Chapter 4

DESIGNS ASSOCIATED WITH CUBIC GRAPHS
4.1 Introduction

In this chapter, we establish a link between PBIBD and cubic graphs through the collection of maximum independent sets. The maximum independent set (MIS) in a graph has important applications and needs exact algorithm to find it. The execution time complexity of the available exact algorithms to find the MIS tend to be an exponential growth function. One algorithm of the exact algorithms is the Modified Wilfs (MW) algorithm. It can’t handle graphs with larger sizes. In [3] introduced a new algorithm, named Finding Maximum Independent Set in a graph (FMIS), to find a MIS set with less run time and complexity and close to $O(n^4)$, for a graph of $n$ nodes. We use the FMIS algorithm to calculate maximum independent sets and independence number of cubic graphs.

With the continuation of the study of $(v, \beta_0, \mu)$-design over a regular graph $G$ particularly, cubic graphs on ten vertices. Since, Petersen graph is one of the cubic graph, which is known to be a strongly regular graph and is associated with PBIBD with two association scheme. Petersen graph is known to possess many important properties of graphs such as being distance transitive, distance regular, being a Moore graph of diameter
two, strongly regular graph and many more. Also, it serves as a counter
eexample to many conjectures. The whole book entitled “Petersen Graph” was written by Sheehan and Holton [1] is devoted to study this graph.

A \((v, \beta_0, \mu)\)-design over a regular graph \(G = (V, E)\) of degree \(d\) is an ordered pair \(D = (V, B)\), where \(|V| = v\) and \(B\) is the set of maximum independent sets of \(G\) called blocks, such that if \(i, j \in V\), \(i \neq j\) and if \(i\) and \(j\) are not adjacent in \(G\) then there are exactly \(\mu\) blocks containing \(i\) and \(j\).

In the Chapter 3, we proved that there exist a \((v, \beta_0, \mu)\)-design over
the Petersen graph with the parameter \(v = 10\), \(\beta_0 = 4\) and \(\mu = 1\). This motivated us to investigate the study of \((v, \beta_0, \mu)\)-design associated with
the other cubic graphs on ten vertices. Here we prove that there does
not exist \((v, \beta_0, \mu)\)-design over cubic graph on ten vertices except for the
Petersen graph.

4.2 Results

We now proceed to determine the existence and non existence of
\((v, \beta_0, \mu)\)-designs over the cubic graphs on ten vertices. There are 21 cubic
graphs on ten vertices which are given in Figure 4.1, 4.2, 4.3. and 4.4.
Figure 4.1: Cubic graphs
Figure 4.2: Cubic graphs continued...
Figure 4.3: Cubic graphs continued...
Theorem 4.2.1. There exist a \((v, \beta_0, \mu)\)-design over the Petersen graph with the parameter \(v = 10\), \(\beta_0 = 4\) and \(\mu = 1\).

Proof: The only maximum independent sets of the Petersen graph \(G_{18}\) are \(\{1, 6, 7, 3\}\), \(\{2, 4, 6, 10\}\), \(\{3, 5, 9, 10\}\), \(\{1, 4, 8, 9\}\), \(\{2, 5, 7, 8\}\). One can easily verify the condition \(vr = bk\) as \(v = 10\), \(b = 5\), \(r = 2\) and \(k = 4\). By considering the above maximum independent sets as blocks and association scheme can be defined with two elements \(\alpha\) and \(\beta\) as the 1st associates if
they are adjacent vertices in $G$ and they are second associates otherwise.

Thus we have Association Scheme

<table>
<thead>
<tr>
<th>Elements</th>
<th>First Associates</th>
<th>Second Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2, 5, 10</td>
<td>3, 4, 6, 7, 8, 9</td>
</tr>
<tr>
<td>2</td>
<td>1, 3, 9</td>
<td>4, 5, 6, 7, 8, 10</td>
</tr>
<tr>
<td>3</td>
<td>2, 4, 8</td>
<td>1, 5, 6, 7, 9, 10</td>
</tr>
<tr>
<td>4</td>
<td>3, 5, 7</td>
<td>1, 2, 6, 8, 9, 10</td>
</tr>
<tr>
<td>5</td>
<td>1, 4, 6</td>
<td>2, 3, 7, 8, 9, 10</td>
</tr>
<tr>
<td>6</td>
<td>5, 8, 9</td>
<td>1, 2, 3, 4, 7, 10</td>
</tr>
<tr>
<td>7</td>
<td>4, 9, 10</td>
<td>1, 2, 3, 5, 6, 8</td>
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<tr>
<td>8</td>
<td>3, 6, 10</td>
<td>1, 2, 4, 5, 7, 9</td>
</tr>
<tr>
<td>9</td>
<td>2, 6, 7</td>
<td>1, 3, 4, 5, 8, 10</td>
</tr>
<tr>
<td>10</td>
<td>1, 7, 8</td>
<td>2, 3, 4, 5, 6, 9</td>
</tr>
</tbody>
</table>

With this association scheme, one can easily verify that the maximum independent sets of $G_{18}$ forms a PBIB design with parameters $v = 10$, $b = 5$, $r = 3$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$. Hence, there exist a $(v, \beta_o, \mu)$-design over the Petersen graph with the parameter $v = 10$, $\beta_o = 4$ and $\mu = 1$. □

**Theorem 4.2.2.** There doesn't exist a $(v, \beta_o, \mu)$-designs over cubic graphs on ten vertices except $G_8$.  

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**Proof**: In proving the graphs listed above, is not a \((v, \beta_o, \mu)\)-design, we only consider three graphs i.e., \(G_1\), \(G_{11}\) and \(G_{12}\) and all others can be proved in the similar fashion. Now, in \(G_1\), there are only eight maximum independent sets (this can be obtained through the AFS algorithm [44]) which are listed below:

\[
\begin{align*}
\{1, 3, 5, 7\}, \{1, 3, 5, 8\}, \{1, 4, 7, 9\}, \{1, 5, 7, 9\}, \{2, 4, 6, 9\}, \{2, 4, 6, 10\}, \\
\{2, 6, 8, 10\}, \{3, 5, 8, 10\}.
\end{align*}
\]

There are two maximum independent sets containing non adjacent vertices 1, 3 where as there is only one maximum independent sets containing 1, 4, which violate the condition of the \((v, \beta_o, \mu)\)-design.

For \(G_{11}\), a bipartite graph, the maximum independent sets are \(\{1, 3, 5, 7, 9\}\) and \(\{2, 4, 6, 8, 10\}\). The non adjacent vertices 1, 3 occur in one maximum independent set, but the non adjacent pair 1, 4 is not occurring in any of the maximum independent set. This violates the definition of \((v, \beta_o, \mu)\)-design.

For \(G_{12}\), the only maximum independent set is \(\{2, 5, 7, 10\}\), which implies no pair of maximum independent set other than \(\{2, 5\}\) \(\{2, 7\}\) \(\{2, 10\}\) \(\{5, 7\}\) \(\{5, 10\}\) \(\{7, 10\}\) occurs in maximum independent set. This fails to satisfy the definition of \((v, \beta_o, \mu)\)-design. \qed

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REFERENCES


