Chapter 2

DESIGNS ASSOCIATED WITH MINIMUM DOMINATING SETS OF A GRAPH
2.1 Introduction

The theory of design of experiments came into being largely through the work of R. A. Fisher and F. Yates in the early 1930's. They were motivated by questions of design of field experiments in agriculture. The applicability of this theory is now very widespread, much of the terminology still bears the stamp of its origins.

R. C. Bose [2] established the relation between graph theory and Partially Balanced Incomplete Block Designs (PBIBD's). He has shown that strongly regular graphs emerge from PBIB designs. F. Harary, R. W. Robinson and N. C. Wormald [4] established the link between isomorphic factorization of graphs and designs. Very recently, Walikar et al. [10] found the relation between the minimum dominating sets of a given graph with the blocks of PBIBDs.

Ionin and Shrikhande [7], introduced the general notion of a \((v, k, \lambda, \mu)\)-design over a regular graph \(G\). In this chapter, we introduce a combinatorial design called \((v, \gamma, \lambda, \mu)\)-design over a regular graph \(G\) as an ordered pair \(D = (V, \mathcal{B})\), where \(|V| = v\) and \(\mathcal{B}\) is the set of minimum dominating sets of \(G\) called blocks such that \(i, j \in V, i \neq j\) whenever there are exactly
A blocks containing \( i \) and \( j \) if \( \{i, j\} \in E(G) \) and exactly \( \mu \) blocks containing \( i \) and \( j \) if \( \{i, j\} \notin E(G) \). The blocks \([7]\) are certain \( k \)-subsets of the points, whereas the blocks in the present chapter are all of size \( k = \gamma(G) \), the domination number of the graph \( G \). Ionin and Shrikhande \([7]\) considered the study of \((v, k, \lambda, \mu)\)-design when the replication number \( r = 2\lambda - \mu \).

We shall consider the \((v, \gamma, \lambda, \mu)\)-designs with \( r = \frac{d\lambda + (v-1-d)\mu}{(\gamma-1)} \).

Further, in this chapter, we shall construct PBIBDs with two association schemes through \((v, \gamma, \lambda, \mu)\)-designs corresponding to the Petersen graph, the Clebsch graph, the Schlӓfli graph and we show the nonexistence of \((v, \gamma, \lambda, \mu)\)-designs over the Shrikhande graph and the three Chang graphs.

A Balanced Incomplete Block Design (BIBD) is a set of \( v \) elements arranged in \( b \) blocks of \( k \) elements each in such a way that each element occurs in exactly \( r \) blocks and every pair of distinct unordered elements occur in \( \lambda \) blocks. The combinatorial configuration so obtained is called a \((v, b, r, k, \lambda)\)-design. Interplay between graphs and designs was first time started with an observation made by C. Berge \([1]\) on maximum independent sets in a graph \( G(\binom{15}{3})_1 \) on a binomial coefficient \( \binom{15}{3} \) and BIBD with the
parameters \((15, 35, 7, 3, 1)\). The graph on binomial coefficient \(\binom{v}{k}\), written as \(G(\binom{v}{k})\), whose vertex set is the set of all \(k\)-subsets of a set \(V\) with \(v\) elements and two vertices are adjacent if and only if the corresponding \(k\)-subsets have more than \(\lambda\) and less than \(k\) elements in common. This observation mentioned above made by C. Berge, motivated J. W. DI Paóla [8] to establish a thorough link between the graphs \(G(\binom{v}{k})\) and BIBD’s whose blocks are the maximum independent sets in \(G(\binom{v}{k})\).

A set \(S \subseteq V\) of vertices in a graph \(G\) is called a dominating set if every vertex \(v \in V\) is either an element of \(S\) or is adjacent to an element of \(S\). A dominating set \(S\) is a minimal dominating set if no proper subset of it is a dominating set. The domination number \(\gamma(G)\) is then defined as the minimum cardinality of a dominating set in \(G\).

Walikar et al. in [10] established a relation between the set of all minimum dominating sets and PBIBD’s for cycles and cubic graphs on ten vertices.

Following simple results Proposition 2.1.1 and Lemma 2.1.2, on strongly regular graphs whose proof uses the two-way counting method and these results are of immediate use.
Proposition 2.1.1. [3] If $G$ is a strongly regular graph with parameters $(n, k, p, q)$, then

$$ (n - k - 1)q = k(k - 1 - p) \quad (2.1.1) $$

Lemma 2.1.2. [5] For any graph $G$ of order $n$, $\gamma(G) \geq \lceil \frac{n}{\Delta+1} \rceil$, where $\Delta$ denotes the maximum degree of $G$.

2.2 $(v, \gamma, \lambda, \mu)$-design over the Petersen graph

The Petersen graph $G$ is a graph whose vertices are 2-element subsets of the set $\{1, 2, 3, 4, 5\}$ where two vertices are adjacent if and only if their intersection is empty and the Petersen graph $G$ is as depicted in Figure 2.1.

![Figure 2.1: Petersen graph](image-url)
The following results were established in [9], in connection with the
minimum dominating sets of the Petersen graph.

**Lemma 2.2.1.** [9] The domination number of the Petersen graph is three.

**Lemma 2.2.2.** [9] Every minimum dominating set in the Petersen graph is
an independent set and further it is an open neighborhood of some vertex.

By Lemma 2.2.2, the minimum dominating sets are

\[
\begin{align*}
\{34,35,45\}, \{13,14,34\}, \{12,15,25\}, \\
\{13,15,35\}, \{23,24,34\}, \{25,35,23\}, \\
\{14,15,45\}, \{25,24,45\}, \{12,14,24\}, \\
\{12,13,23\}
\end{align*}
\]

Thus, the open neighborhood of every vertex in the Petersen graph is a
block in \((v, \gamma, \lambda, \mu)\)-design with the parameter \((10, 3, 0, 1)\).

**Theorem 2.2.3.** There exist \((v, \gamma, \lambda, \mu)\) design over Petersen graph with the parameters \(v = 10, \gamma = 3, \lambda = 0, \mu = 1\)

### 2.3 \((v, \gamma, \lambda, \mu)\) design over the Clebsch graph

The Clebsch graph \(G\) is the graph whose vertices are all subsets of a set
\(\{1, 2, 3, 4, 5\}\) with even cardinality, where two vertices are adjacent when-
ever their symmetric difference has cardinality four. The graph $G$ so obtained is as shown in Figure 2.2.

![Figure 2.2: Clebsch graph](image)

For the sake of convenience, we relabel the vertices of Clebsch graph $G$ as in Figure 2.3.

Firstly, we discuss and highlight some of the structural properties of the Clebsch graph. For any arbitrary vertex $x$, in a Clebsch graph $G$, we partition the vertex set $V(G)$ into two subsets $A_x = N[x] = \{x\} \cup N(x)$, the closed neighborhood of $x$ and $B_x = V - A_x$. By the structure of $G$, $B_x$ induces the Petersen graph for every $x \in V(G)$. As $G$ is a strongly
regular graph with parameters \((16, 5, 0, 2)\), one can make the following observations

**Observation 2.3.1.** \(G\) is a triangle-free 5-regular graph.

**Observation 2.3.2.** Any two nonadjacent vertices of \(G\) are adjacent to two common vertices. Further, for any two nonadjacent vertices \(u\) and \(v\) in \(G\).

(a) If \(u, v \in A_x\), then they are adjacent to two common vertices namely \(x\) and some vertex in \(B_x\).

(b) If \(u = x\) and \(v \in B_x\), then both are adjacent two nonadjacent vertices in \(A_x\).
(c) If both $u$ and $v$ are in $B_x$, then they are adjacent to one common vertex in $B_x$ itself and one other in $A_x$.

Keeping these observations in mind, following results are established, in connection with the minimum dominating sets of the Clebsch graph.

**Proposition 2.3.3.** The domination number of the Clebsch graph $G$ is four.

**Proof:** Let us partition the vertex set $V(G)$ into two subset $A_x = N[x]$ and $B_x = V - A_x$ as above. We prove that $\gamma(G) = 4$. For, if $\gamma(G) = 3$ then, let $S = \{u, v, w\}$ be the minimum dominating set in $G$. As $G$ is triangle free graph, then the possible subgraphs induced by $S$ are given below:

![Figure 2.4.](image)

To prove this proposition, it is sufficient to prove that if $S$ induces one of $G_i$, $1 \leq i \leq 3$, then $S$ is no more a dominating set.

**Case 1:** $(S) = G_1.$
The two vertices $u, v$ of $G$, can cover ten vertices of $G$ and remaining six vertices induces $3K'_2$s in $G$, By taking $w$ to be any of the vertex of $3K'_2$s, there are four vertices in $G$ not covered by $S$ and hence $S$ is not a dominating set in $G$.

**Case 2:** $\langle S \rangle = G_2$.

The two nonadjacent vertices $u, v$ of $G_2$ have one more common vertex say $x$ and by regularity of $G$, the vertices $u, v, w$ can cover 13 vertices, that is $\bigcup_{y \in S} N[y] = 13$, which implies that the remaining three vertices of $G$ are not covered by $S$. There for $S$ is not a dominating set.

**Case 3:** $\langle S \rangle = G_3$.

As $u, v, w$ are mutually nonadjacent, and every pair of these nonadjacent vertices have two common neighbors and by the regularity of $G$, we have $\bigcup_{y \in S} N[y] \leq 12$ and hence there are at least four vertices in $G$ not covered by $S$ and hence $S$ is not a dominating set.

Hence, in all the cases we conclude $S$ is not a dominating set. Thus, $\gamma(G) = 4$ holds. On the other hand, a minimum dominating set in a Petersen graph induced by $B_x$ and a central vertex in $A_x$ forms a minimum dominating set in $G$ which gives us $\gamma(G) = 4$ and hence the result. $\square$
Theorem 2.3.4. If $S$ is a minimum dominating set in the Clebsch graph $G$, then $S$ induces either $\overline{K_4}$ or $C_4$.

Proof: By Proposition 2.3.3 the domination number of a Clebsch graph $G$ is four and hence let $S = \{u_1, u_2, u_3, u_4\}$ be a minimum dominating set in $G$. We prove that this set $S$ induces either $\overline{K_4}$ or $C_4$ in $G$.

The possible subgraphs induced by $S$ are given below:

![Figure 2.5](image_url)

To prove the theorem, it is sufficient to prove that if $S$ induces one of $G_i$, $1 \leq i \leq 5$ then, $S$ is no more a dominating set in $G$. We consider various
cases as below

**Case 1:** \( \langle S \rangle = G_1 \) or \( \langle S \rangle = G_2 \).

The two adjacent vertices \( u_1 \) and \( u_2 \) of \( G_1 \) can cover ten vertices in \( G \) and the remaining six vertices induces \( 3K'_2s \) in \( G \). By taking \( u_3 \) and \( u_4 \), then there are at least two vertices in \( G \) not covered by \( S \), hence \( S \) is not a dominating set.

![Figure 2.6.](image)

**Case 2:** \( \langle S \rangle = G_3 \).

Without loss of generality, assume that \( u_1, u_2, u_4 \in A \) and \( u_3 \in B_x \). By the structure of Clebsch graph \( G \), \( u_2 \) and \( u_4 \) are adjacent to a common vertex \( v \) in \( B_x \). We consider two sub cases depending on whether \( v \) is adjacent to \( u_3 \).

**Subcase 2.1:** \( v \) is adjacent to \( u_3 \).

As \( v \in B_x \), \( v \) must have two more vertices say \( v_1 \) and \( v_2 \) are the neighbors
in $B_x$. As $S$ is a dominating set $v_1$ and $v_2$ must be adjacent to at least one vertex of $S$. If $v_1$ is adjacent to $u_1$, then degree of $u_1$ exceeds five. If $v_1$ is adjacent to any of the vertices $u_2, u_3, u_4$ that forms a triangle in $G$. Both cases are impossible.

![Figure 2.7](image)

**Subcase 2.2:** $v$ is not adjacent to $u_4$.

Let $v_1, v_2, v_3$ be the vertices adjacent to $v$ in $B_x$. All of these vertices cannot be adjacent $u_4$. Since $v$ and $u_4$ (being a non-adjacent vertices) have only one common neighbor in $B_x$. Hence, two of the remaining vertices are adjacent to at least one of the vertices of $u_1, u_2, u_3$. Thus, we arrive at the same contradiction as Subcase 2.1.

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Case 3: $\langle S \rangle = G_4$.

Without loss of generality, assume that $u_2, u_3, u_4$ are in $A_x$ and $u_1$ is in $B_x$.

Again, by the structure of Clebsch graph $G$, the vertices $u_2$ and $u_4$ have a common neighbor $w$ in $B_x$. Clearly, $w \notin N(u_1)$, otherwise $u_2u_1w$ forms a triangle in $G$. Let $v_1, v_2, v_3$ be the neighbors of $w$ in $B_x$.

By the structure of Clebsch graph, we arrive at the same contradiction as in Subcase 2.2.
Case 4: \( \langle S \rangle = G_5 \).

Assume that all the vertices of \( S \) are in \( A_x \) with \( u_1 \) being the central vertex of \( K_{1,3} = G_5 \). As \( u_2 \) and \( u_3 \) are non-adjacent vertices, so they must have common vertex say \( w \) in \( B_x \). Let \( v_1, v_2, v_3 \) be the neighbors of \( w \) in \( B_x \).

![Figure 2.10.](image)

Clearly, none of \( v_i \) is adjacent to \( u_1 \), otherwise it exceeds the degree of \( u_1 \) and also none of them are adjacent to either \( u_2 \) or \( u_3 \). Since it creates a triangle in \( G \). Thus, the only possibility is that all the three vertices \( v_1, v_2, v_3 \) are adjacent to \( u_4 \), which in turn imply that the two non-adjacent vertices \( w \) and \( u_4 \) are adjacent to three common neighbors, namely \( v_1, v_2 \) and \( v_3 \) which is impossible.

To complete the proof it is sufficient to prove that the set \( S = \{ u_1, u_2, u_3, u_4 \} \) with \( \langle S \rangle = C_4 \) and \( \langle S \rangle = \overline{K}_4 \) forms a dominating set in \( G \).
Case 5: \( \langle S \rangle = C_4 = G_7 \).

As \( G \) is a strongly regular, triangle free graph. Therefore \( N(u_i) \cap N(u_j) = \phi \), for every pair \( u_i \) and \( u_j \) of adjacent vertices of \( S \) in \( G \). Therefore, \( \bigcup_{i=1}^{4}N[u_i] = V(G) \) and hence \( S \) is a dominating set in \( G \).

Case 6: \( \langle S \rangle = \overline{K}_4 = G_6 \).

An interesting fact, we prove here is that an arbitrary set \( S \) of four vertices which induces \( \overline{K}_4 \) is not a dominating set. But a particular choice of vertices in \( S \) forms a dominating set. For this we consider three subcases:

Subcase 6.1: \( S \subset A_x \) or \( S \subset B_x \).

If \( S \subset A_x \), then there is some vertex in \( A_x \) which is not adjacent to any vertex of \( S \). If \( S \subset B_x \), then the central vertex of \( A_x \) is not adjacent to any vertex of \( S \).

Subcase 6.2: Two vertices of \( S \) are in \( A_x \) and other two in \( B_x \).

Without loss of generality, assume that \( u_1, u_2 \in A_x \) and \( u_3, u_4 \in B_x \). As \( u_1 \) and \( u_2 \) are non adjacent vertices. They are adjacent to one common neighbor \( y \) in \( B_x \). As \( u_3 \) and \( u_4 \) are non adjacent vertices, they must be adjacent to some vertex (say \( x \)) in \( A_x \), as shown in the Figure 2.11.
By case 5, the vertices $x, u_3, u_4, y$ forms a dominating set in $G$. Therefore, $u_1$ and $u_2$ are adjacent to $y$. But the degree of $y$ is five, so $y$ must be adjacent to one more vertex $w$ in $B_x$. As the set $S$ is dominating set in $G$, the vertex $w$ is adjacent to one of the vertex of $S$, which forms a triangle in $G$, a contradiction.

**Subcase 6.3:** One vertice of $S$, is in $A_x$ and other three are in $B_x$.

Assume that $u_1$ is in $A_x$ and $u_2, u_3, u_4$ are in $B_x$. Without loss of generality assume that $u_1$ is a central vertex in $A_x$. As the independence number of a Petersen graph induced by $B_x$ is four, so if $u_2, u_3, u_4$ is subset of maximum independent set in $B_x$, then there is one vertex in this set which is not adjacent to $u_1$. If $u_2, u_3, u_4$ are not adjacent to a common vertex, then $u_2, u_3, u_4$ forms a dominating set in the Petersen graph induced by $B_x$. 

Figure 2.11.
Thus, any minimum dominating set \( S \) in \( G \) which induces \( K_4 \) is consisting of one central vertex in \( A_x \) and the vertices in a minimum dominating set in \( B_x \). This completes the proof.

With the help of Theorem 2.3.4, and Figure 2.3, following is the list of minimum dominating sets each of which induces \( K_4 \):

\[
\begin{align*}
&\{0,3,8,12\}, \{1,4,9,13\}, \{2,4,6,13\}, \{4,8,9,11\} \\
&\{0,2,9,11\}, \{1,3,5,12\}, \{2,4,7,11\}, \{4,6,7,10\} \\
&\{0,7,8,11\}, \{1,4,8,10\}, \{2,5,9,13\}, \{4,7,8,15\} \\
&\{0,5,9,12\}, \{1,8,9,12\}, \{2,6,7,14\}, \{4,10,11,13\} \\
&\{0,2,5,14\}, \{1,5,6,13\}, \{2,5,6,15\}, \{5,12,13,14\} \\
&\{0,3,7,14\}, \{1,3,6,10\}, \{2,4,9,15\}, \{7,10,11,14\} \\
&\{0,3,5,15\}, \{1,5,9,15\}, \{2,11,13,14\}, \{8,10,11,12\} \\
&\{0,2,7,15\}, \{1,4,6,15\}, \{3,7,8,10\}, \{9,11,12,13\} \\
&\{0,8,9,15\}, \{1,3,8,15\}, \{3,5,6,14\}, \{6,10,13,14\} \\
&\{0,11,12,14\}, \{1,10,12,13\}, \{3,6,7,15\}, \{3,10,12,14\}
\end{align*}
\]

**Minimum dominating sets that induce \( K_4 \).**

Also, the minimum dominating sets that induce \( C_4 \) in \( G \) are listed below:

\[
\{0,1,2,10\}, \{1,2,3,11\}, \{2,10,12,15\}, \{5,11,8,6\}
\]
Minimum dominating sets which induces $C_4$.

Now, we define a 2-association scheme on the vertex set

\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}

of the Clebsch graph as follows.

Two vertices of $G$ are first associates if they are adjacent in $G$, otherwise they are second associates. Let $\mathcal{D}_1=\{D: D$ is a minimum dominating set in the Clebsch graph with $\langle D \rangle = K_4\}$ and $\mathcal{D}_2=\{D: D$ is a minimum dominating set in the Clebsch graph with $\langle D \rangle = C_4\}$.

**Theorem 2.3.5.** For the Clebsch graph $G$,

1. $\mathcal{D}_1$ is the set of blocks of a PBIBD with $(16, 40, 10, 4, 0, 3)$ as the
parameters of the first kind and with parameters of the second kind
given by
\[
P_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}.
\]

2. \( D_2 \) is the set of blocks of a PBIBD with \((16,40,10,4,1,4)\) as the
parameters of the first kind and with parameters of the second kind
given by
\[
P_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 0 \end{bmatrix}.
\]

3. \( D_1 \cup D_2 \) is the set of blocks of a BIBD with parameters \((16,80,10,4,4)\).

**Theorem 2.3.6.** There exists a \((v, \gamma, \lambda, \mu)\)-design over the Clebsch graph
with the parameters \(v = 16, \gamma = 4, \lambda = 0, \mu = 3\)

**Theorem 2.3.7.** There exists a \((v, \gamma, \lambda, \mu)\)-design over the Clebsch graph
with the parameters \(v = 16, \gamma = 4, \lambda = 1, \mu = 4\)
2.4 \((v, \gamma, \lambda, \mu)\)-design over the Schläfli graph

The Schläfli graph \(G\) is a strongly regular graph with the parameters \((27, 10, 1, 5)\). The subgraph of the Shläfli graph formed by the non-neighbors of any vertex is isomorphic to the Clebsch graph. Let \(u\) be any vertex in \(G\) and by the parameters of \(G\), the subgraph induced by \(N(u)\) is just \(5K_2\) and \(H = \langle V(G) - N[u] \rangle\) is the Clebsch graph. Figure 2.12, is the Schläfli graph drawn using the construction of generalized quadrangle \(GQ(2, 4)\) that has appeared in [11].

Figure 2.12: Schläfli graph
Proposition 2.4.1. The domination number of the Schlafli graph is three.

Proof: Let $G$ be the Schlafli graph. By Lemma 2.1.2,

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta+1} \right\rceil = \left\lceil \frac{27}{10+1} \right\rceil = 3$$

Now, if $\gamma(G) = 3$, let $S = \{0, 1, 23\}$ be a minimum dominating set in $G$.

This implies $\gamma(G) \leq 3$. Therefore $\gamma(G) = 3$. □

Theorem 2.4.2. If $S$ is a minimum dominating set in a Schlafli graph then $S$ induces $C_3$

Proof: By Proposition 2.4.1, domination number of Schlafli graph is three and hence $S = \{u, v, w\}$ be a minimum dominating set in $G$. Then, the possible subgraphs induced by $S$ are given in Figure 2.13.

To prove this theorem, it is sufficient to prove that $S$ induces a $G_i$,

$$1 \leq i \leq 4$$ and $S$ is no more a dominating set if $S$ induces one of $G_i$,
$1 \leq i \leq 3$.

**Case 1:** $\langle S \rangle = G_1$.

A vertex $v$ can dominate at the most 10 other vertices of $G$. Let $H = G - N[v]$, which induces the Clebsch graph. From Proposition 2.3.3, $\gamma(H) = 4$, whence any two vertices are not enough to dominate $V(H)$. This means, $S$ is not a dominating set.

**Case 2:** $\langle S \rangle = G_2$.

This case follows from Case 1.

**Case 3:** $\langle S \rangle = G_3$.

Let $S = \{u, v, w\}$ be a dominating set and suppose it induces a path on three vertices. Since $G$ is a strongly regular graph with parameters $(27, 10, 1, 5)$, $|\bigcup_{u \in S} N[u]| \leq 19$. This implies, the remaining vertices of $G$ are not covered by $S$ and, hence, $S$ is not a dominating set.

**Case 4:** $\langle S \rangle = G_4$.

Since $G$ is a strongly regular graph, $|N(u_i) \cap N(u_j)| = 1$, for every pair adjacent vertices $u_i$ and $u_j$ of $S$ in $G$. Therefore, $\bigcup_{i=1}^{3} N[u_i] = V(G)$, whence $S$ is a dominating set in $G$. \qed

With the help of Theorem 2.4.2 and Figure 2.12, following is the list of minimum dominating sets which induce the triangle $C_3$ is given below:
The subgraph induced by $N[u]$ contains $5C_3$. Thus, there are $\frac{5 \times 27}{3} = 45$ dominating sets which induce $C_3$ and they constitute a PBIBD with parameters $(27, 45, 5, 3, 1, 0)$.

**Theorem 2.4.3.** There exists a $(v, \gamma, \lambda, \mu)$-design over the Schlafli graph with the parameters $v = 27$, $\gamma = 3$, $\lambda = 1$, $\mu = 0$

### 2.5 $(v, \gamma, \lambda, \mu)$-design over the Shrikhande graph

S.S. Shrikhande, in 1959, proved that the strongly regular graph $L_2(n)$ is characterized by its parameters except for the case $n = 4$ and showed that for $n = 4$ there is a unique exception, the so called Shrikhande graph
$L_2'(4)$; it is a $(16,6,2,2)$ strongly regular graph which can be described as a Cayley graph as follows:

Let $H$ be a finite group. A Cayley subset $S$ of $H$ is a generating set of $H$ with the property that $s \in S$ whenever $s^{-1} \in S$. A Cayley graph of a group $H$ with respect to $S$, where $S$ is a Cayley subset of $H$, denoted by $Cay(H, S)$ is a graph with vertex set $H$, where $x \in H$ is adjacent to $y \in H$ whenever $xy^{-1} \in S$. Now consider the group $H = Z_4 \times Z_4$ and $Cay(H, S)$ with $S = \{\pm(0,1), \pm(1,1), \pm(1,0)\}$, we denote an arbitrary element $(x,y)$ as $xy$, this case is nothing but the Shrikhande graph and is shown in Figure 2.14.
Proposition 2.5.1. The domination number of the Shrikhande graph is three.

Proof: Let $G$ be Shrikhande graph. By Lemma 2.1.2,

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta(G) + 1} \right\rceil = \left\lceil \frac{16}{6 + 1} \right\rceil = 3.$$ 

To prove $\gamma(G) = 3$, it is sufficient to prove that $\gamma(G) \leq 3$. As $S = \{00, 12, 31\}$ is a dominating set of $G$. This implies $\gamma(G) \leq 3$. Which proves $\gamma(G) = 3$. 

We can find the number of minimum dominating sets which contain 00. Consider the vertex 00 of the graph and the set

$N(00) = \{01, 30, 33, 03, 10, 11\}$. The subgraph $G_1$ induced by the set $V(G) - N[00]$ is as shown in Figure 2.15.

Consider the vertex 12 of $G_1$ and the set $N(12) = \{12, 02, 23, 13, 22\}$. The...
subgraph $G_2$ induced by the set $V(G_1) - N[12]$ is as shown in Figure 2.15.

This implies, $\{00,12,31\}, \{00,12,21\}$ are the only two dominating sets containing the vertices 00 and 12. Similarly doing for all vertices of $G_1$ we get the minimum dominating sets containing 00, which are listed below

$\{00,12,31\}, \{00,12,21\}, \{00,31,23\}, \{00,23,32\}, \{00,32,13\}, \{00,21,13\}$.

By the definition of $(v, \gamma, \lambda, \mu)$-design, the parametric values are constants and hence to show the non-existence of $(v, \gamma, \lambda, \mu)$-design, it is sufficient to show that one of $\gamma$ and $\mu$ is not constant with respect to the blocks of a $(v, \gamma, \lambda, \mu)$-design. Therefore, 00 and 20, being nonadjacent vertices, do not occur in the list of minimum dominating sets containing 00.

**Theorem 2.5.2.** There does not exist any $(v, \gamma, \lambda, \mu)$-design over the Shrikhande graph.
2.6 Nonexistence of \((v, \gamma, \lambda, \mu)\)-design over the Chang graph

S.S. Shrikhande, A.J. Hoffman in the late 1950’s and Chang have contributed to the development of J.J. Seidel’s proof of the classification of strongly regular graphs. Chang proved that \(T(n)\) are uniquely determined by their parameters when \(n\) is not 8. For \(n = 8\) there are exactly three graphs, the so called Chang graphs namely \(T_1(8), T_2(8)\) and \(T_3(8)\). We shall follow Seidel’s description of the three Chang graphs:

Let \(V\) be 2-subsets of the set \(\{0, 1, 2, 3, 4, 5, 6, 7\}\) and we denote an arbitrary 2-subset \(\{a, b\}\) of \(V\) as \(ab\) only.

Define,
\[
V_1 = \{04, 15, 26, 37\}
\]
\[
V_2 = \{01, 12, 23, 34, 45, 56, 67, 07\} \text{ and}
\]
\[
V_3 = \{01, 12, 23, 34, 04, 56, 67, 75\}
\]

Note that the above three sets correspond to three subgraphs of \(K_8\) namely \(4K_2, C_8\) and \(C_5 \cup C_3\).

For \(i \in \{1, 2, 3\}\), let
\[
E_i = \{xy \in V : x, y \in V_i \text{ or } x, y \notin V_i \text{ and } |x \cap y| = 1\} \cup \{xy \in V : x \in V_i, \}
\]
$y \notin V_i$ and $|x \cap y| = 0$.

The graphs $T_i(8) = (V, E_i)$, $i = 1, 2, 3$ are strongly regular graphs with parameters $(28, 12, 6, 4)$ and are called the Chang graphs.

By Lemma 2.1.2, $\gamma(G) \geq \frac{n}{\Delta(G)+1} = \frac{28}{12+1} = 3$. Since $G$ is a 12-regular graph, two vertices are not enough to dominate the whole vertex set. So, $\gamma(G) \leq 3$. This implies $\gamma(G) = 3$.

Now, we are ready to show the non existence of any $(v, \gamma, \lambda, \mu)$-design over $T_i(8)$ for $i = 1, 2, 3$. By the definition of a $(v, \gamma, \lambda, \mu)$-design, the parameters $v, \gamma, \lambda$ and $\mu$ are constants and hence to show the nonexistence of a $(v, \gamma, \lambda, \mu)$-design, it is sufficient to show that $\lambda$ or $\mu$ is not a constant with respect to the blocks of a $(v, \gamma, \lambda, \mu)$-design.

2.6.1 Nonexistence of $(v, \gamma, \lambda, \mu)$-design over $T_1(8)$

Consider the vertex 01 and the set $N(01) = \{02, 03, 05, 06, 07, 12, 13, 14, 16, 17, 26, 37\}$.

Consider the graph $G_1 = V(G) - N[01]$. Let us choose a vertex not adjacent to 01, say 04, and find the set $N(04)$; we have $N(04) = \{23, 25, 27, 35, 36, 56, 57, 67\}$. Now, find the graph $G_2 = V(G_1) - N[04]$.
In $G_2$, select a vertex of maximum degree, $|V(G_2)| - 1$. In this case, 24, 34, 46, 47 are the vertices of maximum degree.

So, $\{01, 04, 24\}$, $\{01, 04, 34\}$, $\{01, 04, 46\}$ and $\{01, 04, 47\}$ are the minimum dominating sets containing 01 and 04.

Similarly, for all $v \in V(G_1)$ we get the following minimum dominating sets containing 01:

$\{01,04,24\}\{01,24,56\}\{01,04,34\}\{01,24,46\}$

$\{01,34,57\}\{01,04,47\}\{01,35,47\}\{01,15,35\}$

$\{01,15,25\}\{01,15,56\}\{01,15,57\}$

Observe that 01 and 04 are two nonadjacent vertices occurring in four blocks whereas the non adjacent vertices 01 and 24 occur in two blocks, violating the definition of a $(v, \gamma, \lambda, \mu)$-design.

2.6.2 Nonexistence of $(v, \gamma, \lambda, \mu)$-design over $T_2(8)$

Consider the vertex 01 and the set

$N(01) = \{07, 12, 24, 25, 26, 27, 35, 36, 37, 46, 47, 57\}$.

Consider the graph $G_1 = V(G) - N[01]$. Let us choose a vertex not adjacent to 01, say 02, and find the set $N(02)$;

we have $N(02) = \{03, 05, 06, 07, 34, 56, 57, 67\}$. Now, find the graph


\[ G_2 = V(G_1) - N[02]. \]

In \( G_2 \), select a vertex of maximum degree, \( |V(G_2)| - 1 \). We see that 14, 15, 16, 17 are the vertices of maximum degree.

So, \{01, 02, 14\}, \{01, 02, 15\}, \{01, 02, 16\} and \{01, 02, 17\} are the minimum dominating sets containing 01 and 02.

Similarly, for all \( v \in V(G_1) \) we get the following minimum dominating sets containing 01:

\[
\begin{align*}
\{01,02,14\}, & \{01,02,15\}, \{01,02,16\}, \{01,02,17\}, \\
\{01,03,15\}, & \{01,03,16\}, \{01,03,17\}, \{01,03,23\}, \\
\{01,04,16\}, & \{01,04,17\}, \{01,05,13\}, \{01,05,17\}, \\
\{01,06,13\}, & \{01,06,14\}, \{01,16,67\}, \{01,23,56\}, \\
\{01,34,67\}, & \{01,34,56\}, \{01,16,67\}.
\end{align*}
\]

Observe that 01 and 02 are two nonadjacent vertices occurring in four blocks whereas the nonadjacent vertices 01 and 04 occur in only two blocks, violating the definition of a \((v, \gamma, \lambda, \mu)\)-design.

2.6.3 Nonexistence of \((v, \gamma, \lambda, \mu)\)-design over \( T_3(8) \)

Consider the vertex 01 and the set

\[ N(01) = \{04, 12, 24, 25, 26, 27, 35, 36, 37, 45, 46, 47\}. \]
Consider the graph $G_1 = V(G) - N[01]$. Let us choose a vertex not adjacent to 01, say 02, and find the set $N(02)$; we have $N(02) = \{03, 04, 05, 06, 34, 45, 56, 67\}$. Now, find the graph $G_2 = V(G_1) - N[02]$.

In $G_2$, select a vertex of maximum degree, $|V(G)| - 1$. We see that 14, 15, 16, 17 are the vertices of maximum degree.

So, \{01, 02, 14\}, \{01, 02, 15\}, \{01, 02, 16\} and \{01, 02, 17\} are the minimum dominating sets containing 01 and 02.

Similarly, for all $v \in V(G_1)$ we get the following minimum dominating sets containing 01:

\[
\begin{align*}
\{01,02,14\}, & \{01,02,15\}, \{01,02,16\}, \{01,02,17\}, \\
\{01,03,15\}, & \{01,03,16\}, \{01,03,17\}, \{01,03,23\}, \{01,05,13\}, \\
\{01,05,14\}, & \{01,06,13\}, \{01,06,14\}, \{01,07,13\}, \\
\{01,07,14\}, & \{01,13,34\}, \{01,23,67\}, \{01,23,57\}, \\
\{01,34,56\}, & \{01,34,57\}, \{01,34,67\}
\end{align*}
\]

Observe that 01 and 02 are two nonadjacent vertices occurring in four blocks whereas the nonadjacent vertices occur in only two blocks, violating the definition of a $(v, \gamma, \lambda, \mu)$-design.

Thus, we have established the following result.
Theorem 2.6.1. There does not exist any $(v, \gamma, \lambda, \mu)$-design over any of the three Chang graphs $T_1(8), T_2(8)$ and $T_3(8)$.

2.7 Conclusion and scope

In this chapter, we have considered $(v, \gamma, \lambda, \mu)$-design over particular class of regular graphs. The study undertaken in this chapter will give new way of constructing PBIBD's through the minimum dominating sets in a regular graph. Walikar et al. [10] has already studied the partially balanced incomplete block designs associated with minimum dominating sets of cycles, paths and cubic graphs on ten vertices. One may also consider the designs whose blocks are subsets of a vertex set of regular graph with given property such as vertex cover, edge-independent sets and many other properties associated with edge set and vertex set of a graph.

Open Problem: Characterize strongly regular graphs $G$, with $\mu_G$ as the set of distinct (up to isomorphism) subgraphs induced by the minimum dominating sets of $G$, such that for each $H \in \mu_G$ the set of minimum dominating sets whose induced subgraphs are isomorphic to $H$ is a PBIBD and $\cup_{H \in \mu_G}PBIBD(H)$ is a BIBD denoted $BIBD(\mu_G)$.
REFERENCES


