CHAPTER-I

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1.1 In this chapter we take a brief general look at the subject and discuss some simple notions and techniques on the fixed point theorems.

The main object of this thesis is to give a unified account of the classical topics in fixed point theory that lie on the borderline of topology and nonlinear functional analysis.

Functional analysis is embodies the abstract approach in analysis where one studies classes of functions rather than an individual function. Functional analysis also provides one of the major links between abstract mathematics and its applications.

The essence of functional analysis lies on the fact that it provides a powerful tool to discover problems and solutions occurring in pure, applied and social sciences. Fixed point theorems hold a fascination for mathematicians and they are applicable to a variety of mathematical and physical approaches.

The origins of the fixed point theory, which deals the later part of the nineteenth century, rest in the use of successive approximations to establish the existence and uniqueness of solutions, particularly to differential equations. This method is associated with the names of such celebrated mathematicians as Cauchy, Liouville, Lipschitz, Peano, Fredholm and, especially, Picard.
In fact the precursors of a fixed point theoretic approach are explicit in the work of Picard. However, it is the polish mathematician Stefan Banach who is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. Around 1922, Banach recognized the fundamental role of 'metric completeness' a property shared by all of the scopes commonly exploited in analysis.

Our view is that analysis itself is basic and that abstract theories to which it leads are primarily of interest as tool which may be used in treating problems in analysis. For many years, activity in metric fixed point theory was limited to minor extensions in Banach's contraction mapping principle and its manifold applications. The theory gained new impetus largely as a result of the pioneering work of Felix Browder in the mid-nineteen sixties and the development of the nonlinear functional analysis as an active and vital branch of mathematics. Pivotal in this development were the 1965 existence theorems of Browder, Gohde, and Kirk and the early metric results of Edelstein.

The presence or absence of a fixed point is an intrinsic property of a map. However, many necessary or sufficient conditions for the existence of such points involve a mixture of algebraic, order theoretic, or topological properties of the mapping or its domain. Metric fixed point theory is a rather loose knit
branch of fixed point theory concerning methods and results that involve properties of an essentially isometric nature.

The first result on fixed point was proved by Henri Poincare [1854-1913] [112] in 1895. The study of fixed point theorems and their applications through initiated long ago, still continue to be a highly interesting and useful area of investigations. It has been unequivocally established that, in many cases, applications of mathematical principle is mandatory for solving equations. Contextually question arises whether a particular equation has any solution or otherwise, it is called existence theorem provide credence to the concept that solutions do exist. Fixed points have long been used in analysis to solve various kinds of differential and integral equations. The work of Cauchy on differential equations is fundamental concept of existence theorems in the field of Mathematics.

Let \( f \) be a function of the real variable \( x \), continuous in the closed interval \([a,b]\) and assuming the values of different signs at its end points. Then the equation

\[
(1.1.1) \quad f(x)=0 \text{ for all values of } x \geq 0, \text{ has at least one solution inside the interval.}
\]

Existence theorems are often expressed in the form of "fixed point principles".
Example (1.1.1):

Let us write the equation (1.1.1) in the form
\[ \lambda f(x) + x = x \] where \( \lambda \) is a positive parameter, writing:

(1.1.2) \[ F(x) = x \] where \( \lambda f(x) + x = F(x) \)

Therefore in geometrical terms, a theorem ensuring the existence of a solution of equation (1.1.2) is formulated as fixed point theorem:

If \( F \) is a continuous function with a closed interval into itself, then the function has at least one fixed point.

A fixed point is said to be stable if it attracts nearby solutions to itself and unstable otherwise.

Frechet [51] introduced the idea of an abstract metric space. Brouwer [11] proved perhaps the first fundamental result in the area of fixed point theory and its applications which states that "if \( C \) denotes a unit closed ball with center at origin in \( \mathbb{R}^n \) and \( T \) be a continuous self mapping of \( C \) onto itself. Then \( T \) has a fixed point in \( C \), or \( Tx = x \) has a solution.

Meanwhile, a Polish mathematician Stefan Banach [5] proved an important result on fixed points, so called "Banach's contraction mapping principle". The statement of his theorem is as follows:

If \( T \) be a contraction map of a complete metric space \( (X,d) \) to itself satisfying the condition:
\[ d(Tx,Ty) \leq k d(x,y) \] for all \( x, y \) in \( X \) and \( 0 \leq k < 1 \);

Then \( T \) has a unique fixed point \( x_0 \). Moreover, for each \( x_0 \in X \),
for each $x \in X$, the Picard sequence $\{T^n(x)\}$ converges to $x_0$.

$$\lim_{n \to \infty} T^n x = x_0$$

and in fact for each $x \in X$. 

$$d\left( T^n x, x_0 \right) \leq \frac{k^n}{1-k} d(x, Tx)$$

for each $x \in X$ and $n \geq 1$.

A mapping $T : X \to X$ is said to be a contraction map if

$$d(Tx, Ty) \leq k d(x, y)$$

for all $x, y \in X$ and $0 \leq k < 1$.

The Banach contraction principle is the simplest and one of the most versatile elementary results in the fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map, it produces approximations of any required accuracy, and moreover, even the number of iteration needed to get a specified accuracy can be determined.

The application of Banach's fixed point theorem is to find the unique solution of linear algebraic equations, differential equations, integral equations as well as to implicit function theorem.

1.2 **Contraction concept for fixed point**:

A Contraction map is continuous, however, a continuous map is not contraction.

**Example (1.2.1):**

Consider a mapping $T : R \to R$ given by

$$T(x) = x + p, p > 0$$

is continuous but not contraction.
If the space $X$ is not complete, then it does not guarantee the fixed point of a contraction map.

It can be seen that a mapping $T : (0,1] \to (0,1]$ defined by

$$T(x) = x/2$$

Although it is a contraction mapping but it has no fixed point as the space $(0,1]$ is not a complete metric space. Also $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = x+5$$

is not a contraction and has no fixed point although $\mathbb{R}$ is complete.

Similarly consider the mappings

$$f : [0,1] \to [0,1]$$

and

$$g : [0,1] \to [0,1]$$

are continuous functions which satisfies

$$fg(x) = gf(x) \quad \text{for all } x \in [0,1].$$

But have not a common fixed point.

In other words, there exist $x_0 \in [0,1]$ such that, the equation

$$f(x_0) = x_0 = g(x_0)^2$$

is not satisfied.

This was main source of inspiration for a decade and several mathematicians tried to resolve the problems related to the fixed point.
There are various generalizations of the Banach theorem in arbitrary complete metric spaces where the contractive nature of the map is weakened.

Brouwer's theorem was extended to infinite dimensional spaces by Schauder [123], who proved the following:

"If \( X \) is a Banach space and \( C \) be a compact convex subset of \( X \). Let \( T: C \rightarrow C \) be a continuous map. Then \( T \) has at least one fixed point in \( C \)."

Then define compact map as follows:

"A mapping \( T: X \rightarrow X \) is called a compact map if \( T \) is continuous and \( T \) maps bounded sets to precompact sets."

A compact map is always continuous, however, a continuous map need not be compact.

**Example (1.2.2):**

An identity function is continuous but it is not compact.

R. Caccioppoli [18] observed that it is possible to replace the contraction property by the assumption of convergence. More precisely the result of Caccioppoli is as follows:

If \( (X, d) \) is a complete metric space and \( T \) is a mapping from \( X \) into itself under the conditions:

\[
\begin{align*}
(1.2.1) & \quad d(Tx, Ty) \leq ||T|| d(x, y) \\
(1.2.2) & \quad \text{(i) if } T^n(x) = T(T^{n-1}(x)) \quad \text{and} \\
& \quad \text{(ii) } T^n(y) = T(T^{n-1}(y)) \;
\end{align*}
\]
Then the sequence $\{T^n(z)\}$ converges to a fixed point $z_0 = T(z_0)$ if
\[ \sum_{n=1}^{\infty} ||T^n|| < \infty \]
where
\[ (1.2.3) \quad d(T^n(x), T^n(y)) \leq ||T^n|| d(x, y) \]

Tychonoff [147] extended the Brouwer's result to a compact convex subset of a locally convex linear topological space.

E. Rakotch [113] proposed another interesting and useful generalization of contraction mappings as follows:

**Theorem (1.2.1):** A mapping $T$ from a complete metric space $(X, d)$ into itself is called Rakotch contraction if there exists a decreasing function $\alpha(t), \alpha: \mathbb{R}^+ \to [0, 1)$ with $\alpha(t) < 1$ and for all $x, y \in X$ such that:
\[ (1.2.4) \quad d(Tx, Ty) \leq \alpha(d(x, y)) d(x, y) \]
Then $T$ has a unique fixed point.

Chu and Diaz [30] proved that if $T$ is a mapping of a complete metric space $X$ into itself is not contraction, it is possible that $T^n$ is contraction, where $n = n(x)$, is some positive integer such that:
\[ (1.2.5) \quad d(T^n x, T^n y) < k d(x, y) \]
where $k < 1$ ;
Then $T$ has a unique fixed point. This result generalized by Sehgal [125] as follows:

Let $X$ be a complete metric space and $T: X \to X$ be continuous mapping satisfying the condition that there exists a number $k < 1$ such that for each $x \in X$, there is positive integer $n = n(x)$ such that:
\[ d(T^n x, T^n y) < k d(x, y) \]
for all \( x \in X \).

Then \( T \) has a unique fixed point.

**Note:** It is easily seen that there is no need to assume that \( T^n \) is a contraction and is defined on a complete metric space. All that is needed is obtained in obtaining the conclusion of the theorem is that \( T^n \) has a unique fixed point.

**Example (1.2.3):**

Let \( T: \mathbb{R} \to \mathbb{R} \) defined by

\[
T x = 1 \quad \text{if } x \text{ is rational} \\
T x = 0 \quad \text{if } x \text{ is irrational}
\]

Then \( T^2 x = 1 \) for all \( x \).

Thus \( T^2 \) is a contraction map and hence \( T^2 \) has a unique fixed point. \([ T^2(1) = 1, \text{and thus } T \text{ has a unique fixed point } ]\)

Bailey [4] extension of Edelstein[44] proved the following:

**Theorem (1.2.2):** Let \( X \) be a compact metric space and \( T: X \to X \) be continuous. If there exists \( n = n(x, y) \) with \( d(T^n x, T^n y) < d(x, y) \) for \( x \neq y \).

Then \( T \) has a unique fixed point.

For the study of complete metric space has a unique fixed point by Kannan [79] proved the following:
**Theorem (1.2.3):** Let $T$ be a mapping from a complete metric space $(X,d)$ into itself satisfying the condition:

$$
(1.2.6) \quad d(Tx,Ty) \leq a[d(x,Tx)+d(y,Ty)] \quad \forall \ x,y \in X, \quad 0 \leq a < \frac{1}{2}.
$$

Then $T$ has a unique fixed point. A mapping given by (1.2.6) is need not be continuous.

**Example (1.2.4):**

$T: [0,1] \to [0,1]$ defined by

- $Tx = x/4$ when $x \in [0,1/2)$
- $Tx = x/5$ when $x \in [1/2,1)$

satisfies (1.2.6) but is discontinuous at $x=1/2$.

**Theorem (1.2.4):** Let $X$ be a complete metric space and $M$ be a bounded subset of $X$. Suppose $f:M \to M$, such that:

$$
(1.2.7) \quad d(fx,fy) \leq \psi d(x,y) \quad \text{for all} \ x,y \in M \ \text{where} \ \psi:[0,\infty) \to [0,\infty) \ \text{monotonic decreasing and} \ \psi(t) < t.
$$

Then $f$ has a unique fixed point. Holmes [62] generalize the result of Sehgal [125], and Chu and Diaz [30] as follows:

**Theorem (1.2.5):** If $T:X \to X$ is a continuous function on a complete metric space $X$ and if for each $x,y \in X$, there exists a $n=n(x,y)$, such that:

$$
(1.2.8) \quad d(T^n x, T^n y) < k d(x,y), \quad \text{where} \ k < 1.
$$

Then $T$ has a unique fixed point.

Reich [115] unified the mappings of Banach and Kannan, in the form studied as follows:
**Theorem (1.2.6)** The mapping $T$ from a complete metric space $(X,d)$ into itself satisfying the condition:

$$(1.2.9) \quad d(Tx, Ty) \leq a \ d(x, y) + b \ d(x, Tx) + c \ d(y, Ty) \quad \text{for all} \quad x, y \in X,$$

and $a, b, c$ are non-negative and $a+b+c < 1$.

Then $T$ has a unique fixed point in $X$.

Since the Reich's theorem is stronger than Banach's and the Kannan's theorem, can be seen from the following example:

**Example (1.2.5):**

Let $X = [0,1]$ and $T: X \to X$ be defined by

$$Tx = \begin{cases} 
\frac{x}{3}, & 0 < x < 1 \\
\frac{1}{6}, & x = 1
\end{cases}$$

Clearly $T$ is not continuous and hence does not satisfy the contraction condition. Kannan's condition (1.2.6) is also not satisfied because if $x=0, y=1/3$, then

$$d(T_0, T_{1/3}) \leq \frac{1}{2} [d(0, T_0) + d(1/3, T_{1/3})].$$

However Reich's condition (1.2.9) is satisfied, by putting $a=1/6, b=1/9$ and $c=1/3$.

The following theorem proved by Chatterjee [24] as follows:

**Theorem (1.2.7)** A mapping $T$ from a complete metric space $(X,d)$ into itself satisfying the following:

$$(1.2.10) \quad d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$, where $0 \leq k < \frac{1}{2}$, and obtained a unique fixed point of $T$. 
Hardy and Rogers [58] studied a more general contractive condition as follows:

A mapping $T$ from a complete metric space into itself is said to be a generalized contraction if

\[(1.2.11) d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y)\]

for all $x, y \in X$ with $a_i \geq 0$ (i = 1, 2, 3, 4, 5) and $\sum_{i=1}^{5} a_i < 1$.

Ciric [35] considered a generalized contraction condition defined as follows:

Let $T$ be a mapping from a complete metric space $(X, d)$ into itself satisfying the condition :

\[(1.2.12) d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\]

for all $x, y \in X$ where $q \in [0, 1]$ is said to be a generalized contraction condition.

The following theorem proved by Hussain and Sehgal [65] which generalizes the Kannan [79] Reich [115] and Ciric [34] type of generalized contraction mapping theorems.

**Theorem (1.2.8)**: Let $T$ be a mapping from a complete metric space $(X, d)$ into itself satisfying the condition :

\[d(Tx, Ty) \leq \phi(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y))\]

for all $x, y \in X$, where $\phi: [R^+]^5 \rightarrow [R^+]$ be continuous and non decreasing in each coordinate variable and $\phi(t, t, a_1 t, a_2 t, t) < t$ for $t > 0$ and $a_i \in \{0, 1, 2\}$ with $a_1 + a_2 = 2$, then there exists a unique $u$ such that $Tu = u$. 

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1.3 Expansion concept of fixed point:

Fisher [47] established following expansion condition of mapping:

**Theorem (9):** Let $T$ be a mapping from a metric space $(X, d)$ into itself satisfying

$$d(Tx, Ty) \geq (1/2) [d(x, Tx)+d(y, Ty)] \quad \text{for all } x, y \in X.$$ 

Then $T$ is identity mapping.

Fisher used expansion mapping condition in his following result:

**Theorem (1.3.1):** Let $X$ be a compact metric space, $T: X \rightarrow X$ and satisfying (1.3.1) for each distinct pair $x, y \in X$, then $T^r$ has a fixed point for some positive integer $r$, and $T$ is invertible.

Wang, Gao and Iseki[151] obtained some fixed point theorems for expansion mappings in complete metric space $(X, d)$ in a different fusion and proved:

**Theorem (1.3.2):** Let $T$ be surjective mapping from a complete metric space $(X, d)$ into itself satisfying the condition:

$$(1.3.2) \quad d(Tx, Ty) \geq h \cdot d(x, y)$$

for all $x, y \in X$, where $h>1$. Then $T$ has a unique fixed point in $X$.

**Theorem (1.3.3):** Let $T$ be surjective mapping from a complete metric space $(X, d)$ into itself. If there exist non-negative real numbers $a, b, c$ with $a+b+c > 1$ and $c<1$, such that:

$$(1.3.3) \quad d(Tx, Ty) \geq ad(x, Tx)+bd(y, Ty)+cd(x, y)$$

for all $x, y \in X$.

Then $T$ has a unique fixed point in $X$. 
In 1992 following examples due to Lahiri [93] compare the condition of Banach and Kannan:

**Example (1.3.1):**

Let $X = [0, 1]$

$T(x) = x/4$ for $x \in [0, 1/2)$

$T(x) = x/5$ for $x \in (1/2, 1]$

Here $T$ is discontinuous at $x = 1/2$, and so Banach contraction condition is not satisfied, but if $a = \frac{4}{9}$, it satisfies Kannan's contraction conditions.

**Example (1.3.2):**

Let $X = [0, 1]$

$T(x) = x/3$ for $x \in [0, 1]$.

Here Banach's contraction condition is satisfied but it does not satisfy the Kannan's contractive condition for $x = 1/3$ and $y = 0$.

All above results rigorously demonstrate a trend of perpetual change in contraction conditions, not with standing the last four results which deals with expansion conditions.

Banach's contraction principle gives us an inspiration to a number of researchers to work over the field of fixed points and in this direction Chu and Diaz [30] generalized the Banach's contraction principle in the following way:
Theorem (1.3.4) If \( T \) is a mapping of a complete metric space \( X \) into itself such that \( T^n \) , for some positive integer \( n \) is a contraction , then \( T \) has a unique fixed point.

1.4 Hilbert space concept for fixed point:

In chapter -2 we provide a contraction type mapping and proved the results concerning the existence of fixed points and common fixed points satisfying generalized contractive conditions by using functional inequality in Hilbert space.

Ciric [34] introduced the notion of generalized contraction mapping and proved fixed point theorems .

Das and Gupta [37] present the fixed point theorems in reflexive Banach spaces. The most of fixed point theorems in metric spaces , satisfying different contraction conditions which may be extended to the abstract spaces like Hilbert spaces, Banach spaces, locally convex spaces etc. with some modifications.

We know that a complete norm linear space is called a Banach space and Complete inner product space is called Hilbert space . Every Hilbert space is a Banach space but the converse is not true always. A set \( C \) in a Banach space \( X \) is said to be convex if

\[ \alpha x + (1 - \alpha) y \in C \]

whenever \( x, y \in C \) and \( 0 \leq \alpha < 1 \).

The intersection of all closed convex sets containing a set \( G \) is called closed convex which contains \( G \) and which is contained in every closed convex set containing \( G \).
We recall that a mapping $T: X \rightarrow X$ is said to be non-expansive if

$$ ||Tx - Ty|| \leq ||x-y|| $$

for all $x,y \in X$.

Browder and Petryshyn [15] proved the following result:

Let $C$ be a closed subset of a Hilbert space $H$ and $T$ be a self map of $C$ satisfying the condition:

$$(1.4.1) \quad ||Tx-Ty||^2 \leq a||x-y||^2$$

for all $x,y \in C, x \neq y$, where $0 < a < 1$.

Then $T$ has a unique fixed point in $C$.

Koparde and Waghmode (88) proved common fixed point theorem for a sequence $\{T_n\}$ of a mapping satisfying the condition:

$$(1.4.2) \quad ||Tx-Ty||^2 \leq b(||x-Tx||^2 + ||y-Ty||^2) \quad \text{for all } x,y \in C, x \neq y, \text{ where } 0 \leq b < 1/2$$

Then $T$ has a unique fixed point.

Pandhare and Waghmode (105) proved common fixed theorem for a sequence $\{T_n\}$ of the mapping satisfying the condition:

$$(1.4.3) \quad ||Tx-Ty||^2 \leq a||x-y||^2 + b(||x-Tx||^2 + ||y-Ty||^2) \quad \text{for all } x,y \in C, x \neq y, \text{ where } a+2b < 1 \text{ and } a \geq 0, 0 \leq b < 1$$

Then $T$ has a unique fixed point.

It is well known that the fundamental properties of contraction mappings do not extend to non-expansive mappings. It is of great importance in applications to find out a non-expansive
mappings have fixed points. In order to obtain existence of fixed points for such mappings some restriction has to be made on the domain of the mapping or on the mapping itself. This result is due to Kirk[87].

1.5 2-metric space concept for fixed point:

In chapter -3 our study is devoted to a series of papers [52-54] Gahler introduced a notion of 2-metric space as below:

A 2-metric space is a space $X$ with a real valued function $d$ on $X \times X \times X$ satisfying the following conditions:

(M1) For distinct points $x, y \in X$, there exists a point $z \in X$, such that $d(x, y, z) \neq 0$;

(M2) $d(x, y, z) = 0$ when at least two of $x, y, z$ are equal.

(M3) $d(x, y, z) = d(y, z, x) = d(x, z, y)$,

(M4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$,

for all $x, y, z, u \in X$.

While investigating the above concept Gahler [52] give an idea for the study of 2-metric on the real valued function on a set $X$. Which has abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space. According to Gahler [53] a given 2-metric has natural topology associated with area function.

Firstly studied by Sharma, Sharma and Iseki[135]. The contraction type mappings in 2-metric space. Iseki [67] generalized Ciric's result in 2-metric space. During the last three decades
several authors Rhoades[116], Singh[136], Singh, Tiwari and Gupta[141], Khan and Swaleh[85], Kubiak [91], Naidu and Prasad[99] gives an idea of the aspects of the fixed point theory in the setting of 2-metric spaces and motivated us by various concepts already of metric spaces which analogous of such concepts in the framework of the 2-metric spaces. Khan [82], Murthy-Chang-Cho-Sharma[96] and Naidu-Prasad [98] introduced the concepts of weakly commuting pair of self mappings, compatible pairs of self mappings of type(A) in a metric space and the weak continuity of a 2-metric space respectively, of course commuting mappings are weakly commuting and compatibility of type(A), but the converse is not true.

Example(1.5.1):
Every commuting pair of self mappings are compatible of type(S), compatibility of type(A) and compatibility of type(P) but converse is not necessary true.

Let \( X = \{1,2,3,4\} \) and define 2-metric \( d: X \times X \times X \to \mathbb{R} \) as follows:

\[
d(x,y,z) = 0 \text{ if } x = y \text{ or } y = z \text{ or } z = x \text{ and } \{x,y,z\} = \{1,2,3\} \\
= \frac{1}{2} \text{ if } x,y,z \text{ are distinct and } \{x,y,z\} \neq \{1,2,3\} 
\]

define two mappings from 2-metric space \( (X,d) \) into itself by

\[
S(1) = S(2) = S(3) = 2, \quad S(4) = 1, \\
\text{and} \\
T(1) = T(2) = T(3) = 2, \quad T(4) = 1.
\]
Since
\[ ST(4) = TS(4) = TS(4) = SS(4) = SST(4) = STS(4) = 2 \]
This shows that the pair \((S, T)\) is commuting on \(X\),
but
\[ d(STS(4), STS(4), z) = 0 \]
and
\[ d(TST(4), TTS(4), z) = 0 \]
for all \(z \in X\), where \(S(4) = T(4) = 1\).
Also \( d(TS(4), SS(4), z) = 0 \) for all \(z \in X\), where \(S(4) = 1\) and \(S(1) = T(1) = 2\).
Clearly pair \((S, T)\) is commuting on \(X\). So it is compatible mappings of type\((S)\), compatible mappings of type\((A)\) and compatible mappings of type\((P)\).

1.6 **L-space concept for fixed point:**

In chapter-4 our work is devoted to the study of fixed points and common fixed points of four self mappings satisfying a contractive type condition in L-space.

It observed by Kasahara [81] that a generalizations of Banach's contraction principle is derived without using the notion of metric spaces. In particular without using the axiom of triangular inequality Kasahara [80] introduced the concept of L-space in fixed point theory. Later on Yeh [155] obtained some fixed point theorems in L-space and after that Iseki [67] has used the fundamental idea of Kasahara to investigate the generalizations of
some known theorems in L-space. During past few years many authors worked on L-space, i.e. Singh[136] Pachpatte[103] and Park[107], Rhoades[116] are worth mentioning. Som and Mukharjee[145] gives some results on fixed points in L-space, using the mapping satisfying the conditions of the type Rhoades[116] and Jaggi[70]. Similarly Pathak and Dubey[105] obtained the interesting results on fixed points in d-complete space and separated L-space with three continuous and pair-wise commuting satisfying a rational expression of Jaggi[70].

Our object in this chapter is to extend and generalize the result of Sharma and Agrawal[134] which is as under:

**Theorem (1.6.1)** Let \((X, \rightarrow)\) be a separated L-space which is d-complete for a non negative real valued function \(d\) on \(X \times X\) with \(d(x,x)=0\) for each \(x\) in \(X\). Let \(E, F\) and \(T\) be three continuous self mappings of \(X\) satisfying the following condition:

(i) \(ET=TE, \ FT=TF,\)

(ii) \(E(X) \subset T(X), \ F(X) \subset T(X),\)

and

(iii) \(d(Ex,Fy) \leq \alpha \frac{d(Ty,Fy) [1+d(Tx,Ex)]}{1+d(Tx,Ty)} + \beta(Tx,Ty)\)

for all \(x, y \in X\) with \(Tx \neq Ty\) and \(\alpha, \beta \geq 0, \alpha + \beta < 1\). Then \(E, F\) and \(T\) has a unique common fixed point.
1.7 **Commutivity concept for fixed point**:

Let $A$ and $B$ be mapping of a metric space $(X,d)$ into itself, then

(i) $A$ and $B$ are said to commute if $ABx = BAx$, $\forall x \in X$,

and

(ii) $t \in X$ is said to be the common fixed point of $A$ and $B$ if $At = Bt = t$.

Let $B$ be a family of self maps on a set $A$. It is useful to know when the members of $B$ have a common fixed point in $A$. We shall show that every member of $B$ has a fixed point in $A$, there may not exist a common fixed point of all members of $B$.

**Example (1.7.1)**

Let $A = [-1,1]$ define

$T(x) = \frac{x+1}{2}$

$U(x) = \frac{x-1}{2}$ for $x \in [-1,1]$

Then $1$ is the unique fixed point of $T$, and $-1$ is the unique fixed point of $U$. Hence $T$ and $U$ have no common fixed point.

It has been convincingly demonstrated that the scope of using commuting condition is limited to only those fixed point results which include more than one self mappings. The reason is results containing more than one self-mapping normally gives coincidence points. In fact commuting condition facilitates obtaining fixed point from such coincidence points which are common to those self-mappings. This aspect has provided ample stimulation for the authors to use and investigate commuting condition for
their fixed point results. Consequently commuting condition was further weakened by Weak commutativity[126] 

\[ d(STx, TSx) \leq d(Sx, Tx) \]

compatibility(71)

\[ d(STx_n, TTx_n) = 0; \text{ and } d(TSx_n, SSx_n) = 0 \]

compatibility of type(A) [96]

\[ d(STSx_n, SSTx_n, a) = 0; \text{ and } d(TSTx_n, TTSx_n, a) = 0 \]

where \( \lim_{n \to \infty} Sx_n = t, \text{ and } \lim_{n \to \infty} Tx_n = t \), for some \( t > 0 \),

to obtain fixed point.


Also Sessa [126] weakening the notion of commutativity for point to point mappings established the idea of weak commutativity for two self mappings \( f \) and \( g \) of a metric space \((X,d)\) i.e. 

\[ d(fgx, gfx) \leq d(fx, gx) \]

for all \( x \in X \).

According to this concept he extended the result of Jungck[71] 

\[ d(STx_n, TTx_n) = 0, \text{ and } d(TSx_n, SSx_n) = 0 \].
Example (1.7.2)

Let \( X = [0, \infty) \) define
\[
S_x = 2x^2 \quad \text{for any } x \in X,
\]
\[
T_x = x^2 \quad \text{for any } x \in X;
\]
it is not commuting, but it is compatible.

Example (1.7.3)

Let
\[
A_x = 1 \quad \text{if } x = 0
\]
\[
= 1/x \quad \text{if } x \neq 0
\]

\[
B_x = \frac{1}{4} \quad \text{if } x = 0
\]
\[
= x/2 \quad \text{if } x \neq 0
\]
\[
T_x = x/2 \quad \text{for any } x \in X.
\]

which satisfied the condition of weakly commuting.

Example (1.7.4)

Let \( X = [-2,2] \)
\[
A_x = x^2 /4
\]
\[
B_x = x^2
\]

It is clear that \( AB \neq BA \), but it is weakly commuting.

Example (1.7.5)

Let \( X = \mathbb{R} \)
\[
A_x = 2 - x^2
\]
\[
B_x = x^2
\]
\[
x_n = 1 + (1/n) \quad n = 1, 2, 3 \ldots
\]

It is clear that \( A, B \) are not commuting and not weakly commuting.
Example (1.7.6)

Let \( X = \mathbb{R} \)

\[
Ax = x^3 \\
Bx = 2x^2 \\
x_n = \frac{1}{n} \quad n = 1, 2, 3 \ldots
\]

It is clear that \( AB \neq BA \) and not weakly commuting.

He and others gave some common fixed point theorems for weakly commuting mappings in metric spaces and 2-metric spaces (see e.g. Bhaskaran and Subrahmanyam [7], Fisher and Sessa [50], Naidu and Prasad [98], Rhoades and Sessa [122], Sessa [126], Sessa, Mukherjee and Som [126], Fisher and Sessa [50], Singh, Ha and Cho [140]. Recently Jungck [72], proposed further weak version of the concepts of commuting mapping and weakly commuting mappings which is called compatible mappings and others proved a common fixed point theorems using this concept. They are Jungck [72], Kaneko and Sessa [76], Kang, Cho and Jungck [77], Rhoades, Park and Moon [121], Murthy, Chang, Cho and Sharma [96], with the following condition:

\[
d(SSTx_n, STSx_n, a) = 0 ;
\]

and

\[
d(TSTx_n, TTSx_n, a) = 0 ,
\]

where \( \lim_{n \to \infty} S x_n = t \), and \( \lim_{n \to \infty} T x_n = t \), for some \( t > 0 \).
The concept of compatible mappings of type(A) in 2-metric spaces and gave some fixed point theorems for three mappings concept further extended in several spaces studied in this chapter.

Example(1.7.7)

Let $X = [0,1]$ with the metric $d$ defined by

$$d(x,y) = |x - y| \quad \text{for each } z \text{ in } (0, \infty)$$

define $d(x,y,z) = \frac{z}{z + |x - y|}$ for $x, y, z \text{ in } X$. Then

$$d(x,y,0) = \frac{0}{z + |x - y|} = 0$$

Clearly $(X,d)$ is a 2-metric space.

Consider $A, B : X \to X$ by

$$Ax = x/16$$

$$Bx = 1-x/3$$

Then $d(ABx, BBx, z) \neq 0$,

$$d(BAx, AAx, z) \neq 0$$

and $d(ABx, BAx, z) = 0$

$$d(ABx, BAx, z) \leq d(Ax, Bx, z)$$

These shows that pair $(A, B)$ is commuting, weakly commuting and compatible, but not compatible of type(A)
1.8 Fuzzy concept for fixed point:

The notion of fuzzy set introduced by Zadeh [157] in 1965 became very interesting for both pure and applied mathematicians. Later on Kalewa and Sheikkala [75] proposed the concept of fuzzy metric space. It is also raised enthusiasm among engineers, biologists, psychologists, economists and others. Among the branches of mathematics, general topology was the first branch in which fuzzy sets theory been applied systematically by Chang [19] in 1968. Still there are my new point regarding the notion of metric spaces in fuzzy topology. Those can be divided into two separate lines. The first line is in the form of fuzzy metric space on a set X is treated as a map \( d: X \times X \rightarrow R^+ \), where \( X \subseteq I^X \).

Along the above line investigations were done by Deng [40], Erceg [45] and Hu [63] satisfying some collection of axioms that are analogous to the ordinary metric space. In such an approach numerical distance are set up between fuzzy objects. The authors of this approach are interested in fuzzy topology [40, 46, 63] and separation properties in metric space. It has been found that fuzzy metric spaces are a kin to probabilistic metric spaces. S. Heilpern [59] first introduced the concept of fuzzy mappings (mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space) and proved a fixed point theorem for fuzzy contraction mapping which is a fuzzy analogue of the fixed point theorems for multivalued mapping:
\[ d(Fx, Fy) \leq kd( x, y), \quad k \in (0, 1) \] for all \( x, y \in M \), where \( F \) is fuzzy mapping and \( M \) is complete metric space.

In this connection Chen and Shih [26] Husain and Latif [64] Nadler, Jr. [97] and the well-known Banach fixed point theorem) Butnariu [16] discussed the existence of fixed points for 'convex and closed fuzzy mappings' which can viewed as the generalization of 'convex mappings' and 'closed point to set mappings'. Bose and Sahani [8] extended the result of Heilpern [59] for a generalized fuzzy contraction mapping. Som and Mukharjee [145] proved a fixed point theorem for a weakly dissipative fuzzy mapping and extended the results of Bose and Sahani [8], Heilpern [59] to prove the certain fixed point theorems for non-expansive fuzzy mappings on metric space. Chang and Hung [22] obtained a coincidence theorem for fuzzy mappings on metric spaces.

Also Kramosil and Michalek [90] introduced the concept of fuzzy metric space using the idea of 2-metric space. Sharma [133] defined fuzzy 2-metric space as follows:

A fuzzy 2-metric space is 3-tuple \((X, M, *)\) where \( X \) is a nonempty set \(*\) is a continuous norm and \( M \) is a fuzzy set \( X^3 \times [0, \infty) \) satisfying the conditions:
(i) to each pair of distinct points $x,y,z$ in $X$ there exists a point $t$

in $[0,1]$, such that :

$$M(x,y,z,t) 
eq 0,$$

(ii) $M(x,y,z,t) = 1$ when at least two of $x,y,z$ are equal,

(iii) $M(x,y,z,t) = M(y,z,x,t) = M(x,z,y,t)$,

(iv) $M(x,y,z,t_1 + t_2 + t_3) \geq M(x,y,w, t_1) \star M(x,w,z, t_2) \star M(w,y,z, t_3)$,

for all $w$ in $X$ and $t_1, t_2, t_3$ in $[0,1]$.

It is easily seen that $M$ is non-negativity and a function value $M(x,y,z,t)$ may be interpreted as the probability that the area of triangle is less than $t$ and also the function $M(x,y,z,.):[0, \infty) \rightarrow [0,1]$ is left continuous.

Example (1.8.1)

Let $X = [2,20]$ with the metric $d$ defined by $d(x,y) = |x-y|$ for each $t$ in $(0, \infty)$

define $M(x,y,z,t) = \frac{t}{1+|x-y|}$ for $x,y,z$ in $X$.

$M(x,y,z,0) = \frac{0}{1+|x-y|} = 0$

Clearly $(X,M, *)$ is a fuzzy 2-metric space .

Then consider $A, B :X \rightarrow X$ by

$$Ax = 6 \quad \text{if} \quad 2 \leq x \leq 5$$

$$= 6 \quad \text{if} \quad x = 6$$

$$= 10 \quad \text{if} \quad x > 6$$

$$= \frac{x-1}{2} \quad \text{if} \quad x \in (5,6)$$

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\[ Bx = \begin{cases} 2 & \text{if } 2 \leq x \leq 5 \\ (x+1)/3 & \text{if } x > 5 \end{cases} \]

Choose \( x_n = \{ 5 + 1/2^n : n \in \mathbb{N} \} \)

These shows that:

\[
\begin{align*}
M(ABx_n, BAx_n, z, t) &= 1 \\
M(ABx_n, BBx_n, z, t) &\neq 1 \\
M(BAx_n, AAX_n, z, t) &\neq 1 \\
M(ABx_n, BAx_n, z, t) &\geq M(Ax_n, Bx_n, z, t)
\end{align*}
\]

Hence the pair \((A, B)\) is compatible and weakly compatible but not compatible of type \(A\)

### 1.9 Saks concept for fixed point:

Let \( X \) be a linear set and suppose that \( N_1 \) and \( N_2 \) are \( B \)-norm and \( F \)-norm on \( X \) respectively. Let \( X_s = \{ x \in X, N_1(x) < 1 \} \) and define

\[ d(x, y) = N_2(x - y) \]

for all \( x, y \) in \( X_s \). Then \( d \) is metric on \( X_s \) and the metric space \((X_s, d)\) will be called a Saks set.

**Definition:** A Saks set \((X_s, d)\) is said to be a Saks space, if it is complete and is denoted by \((X, N_1, N_2)\).

The Saks space and its related spaces are extensively developed by Orlicz[101], Orlicz and Ptak[102], Alexiewicz[2], Alexiewicz and Semandi [3] in the simplest form.

After a wide gap, Cho and Singh[29] introduced the Jungck's contraction condition on a Saks space to obtain common fixed
points for a pair of commuting self mappings. Recently Cho and Singh[29] proved some coincidence point theorems in saks space for three mapping satisfying a new contractive condition of Ding[41].

The thesis is organized as follows:

In chapter-1: It contains a brief survey of the important results which have their relevance with the subject matter of the Thesis.

In chapter-2: We have considered the contraction type mappings and proved the results concerning the existence of fixed point and common fixed points, weakly commuting pairs of mapping, compatible mapping of type(A) satisfying generalized contractive conditions by using functional inequalities in Hilbert space.

In chapter-3: Our work is devoted to common fixed points theorem so the most of results in this chapter are extension and generalization of Naimpally and Singh[100], Ciric[33] result, Fisher [48] theorem.

In chapter-4: Our study is restricted on fixed points and common fixed points of pair wise commuting mappings, compatible mapping of type(A), iteration and semi continuity satisfying a new contractive type condition in L-space.
In chapter 5: We have introduced weakly commuting pair of mapping and obtain common fixed point theorem compatible type (A), weakly commuting for fuzzy metric space, fuzzy 2-metric space which is an expansion of Delbosco [39], Rakotch [113].

In chapter 6: Our result for this chapter are an extension of Cho and Singh [28], Naimpally and Singh [100] results as:

\[ d(Tx, Ty) \leq q \max\{cd(x, y), d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx) \} \]

where \( c > 0 \) and \( 0 \leq q < 1 \), satisfying weakly uniformly contraction and Mann iteration type condition in saks space.

The application of Banach fixed point theory is to find the unique solution of linear algebraic equations, differential equations, Integral equations as well as in implicit function theorem.

Last but not the least the thesis contains adequate and pertinent bibliography containing 157 references and at the end I have attached reprints and Xerox copies of same of my published research papers.