Chapter 5

NUMBER OF DOMINATING SETS OF CIRCULANT GRAPH
5.1 Introduction

Circulant graphs have a vast number of uses and applications in telecommunication networks, VLSI design, parallel and distributed computing, main interest in circulant graphs lies in the role they play in the design of networks. In this chapter, we obtain the number of minimum dominating sets of certain class of circulant graphs. In particular, we found the total number of minimum dominating sets of circulant graph of order $6k + i$, where $1 \leq i \leq 5$ and regularity $2k$.

Bosech and Tindell 1984 [1] defined circulant graph as follows, for an integer $n \geq 3$ and a subset $S$ of $\{1, 2, \ldots, \lceil \frac{n+2}{2} \rceil \}$, the circulant graph $C_n(S)$ is a graph on $n$ vertices $u_1, u_2, \ldots, u_n$ such that each vertex $u_i$ is adjacent to the vertices $u_{i+s}$ for $s \in S$, where the subscripts are taken modulo $n$. Certainly, $C_n(\{1\})$ is isomorphic to the cycle $C_n$ and $C_n(\{1, 2\})$ is isomorphic to the square $C_n^2$ of $C_n$. It is easy to observe that circulant graphs are vertex-symmetric.

Circulant matrix is obtained by taking an arbitrary first row, and shifting it cyclically one position to the right in order to obtain successive rows. Formally, if the first row of the $n$-by-$n$ circulant matrix is $a_0, a_1, \ldots, a_{n-1}$
then the \((i,j)^{th}\) element is \(a_{j-i}\) where the subscripts are taken modulo \(n\).

The term circulant arises from the fact that the adjacency matrix of such a graph is a circulant matrix. Figure 5.1, shows the circulant graph of order twelve with connection set \(S = \{1, 2\}\)

**Example**

![Circulant graph](image)

**Figure 5.1: Circulant graph \(C_{12}(\{1, 2\})\)**

Topics on domination number and related parameters have long attracted graph theorists for their applications and theoretical interest. The general problem of determining \(\gamma(G)\) for a given graph \(G\), and of finding a dominating set \(S\) of \(G\) of this minimum cardinality, has been an active area of research for many years. Although, the problem of finding \(\gamma(G)\) domination number of a graph \(G\), has been extensively studied. Very less
research with respect to counting the number of minimum dominating sets has been done. In this chapter, we determine largest number of minimum dominating sets of some circulant graphs.

In [2, 3] Chelvam et al., have obtained the domination number and independent domination number for certain Cayley graph. As circulant graphs are Cayley graphs but the converse is not true, so we are using the results of [2, 3] and calculate domination number of circulant graphs of order $6k + i$ where $0 \leq i \leq 5$ of regularity $2k$ where $k \geq 2$.

A set $S \subseteq V$ of vertices in a graph $G$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A dominating set $S$ is a minimum dominating set if no proper subset is a dominating set. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set in $G$. The number of minimum dominating sets is denoted by $\gamma_D$. Following preliminary results are used to determine the number of minimum dominating sets of circulant graphs.

**Theorem 5.1.1.** [4] For any graph $G$ with $n$ vertices and maximum degree $\Delta$, we have $\left\lceil \frac{n}{\Delta(G)+1} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.

**Theorem 5.1.2.** [2] Let $G = C_n(S)$ be a circulant graph, where $S \subseteq$
\( \{1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \} \) then \( \gamma(G) = \left\lfloor \frac{n}{|S|+1} \right\rfloor \).

Theorem 5.1.3. [5] \( \gamma_D(P_n) = \begin{cases} 
1, & \text{if } n = 3k, k \geq 1; \\
\frac{k^2 + 5k + 2}{2}, & \text{if } n = 3k + 1, k \geq 0; \\
k + 2, & \text{if } n = 3k + 2, k \geq 0;
\end{cases} \)

Theorem 5.1.4. [5] \( \gamma_D(C_n) = \begin{cases} 
3, & \text{if } n = 3k, k \geq 1; \\
\frac{(3k+1)(k+2)}{2}, & \text{if } n = 3k + 1, k \geq 1; \\
3k + 2, & \text{if } n = 3k + 2, k \geq 1;
\end{cases} \)

Remark 5.1.5. When \( |S| = n - 1 \), then \( G = C_n(S) \) is complete and hence \( \gamma(G) = 1 \).

Remark 5.1.6. \( |S| \) is odd. For example, consider \( C_8(S) \) where \( |S| = 3 \). Note that the only generating sets of cardinality 3 are \( \{1, 4, 7\} \) and \( \{3, 4, 5\} \). When \( |S| = \{1, 4, 7\} \), \( \gamma(C_8(S)) = 3 = \left\lfloor \frac{8}{3+1} \right\rfloor + 1 \).

5.2 Results

Theorem 5.2.1. \( \gamma(G) = \begin{cases} 
3, & \text{for } n = 6k + i, i = 0, 1, 2, 3 \\
4, & \text{for } n = 6k + i, i = 4, 5
\end{cases} \)

Proof: Let \( G \) be a circulant graph. We prove this Theorem in two cases,

Case 1: For \( n=6k+i, i=0,1,2,3 \)

108
As $G$ is a circulant graph of order $n = 6k + i$, $k \geq 2$ and $0 \leq i \leq 3$ be a positive integers, with regularity $2k$. By the Theorem 5.1.1,

$$\gamma(G) = \left\lceil \frac{n}{|S| + 1} \right\rceil = \left\lceil \frac{6k + i}{2k + 1} \right\rceil = 3$$

for any $k$, where $0 \leq i \leq 3$ and $|S|$ is the regularity of $G$.

**Case 2:** For $n=6k+i$, $i=4,5$

As $G$ is a circulant graph of order $n = 6k + i$, $k \geq 2$ and $i = 3,4$ be a positive integers, with regularity $2k$. By the Theorem 5.1.1

$$\gamma(G) = \left\lceil \frac{n}{|S| + 1} \right\rceil = \left\lceil \frac{6k + i}{2k + 1} \right\rceil = 4$$

for any positive integer $k$ and $i = 3,4$, where $|S| = 2k$ is the regularity of $G$.

\[ \square \]

Now, we prove the main Theorem of this chapter, which gives the total number of minimum dominating sets of circulant graphs.

**Theorem 5.2.2.** Let $G$ be a circulant graph of order $n$, with regularity $r = 2k$ for any positive integer $k \geq 2$ then,
\( \gamma_D(G) = \begin{cases} 
20k, & \text{for } n = 6k; \\
2(6k + 1), & \text{for } n = 6k + 1; \\
6k + 2, & \text{for } n = 6k + 2; \\
2k + 1, & \text{for } n = 6k + 3; \\
\frac{(6k+4)35}{4}, & \text{for } n = 6k + 4. \\
(k - 1)(2k + 1)(6k + 5), & \text{for } n = 6k + 5. 
\end{cases} \)

**Proof:** Let \( G \) be a circulant graph of order \( n \), with regularity \( r = 2k, k \geq 2 \) is a positive integer. We consider the following cases

**Case 1:** Let \( n = 6k, k \geq 2 \) be a positive integer.

Let \( D_i = \{0, 2j + i, 4j + (i + 2)\} \) for a integer \( i = 1, 2, j \geq 1 \) and \( D_i = \{0, 2j + (i - 2), 4j + i\} \), for a integer \( i = 3, 4, j \geq 1 \) are the only generating minimum dominating sets (\( \gamma \)-sets) of circulant graph of order \( 6k \), regularity \( 2k \). For \( D_i, i = 1, 2, 3 \) three generating minimum dominating sets generates total \( 3 \times 6k \) number of minimum dominating sets and the minimum dominating sets \( D_4 \) generates \( 2k \) number of minimum dominating sets. Therefore, \( D_i, 1 \leq i \leq 4 \) generates \( 3 \times 6k + 2k = 20k \) \( \gamma \)-sets.

**Case 2:** If \( n = 6k + 1, k \geq 2 \) be a positive integer.

Let \( D_i = \{0, 2j + i, 4j + 4\} \) where \( i = 1, 2, j \geq 1 \) are the only two generating
\(\gamma\)-sets of circulant graph of order 6\(k+1\) and of regularity 2\(k\). This implies \(D_i\) generates total \(2 \times (6k + 1)\) number of minimum dominating sets.

**Case 3:** If \(n = 6k + 2\), \(k \geq 2\), be a positive integer.

Let \(D = \{0, j + 4, 4j + 5\}\) is the only generating \(\gamma\)-set of the circulant graph of order 6\(k + 2\). Therefore total number of minimum dominating sets generated by the \(\gamma\)-set \(D\) are 6\(k + 2\).

**Case 4:** If \(n = 6k + 3\), \(k \geq 2\) be a positive integer.

Then, \(D = \{0, 2j + 3, 4j + 6\}\), \(j \geq 1\) is the only \(\gamma\)-set which generates total \(2k + 1\) number of minimum dominating sets.

**Case 5:** If \(n = 6k + 4\), \(k \geq 2\) be a positive integer.

Let us label the vertex set of circulant graph as \(v_0, v_1, \ldots, v_{6k+3}\).

The vertices \(v_0, v_1\) dominates 2\((k + 1)\) vertices, the remaining 2\((2k + 1)\) vertices dominated by two independent vertices which dominates 2\(k + 1\) vertices respectively. Therefore \(v_0, v_1\) contained in only one minimum dominating sets. Hence combination of \(v_0, v_1, v_{2i+4}, v_{4i+7}\) is the minimum dominating set of the circulant graph. Similarly, \(v_0, v_2\) contained in three minimum dominating sets, \(v_0, v_3\) contained in six minimum dominating sets, \(v_0, v_4\) contained in ten minimum dominating sets, \(v_0, v_5\) contained in fifteen minimum dominating sets. Thus, the vertex \(v_0\) contained in 35
number of minimum number of dominating sets. Clearly, the total number of minimum dominating sets are \( \frac{35 \times (6k+4)}{4} \).

**Case 6:** If \( n = 6k + 5, \ k \geq 2 \) be a positive integer.

Let us label the vertex set of circulant graph as \( v_0, v_1, \ldots, v_{6k+3} \).

The vertices \( v_0, v_1 \) dominates \( 2(k + 1) + 1 \) vertices, the remaining \( 6k + 5 - (2(2k + 1) + 1) \) vertices dominated by two independent vertices which dominates \( 2k + 1 \) vertices respectively. Therefore \( v_0, v_2 \) contained in only one minimum dominating sets. Hence, the combination of \( v_0, v_2, v_{2i+5}, v_{4i+8} \) is the minimum dominating set of the circulant graph. Similarly, counting as in case 5, we get \( (k - 1)(2k + 1)(6k + 5) \) total number of dominating sets of circulant graph of order \( 6k + 5 \).

\[ \square \]
REFERENCES


