \( n \to \infty \) (by lemma 3.1.2 and taking \( \beta = 1 \)). Now

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by the contractive condition, we get,

$$ F_{Lz, ABz}(kx) = F_{Lz, Myn}(kx) $$

$$ \geq \min\{F_{ABz, Lz}(z), F_{STyn, Myn}(z), F_{STyn, Lz}(\beta x), F_{ABz, Myn}((2 - \beta)x)\}, $$

which implies, $Lz = ABz$ as $n \to \infty$ (by Lemma 3.1.2 and taking $\beta = 1$). Thus $ABLx_n \to ABz$ and $LABx_n \to ABz = Lz$ as $n \to \infty$. Therefore, pair of maps $(L, AB)$ is reciprocal continuous in $(X, F, t)$.

**Part-II:** Suppose that $L$ is continuous in the compatible pair of mappings $L$ and $AB$ then we show that $(L, AB)$ is reciprocal continuous.

For this let $\{x_n\}$ be any sequence in $X$ such that $Lx_n \to z$ and $LABx_n \to z$ for some $z \in X$ as $n \to \infty$.

We need to show that $LABx_n \to Lz$ and $ABLx_n \to ABz$ as $n \to \infty$. Since $L$ is continuous, we have $LLx_n \to Lz$, $LABx_n \to Lz$ as $n \to \infty$.

Now compatibility of $L$ and $AB$ give us $ABLx_n \to Lz$ as $n \to \infty$. Then by routine calculation (using contractive condition), we get $Lz = ABz$ which implies that $ABLx_n \to Lz = ABz$ as $n \to \infty$. Therefore $L$ and $AB$ are reciprocal continuous in $(X, F, t)$. \(\square\)

In the above theorem we have shown that in the setting of the theorem 3.1.1 of Singh et al [132] continuity of one of the mappings in compatible pair implies their reciprocal continuity. Therefore the condition of continuity of one of the mapping in compatible pair $(L, AB)$ can be further replaced by the weaker notion of reciprocal continuity which still assure the existence of common fixed point for maps but does not force the maps to be continuous even at common fixed point.
3.2 A Common Fixed Point Theorem in Menger Space

Now as an application of the relationship between continuity of the mappings and reciprocal continuity established in the above Theorem 3.1.1, we prove the following theorem which improves the result of Singh et al [132] and also give an example which illustrate our Theorem 3.2.1 and demonstrates that the notion of reciprocal continuity of mappings is weaker than the continuity of maps.

The following theorem was proved by Singh & Jain [132]

Theorem 3.2.1 : Let A, B, S, T, L and M be self maps on a complete Menger Space \((X, F, t)\) where \(t\) is any continuous \(t\)-norm such that \(t(a, a) \geq a \) for all \(a \in [0,1]\) satisfying:

(a) \(L(X) \subseteq ST(X), M(X) \subseteq AB(X)\),
(b) \(AB = BA, ST = TS, LB = BL, MT = TM\)
(c) either \(AB\) or \(L\) is continuous
(d) \((L, AB)\) is compatible and \((M, ST)\) is weakly compatible,
(e) there exists \(k \in (0,1)\) such that

\[
F_{Lp,Mq}(kx) \geq \min \{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2 - \beta)x), F_{ABp,STq}(x)\},
\]

for all \(p, q \in X, \beta \in (0,2)\) and \(x > 0\). Then \(A, B, S, T\) and \(M\) have a unique common fixed point in \(X\).

Our result states that;
Theorem 3.2.2 : Let $A, B, S, T, L$ and $M$ be self maps on a complete Menger Space $(X, F, t)$ where $t$ is any continuous $t$-norm such that $t(a, a) \geq a$ for all $a \in [0,1]$ satisfying:

(i) $L(X) \subseteq ST(X), M(X) \subseteq AB(X),$

(ii) $AB = BA, ST = TS, LB = BL, MT = TM$

(iii) $(M, ST)$ is weakly compatible,

(iv) there exists $k \in (0,1)$ such that

$$F_{Lp,Mq}(kx) \geq \min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2-\beta)x),$$

$$F_{ABp,STq}(x)\},$$

for all $p, q \in X, \beta \in (0,2)$ and $x > 0$. If mapping pair $(L, AB)$ is reciprocally continuous and compatible then $A, B, S, T, L$ and $M$ have a unique common fixed point in $X$.

Proof: Let $x_0$ be an arbitrary element of $X$, then from condition (i) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, 3, \ldots$. Now putting $p = x_{2n}, q = x_{2n+1}$ and $\beta = 1 - h$ with $k \in (0,1)$ in contractive condition (iv), we get for $x > 0$,

$$F_{Lx_{2n},Mx_{2n+1}}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},Lx_{2n}}((1-h)x),$$

$$F_{ABx_{2n},STx_{2n+1}}((1+h)x), F_{ABx_{2n},STx_{2n+1}}(x)\}$$

$$F_{y_{2n-1},y_{2n+1}}(kx) \geq \min\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), 1, F_{y_{2n-1},y_{2n+1}}((1+h)x),$$

$$F_{y_{2n-1},y_{2n}}(x)\},$$

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As $t$-norm is continuous, letting $h \to 1$ we get,

$$F_{y_{2n}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n}, y_{2n+1}}(hx)\} \geq \min\{F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n}, y_{2n+1}}(hx)\}.$$

Thus, $F_{y_{2n}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x)\}$. Similarly $F_{y_{2n+1}, y_{2n+2}}(kx) \geq \min\{F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n+1}, y_{2n+2}}(x)\}$. Thus, for all $n$ even or odd we have:

$$F_{y_{n}, y_{n+1}}(kx) \geq \min\{F_{y_{n-1}, y_{n}}(x), F_{y_{n}, y_{n+1}}(x)\}$$

Consequently,

$$F_{y_{n}, y_{n+1}}(x) \geq \min\{F_{y_{n-1}, y_{n}}(k^{-1}x), F_{y_{n}, y_{n+1}}(k^{-1}x)\}.$$

By repeated application of above inequality, we get:

$$F_{y_{n}, y_{n+1}}(x) \geq \min\{F_{y_{n-1}, y_{n}}(k^{-m}x), F_{y_{n}, y_{n+1}}(k^{-m}x)\}.$$

Since $F_{y_{n}, y_{n+1}}(k^{-m}x) \to 1$ as $n \to \infty$, it follows that $F_{y_{n}, y_{n+1}}(kx) \geq F_{y_{n-1}, y_{n}}(x)$, for all $n \in \mathbb{N}$ and for all $x > 0$. Therefore, by Lemma 3.1.1, $\{y_{n}\}$ is a Cauchy sequence in $X$. Since $X$ is complete hence $\{y_{n}\}$ converges to a point $z$ (say) in $X$ and hence its subsequences $\{mx_{2n+1}\}$, $\{STx_{2n+1}\}$, $\{Lx_{2n}\}$ and $\{ABx_{2n+1}\}$ converges to the same point $z \in X$.

Now reciprocal continuity and compatibility of the pair $(L, AB)$ gives us that

$$LABx_{2n} \to Lz, \quad ABLx_{2n} \to ABz \quad \text{and} \quad F_{LABx_{2n}, ABLx_{2n}} \to 1, \quad \text{as} \quad n \to \infty.$$
Hence $Lz = ABz$. Now putting $p = z$, $q = x_{2n+1}$ with $\beta = 1$ in contractive condition (iv) we get,

$$F_{Lz,Mx_{2n+1}}(kx) \geq \min\{F_{ABz,Lz}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},Lz}(x), F_{ABz,Mx_{2n+1}}(x), F_{ABz,STx_{2n+1}}(x)\}.$$ 

Letting $n \to \infty$, we get;

$$F_{Lz,z}(kx) \geq \min\{F_{z,Lz}(x), F_{z,z}(x), f_{z,Lz}(x), F_{Lz,z}(x), F_{z,Lz}(x)\}$$

i.e. $$F_{Lz,z}(kx) \geq F_{Lz,z}(x),$$

which gives $Lz = z$, thus we have $ABz = Lz = z$. Putting $p = Bz$, $q = x_{2n+1}$ with $\beta = 1$ in contractive condition, we get,

$$F_{LBz,Mx_{2n+1}}(kx) \geq \min\{F_{ABBz,LBz}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},LBz}(x), F_{ABBz,Mx_{2n+1}}(x), F_{ABBz,STx_{2n+1}}(x)\}.$$ 

As $BL = LB; \ AB = BA$, we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = R(ABz) = Bz$. Letting $n \to \infty$, we get,

$$F_{Bz,z}(kx) \geq \min\{F_{Bz,Bz}(x), F_{z,z}(x), F_{z,Bz}(x), F_{Bz,z}(x), F_{Bz,z}(x)\},$$

i.e. $$F_{Bz,z}(kx) \geq F_{Bz,z}(x).$$

Hence by lemma 3.1.2 we have, $Bz = z$ and $ABz = z \Rightarrow Az = z$. Thus $z = Az = Bz = Lz$. Since $L(X) \subset ST(X)$, there exists $u \in X$ such that $z = Lz = STu$. Putting $p = x_{2n}$, $q = u$ with $\beta = 1$ in condition (iv), we get,

$$F_{Lx_{2n},Mu}(kx) \geq \min\{F_{ABz_{2n},Lx_{2n}}(x), F_{STu,Mu}(x), F_{STu,Lx_{2n}}(x), F_{ABz_{2n},Mu}(x), F_{ABz_{2n},STu}(x)\}.$$
Letting \( n \to \infty \) we get,
\[
F_{z,Mu}(kx) \geq \min\{F_{z,z}(x), F_{z,Mu}(x), F_{z,z}(x), F_{z,Mu}(x), F_{z,z}(x)\}
\geq F_{z,Mu}(x).
\]

Thus by lemma 3.1.2 \( Mu = z \). Hence \( STu = z = Mu \). Since the pair of maps \((M, ST)\) is weakly compatible, we have \( STMu = MSTu \) thus \( STz = Mz \). Now putting \( p = x_{2n}, \quad q = z \) with \( \beta = 1 \) in condition (iv), we have,
\[
F_{Lx_{2n},Mz}(kx) \geq \min\{F_{ABx_{2n},Lz}(x), F_{STz,Mz}(x), F_{STz,Lz_{2n}}(x), F_{ABz_{2n},Mz}(x), F_{ABz_{2n},STz}(x)\}.
\]

Letting \( n \to \infty \), we get,
\[
F_{z,Mz}(kx) \geq \min\{F_{z,z}(x), F_{STz,Mz}(x), F_{STz,z}(x), F_{z,Mz}(x), F_{STz}(x)\},
\]
i.e.
\[
F_{z,Mz}(kx) \geq \min\{F_{z,z}(x), F_{Mz,Mz}(x), F_{Mz,z}(x), F_{z,Mz}(x), F_{z,Mz}(x)\},
\]
i.e.
\[
F_{z,Mz}(kx) \geq F_{z,Mz}(x).
\]
Thus we have \( z = Az = Bz = Lz = Mz \). Similarly putting \( p = x_{2n}; \quad q = Tz \) with \( \beta = 1 \) and using condition (ii) we can easily show that \( Tz = z \), since \( STz = z \) thus \( Sz = z \) and finally we have \( z = Az = Bz = Sz = Tz = Lz = Mz \). Hence \( z \) is a common fixed point of \( A, B, S, T, L \) and \( M \).

**Uniqueness:** Let \( z' \) be another fixed point of mappings \( A, B, S, T, L \) and \( M \), then by contractive condition, with \( \beta = 1 \) we get,
\[
F_{Lz,Mz'}(kx) \geq \min\{F_{ABz,Lz}(x), F_{STz,Mz'}(x), F_{STz',Lz}(x), F_{ABz,Mz}(x), F_{ABz,Tz'}(x)\},
\]
i.e.
\[
F_{z,z'}(kx) \geq \min\{F_{z,z}(x), F_{z',z'}(x), F_{z',z}(x), F_{z,z}(x), F_{z,z}(x)\},
\]
i.e.
\[
F_{z,z'}(kx) \geq F_{z,z'}(x).
\]
by lemma 3.1.2 we have, \( z = z' \). Thus \( z \) is a unique common fixed point of \( A, B, S, T, L \) and \( M \).
We now give an example, which not only illustrates our Theorem 3.2.1 but also shows that the notion of reciprocal continuity is weaker than the continuity condition of related maps.

Example 3.2.1: Let \((X,d)\) be a metric space where \(X = [0,3]\) and \((X,F,t)\) be the induced Menger space with \(F_{pq}(\epsilon) = H(\epsilon - d(p,q))\), for all \(p, q \in X\) and for all \(\epsilon > 0\), \(t(a,b) = \min(a, b)\), for all \(a, b \in [0,1]\). Define self maps \(A, B, S, T, L\) and \(M\) on \(X\) as follows:

\[
Lx = 1 \text{ if } 0 < x < 2 \text{ and } 2 < x \leq 3, L2 = 0;
Mx = 0 \text{ if } 0 < x < 1, 1 < x < 2, 2 < x \leq 3, M1 = M2 = 1;
(A = B)Ax = 0 \text{ if } 0 \leq x < 1, 1 < x < 2, 2 < x \leq 3, A1 = A3 = 1, A2 = 2;
Sx = 0 \text{ if } 0 < x < 1, 1 < x < 2, 2 < x \leq 3, S1 = 1, S2 = 0, S0 = S3 = 2;
Tx = 0 \text{ if } 0 < x < 1, 1 < x < 2, 2 < x \leq 3, T1 = 1, T2 = 2.
\]

Then the maps \(A(=B), S, T, L, M\) satisfy all the conditions of the above Theorem 3.2.1 with \(k \in (1/2,1)\) and \(\beta = 1\) and have a unique common fixed point \(x = 1\). It may be noted that in this example \(L(X) = \{0,1\} \subseteq ST(X) = \{0,1,2\}, M(X) = \{0,1\} \subseteq AB(X) = \{0,1,2\}\) and the pair \((L,AB)\) is reciprocally continuous for a sequence \(\lim_{n \to \infty} x_n = 1\) in \(X\). Also \((L,AB)\) is commuting pair of maps and hence compatible. But neither \(L\) nor \(AB\) is continuous at the fixed point.

Remark 3.2.1 The maps \(A(=B), S, T, L, M\) are discontinuous even at the common fixed point \(x = 1\).

Remark 3.2.2 The known common fixed point theorems involving a collection of maps in Menger spaces as well as fuzzy metric spaces require one of the maps in compatible pair to be continuous. For example, main theorems of Singh et al [132]-[136] assumes at
least one of the maps to be continuous in compatible pair of maps. Likewise, theorem 3.1 of Kutukcu [73] assumes either $AB$ or $L$ to be continuous maps. One more theorems of Kutukcu [74] assumes the mappings $S$ to be continuous and $(S, T_n)$ to be commuting pair of maps in Menger spaces. Similarly, the main theorem of Chug et al [23] assumes one of the mappings $A, B, S$ or $T$ to be continuous in fuzzy metric spaces. The present theorem however does not require any of the mappings to be continuous and hence all the results mentioned above can be further improved and generalized in the spirit of our Theorem 3.2.1. Further, since every metric space induces a Menger space, our result extends the results of Pant [97, 98], Fishrer [35], Jungck [61], Jachymski [56] for six mappings in metric spaces.

**Remark 3.2.3** It is obvious that in most of the fixed point theorems in Menger spaces as well as fuzzy metric spaces to prove the sequence of iterates of a point is a Cauchy sequence, a particular class of $t$-norm is required. In our Theorem 3.2.1 above we have assumed the $t$-norm as min norm, however, adopting the approach of Liu et al [79] one can easily replace the condition of min norm by a larger class of $t$-norm called Hadzic type $t$-norm (in short H type $t$-norm).

### 3.3 Another Common Fixed Point Theorem in Menger Space

Next we prove the following theorem which improves the result of Singh et al [132] and give an example which illustrate our theorem.

**Theorem 3.3.1** : Let $(X, F, t)$ be a Menger space with $t(a, a) \geq a$ for all $a \in [0, 1]$ and let $A, B, S, T, L$ and $M$ be self maps of $(X, F, t)$ such that

(i) $L(X) \subset ST(x), \ M(X) \subset AB(X)$;

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(ii) there exists $k \in (0,1)$ such that
\[
F_{Lp,Mq}(kx) \geq \min\{F_{AB,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2 - \beta)x), F_{ABp,STq}(x)\},
\]
for all $p, q \in X, \beta \in (0,2)$ and $x > 0$. If one of $(L(X), F, t)$ or $(ST(X), F, t)$ or $(M(X), F, t)$ or $(AB(X), F, t)$ is complete subspace of $X$ then

(a) the pair of maps $(L, AB)$ and $(M, ST)$ have a coincidence point;

(b) further, if the pairs $(L, AB)$ and $(M, ST)$ are coincidentally commuting then $AB, ST, L$ and $M$ have a unique common fixed point say $z$.

(c) More over if the pairs $(A, B), (S, T), (L, B), (M, S), (M, T)$ commute at $z$ then $z$ also remains the unique common fixed point of $A, B, S, T, L$ and $M$.

**Proof:** Let $x_0 \in X$ be arbitrary. From condition (i) we can construct a sequence $\{y_n\}$ in $X$ such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$. By a routine calculation we can easily show that $\{y_n\}$ is a Cauchy sequence in $X$. Now suppose $ST(X)$ is complete then the sequence $\{y_n\}$ must have limit (say) $z$ in $ST(X)$ so there exists $u \in ST(X)$ then $STu = z$. Now we will prove $Mu = z$. For this put $p = x_{2n}, q = u & \beta = 1$ in condition (ii) to get
\[
F_{Lx_{2n},Mu}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STu,Mu}(x), F_{STu,Lx_{2n}}(x), F_{ABx_{2n},Mu}(x), F_{ABx_{2n},STu}(x)\}.
\]
Letting $n \to \infty$, we get, $F_{z,Mu}(kx) \geq F_{z,Mu}(x)$. Therefore by lemma 3.1.2 we get $z = Mu$. Combining all this we have that pairs $(M, ST)$ has a coincidence point. Since $M(X) \subseteq AB(X)$ there exists a point $v \in X$ such that $ABv = Mu$. Now we prove $Lv = Mu$. For this put $p = v, q = u$ with $\beta = 1$ in condition (ii) and using similar
argument as we prove \( Lv = Mu \). Thus \( Lv = Mu = ABv = STu = z \), which shows that \((L, AB)\) also have coincidence point. If we assume \( AB(X) \) is complete then by the same argument as previous one may establishes (a) and (b) as well. The remaining two cases pertain essentially to the previous cases. Indeed, if \( L(X) \) is complete, then \( z \in L(X) \subseteq ST(X) \).

Similarly if \( M(X) \) is complete, then \( z \in M(X) \subseteq AB(X) \). Thus in each case (a) and (b) are completely established. To prove (c), note that \((L, AB)\) and \((M, ST)\) are coincidentally commuting. So we have \( STMu = MSTu = 2 \), which shows that \((L, AB)\) also have coincidence point. If we assume \( AB(X) \) is complete then by the same argument as previous one may establishes (a) and (b) as well. The remaining two cases pertain essentially to the previous cases. Indeed, if \( L(X) \) is complete, then \( z \in L(X) \subseteq ST(X) \).

Now we shall show \( Lz = z \). For this, putting \( p = z, q = x_{2n+1} \) with \( \beta = 1 \) in condition (ii) we get,

\[
F_{Lz, Mx_{2n+1}}(x) \geq \min\{F_{ABz, Lz}(x), F_{STx_{2n+1}, Mx_{2n+1}}(x), F_{STx_{2n+1}, Lz}(x), F_{ABz, STx_{2n+1}}(x)\}.
\]

Letting \( n \to \infty \), we have, \( F_{Lz,z}(x) \geq F_{Lz,z}(x) \). Therefore by lemma 3.1.2 we get \( Lz = z \). Similarly we may prove \( Mz = z \). Thus \( ABz = Lz = z = Mz = STz \), i.e. \( z \) is a common fixed point of \( AB, ST, L \) and \( M \).

If the pairs \((A, B), (S, T), (L, B), (L, A), (M, T), (M, S)\) commute at \( z \), then

\[
\begin{align*}
A(AB) &= A(BA) = AB(Az) = A(Lz) = L(Az), \\
B(AB) &= B(BA) = BA(Bz) = AB(Bz) = B(Lz) = L(Bz), \\
S(Tz) &= T(ST) = TS(Tz) = ST(Tz), \quad Tz = T(Mz) = M(Tz),
\end{align*}
\]

which shows that the pair \((AB, L)\), where as \(Sz \) and \( Tz \) are common fixed point of the pair \((ST, M)\). Now due to the uniqueness of common fixed point of both the pairs one gets, \( z = Az = Bz = Sz = Tz = Mz = Lz \). \( \Box \)
Now we give an example to illustrate our Theorem 3.3.1:

Example 3.3.1: Let \((X,d)\) be a metric space where \(X = [0,4]\) and \(d\) is the usual metric. \((X,F,t)\) be the induced Menger space with \(F_{pq}(\epsilon) = H(\epsilon - d(p,q))\), for all \(p, q \in X\) and for all \(\epsilon > 0\). Define self maps \(A, B, S, T, L, M\) as follows:

\[
Lx = 1 \text{ if } x \in [0,1] \cup (2,4], \quad Lx = 0 \text{ if } x \in (1,2]
\]

\[
Mx = 0 \text{ if } x \in (0,1) \cup (1,2], \quad M3 = 0, \quad M1 = 1 = M2, \quad Mx = 1 \text{ if } x \in (2,3) \cup (3,4],
\]

\[
A0 = 0, \quad Ax = 1 \text{ if } x \in (0,1] \cup (2,4], \quad A2 = 2, \quad Ax = (x + 1)/2 \text{ if } x \in (1,2).
\]

\[
B0 = 0, \quad Bx = 2 \text{ if } x \in (0,1) \cup (1,3], \quad B1 = 1, \quad Bx = x/3 \text{ if } x \in (3,4].
\]

\[
S0 = 2, \quad Sx = 1 \text{ if } x \in (0,1] \cup (2,4], \quad Sx = (x + 1)/2 \text{ if } x \in (1,2), \quad S2 = 0.
\]

\[
T0 = 0 \text{ if } x \in [0,1) \cup (1,2) \cup (2,3], \quad T1 = 1, \quad T2 = 2, \quad Tx = x/3 \text{ if } x \in (3,4].
\]

Then the maps \(A = B), S, T, L, M\) satisfy all the conditions of the above theorem 2 with \(k \in (1/2,1)\) and \(\beta = 1\) and have a unique common fixed point \(x = 1\). To show that \((L,AB)\) and \((M,ST)\) are non-compatible, we may consider the sequence \(\{x_n\} = 3 + 1/n; \ n = 1,2,3 \ldots\)

Remark 3.3.1: The known common fixed point theorems involving a collection of mappings assume explicitly continuity and compatibility condition on maps besides a condition of completeness of the space. For example, Singh et al [132] in their main result 3.1 assumes one of the mappings to be continuous in compatible (or compatible pair of type-(A), or semi-compatible) pair of mappings \((L,AB)\). Theorem 4.1 of Cho [21] assumes one of \(A, B, S\) and \(T\) to be continuous with \((A,S)\) and \((B, T)\) to be compatible of type-(A). Likewise, the main theorem of Mishra et al [92] assume \((A, S)\) and \((B, T)\) to be compatible and \(S\) and \(T\) to be continuous. In the above mentioned theorem the space was considered as a complete Menger space. The present theorem however does not require any of these conditions. Thus our theorem improves the many known
common fixed point theorems in three respects:

(i) We require the notion of (non-compatible) weakly compatible maps instead of compatible maps.

(ii) We do not require any mapping to be continuous even at the common fixed point.

(iii) We require only range of any one of the mappings to be complete.