Chapter 3

DESIGNS ASSOCIATED WITH
MAXIMUM INDEPENDENT SETS OF A
GRAPH
3.1 Introduction

The purpose of this chapter is to demonstrate the relation of design theory to certain concepts of graph theory. In this chapter, we construct PBIBDs with two association schemes through \((v, \beta_o, \mu)\)-designs corresponding to the graphs \(K_n \times K_n\), \(T(n)\)-triangular graphs, \(L_2(n)\)-square lattice graph, the Petersen graph, the Shrikhande graph, the Clebsch graph and the Schlafli graph.

A \((v, \beta_o, \mu)\)-design over a regular graph \(G = (V, E)\) of degree \(d\) is an ordered pair \(D = (V, B)\), where \(|V| = v\) and \(B\) is the set of maximum independent sets of \(G\) called blocks, such that if \(i, j \in V, i \neq j\) and if \(i\) and \(j\) are not adjacent in \(G\) then there are exactly \(\mu\) blocks containing \(i\) and \(j\).

The \((v, \beta_o, \mu)\)-design is similar to the \((v, k, \lambda, \mu)\)-design over a regular graph \(G\) introduced by Y. J. Ionin and M. S. Shrikhande [11]. The blocks in [11] are certain \(k\)-subsets of the points, whereas the blocks in the present chapter are of size \(k = \beta_o\), the independence number of the graph \(G\). Ionin and Shrikhande [11] considered the study of \((v, k, \lambda, \mu)\)-design when the replication is \(r = 2\lambda - \mu\), whereas we consider the designs with \(r = \frac{(v-d-1)\mu}{\beta_o - 1}\) (we get this relation by the structure of \((v, \beta_o, \mu)\)-design described in the next
Furthermore, it is well known that a given PBIBD with two association schemes having the parameters \((v, b, r, k_1, k_2)\). One can easily construct a strongly regular graph \(G\) with the parameters \((v, n_1, p_{11}^1, p_{11}^2)\).

(We discuss about \((v, n_1, p_{11}^1, p_{11}^2)\) in the next section). On the other hand, it is also known that given a strongly regular graph \(G\), there is a PBIBD associated with it (not necessarily unique; see for details Raghavarao [14]). But given a strongly regular graph with parameters \((v, n_1, p_{11}^1, p_{11}^2)\) it is almost impossible to construct a PBIBD with two association schemes; however some efforts have been made to do so in Y. J. Ionin and M. S. Shrikhande [11] in the case of strongly regular graphs, namely the triangular graph \(T(n)\), the square lattice graph \(L_2(n)\), the Petersen graph, the Clebsch graph, the Shrikhande graph and the Schláfli graph.

A set \(S\) of vertices in a graph \(G\) is said to be an independent set, if no two vertices in \(S\) are adjacent in \(G\). The independence number \(\beta_0(G)\), is the cardinality of a largest independent set in \(G\). A subset \(M\) of an edge set \(E = E(G)\) of a graph \(G\) is called a matching of \(G\) if no two edges in \(M\) have a common vertex. If \(|M| = k\), then the set \(M\) is called a \(k\)-matching in \(G\) and the largest value of \(k\) in a \(k\)-matching of \(G\) is called the edge independence number of \(G\), denoted \(\beta_1(G)\).
A strongly regular graph $G$ is said to be primitive if neither $G$ nor its complement $\overline{G}$ is disconnected. Following will be used in the proof of next results.

**Proposition 3.1.1.** [6] A graph $G$ is a primitive strongly regular graph if and only if it has exactly three distinct eigenvalues.

Note that, by the eigenvalues of a graph, we mean, the eigenvalues of its adjacency matrix.

If $G$ is a graph of order $n$, then $G$ has $n$ eigenvalues $\theta_1, \theta_2, \ldots, \theta_n$ which are usually taken to be so labeled that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$.

**Definition 3.1.2.** [5] Let $G$ be a graph on $n$ vertices. Let $m(G, k)$ be the number of $k$–matchings of $G$, $1 \leq k \leq \beta_1(G)$, and set $m(G, 0) = 1$. Then the matching polynomial of $G$ is

$$M(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k}. \quad (3.1.1)$$

For details on matching polynomials, one can refer to E. J. Farrel [5].

The following results on matching polynomial are useful in calculation of number of maximum independent sets in the triangular graph $T(n)$ and the square lattice graph $L_2(n)$ which we consider in the next section.
Theorem 3.1.3. [5] Let $G = (V, E)$ be a graph with $n \geq 3$ vertices and $m$ edges. If $e = uv \in E$, then
\[ m(G, x) = m(G - e, x) - m(G - u - v, x). \] (3.1.2)

Corollary 3.1.4. [5] Let $G = (V, E)$ be a graph on $n \geq 3$ vertices. Let $u \in V$ be a vertex of degree $l$ and suppose $N(u) = \{v_1, v_2, \ldots, v_l\}$. Then,
\[ m(G, x) = x m(G - u, x) - \sum_{i=1}^{l} m(G - u - v_i, x) \] (3.1.3)
where the empty sum (corresponding to the case when the degree of $u$, $\text{deg}(u) = 0$) is zero.

The following two results give bounds on the independence number $\beta_0(G)$ of a graph in terms of its eigenvalues. The first one is due to an unpublished result of A.J. Hoffman and the second one is due to D. M. Cvetkovic [3].

Theorem 3.1.5. [9] An independent set $S$ is maximal independent if and only if it is independent and dominating.

Theorem 3.1.6. [9] Every maximal independent set in a graph $G$ is a minimal dominating set of $G$. 

62
Theorem 3.1.7. [3] If $G$ is regular graph of order $n$, then

$$
\beta_o(G) \leq n \frac{-\theta_n}{\theta_1 - \theta_n},
$$

and if the cardinality of the maximum independent set $S$ attains this bound, then every vertex not in $S$ is adjacent to precisely $-\theta_n$ vertices of $S$.

Theorem 3.1.8. [3] $\beta_o(G) \leq |\{i/\theta_i \geq 0\}|$ or $\beta_o(G) \leq |\{i/\theta_i \leq 0\}|$.

Theorem 3.1.9. [3] Let $G_1$ and $G_2$ be two graphs of orders $n_1$ and $n_2$ respectively. Let $\theta_1, \theta_2, \ldots, \theta_{n_1}$ be the eigenvalues of $G_1$ and $\gamma_1, \gamma_2, \ldots, \gamma_{n_2}$ and be the eigenvalues of $G_2$. Then,

1. The eigenvalues of $G_1 \times G_2$ are $\theta_i + \gamma_j$, for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.

2. The eigenvalues of $G_1 \oplus G_2$ are $\theta_i \gamma_j$, for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$. 
3.2 \((v, \beta_0, \mu)\)-designs over a regular graphs

We begin this section by considering the graph \(G = K_3 \times K_3\), the Cartesian product of \(K_3\) with itself as labeled as in the Figure 3.1.

Figure 3.1: \(G = K_3 \times K_3\)

Clearly, the size of a maximum independent set in this graph is 3 and the only maximum independent sets in \(G\) are

\[
\{0, 4, 8\}, \{0, 5, 7\}, \{1, 3, 8\}, \{1, 5, 6\}, \{2, 4, 6\} \text{ and } \{2, 3, 7\}.
\]

The set of these independent sets forms a \((v, \beta_0, \mu)\)-design, since it is \((7, 3, 1)\)-design. Whereas for the graph \(G = C_7\) as labeled in the Figure 3.2, no \((v, \beta_0, \mu)\)-design of this kind exists.

Figure 3.2: \(G = C_7\)
Since the sets \{0,2,4\}, \{0,2,5\}, \{0,3,5\}, \{1,3,5\}, \{1,3,6\}, \{1,4,6\} and 
\{2,4,6\} in which the pair of nonadjacent vertices 0,2 occurs in two inde­
pendent set, whereas the pair of nonadjacent vertices 0,3 occurs in only 
one independent set. It violates the definition of a \((v, (\beta_0, \mu))-\text{design}\). The 
following proposition establishes the relation between \((v, \beta_0, \mu))-\text{design} and 
the regular graph \(G\) of order \(v\), of degree \(d\) and the independence number 
\(\beta_0\), if the design exists over \(G\).

**Proposition 3.2.1.** Let \(D = (V, B)\) be \((v, \beta_0, \mu))-\text{design} over a regular 
graph \(G\) with the degree of valency \(d\). Then there exists an integer \(r\) (called 
the replication number of \(D\)) such that any vertex \(x \in V\) is contained in 
extactly \(r\) blocks and such that the following three conditions hold:

\[(v - d - 1)\mu = r(\beta_0 - 1) \quad (3.2.1)\]

\[vr = b\beta_0 \quad (3.2.2)\]

\[r \geq \mu \quad (3.2.3)\]

where \(b = |B|\)
**Proof:** To prove (3.2.1), choose an arbitrary vertex \( u \) in \( G \) and

\[
N[u] = u \cup \{v \in V/uv \in E\}
\]

(called the closed neighborhood of \( u \) in \( G \)) and \( B_u = \{b \in B/u \in B\} \). By two way counting method between the sets \( V - N[u] \) and \( B_u \), the equation (3.2.1) follows.

To prove (3.2.2), consider the set of pairs of nonadjacent vertices which are \( \binom{v}{2} - \binom{v-1-d}{2} \) in number and the set of blocks \( B \), again by counting in two ways, we have

\[
\binom{v}{2} - \binom{v-1-d}{2} = b_\beta. \quad \text{On simplification of this and using the equation (3.2.1), we have the desired result. The inequality (3.2.3) follows from the definition of } (v, \beta_0, \mu)\text{-design itself.} \quad \square
\]

Next proposition shows the existence of a \((v, \beta_0, \mu)\)-design in case the equality in (3.2.3) of Proposition 3.2.1.

**Proposition 3.2.2.** Let \( D = (V, B) \) be a \((v, \beta_0, \mu)\)-design over a regular graph \( G \) of degree \( d \). Then, \( D \) is a \((v, \beta_0, \mu)\)-design over \( G \) with \( r = \mu \) if and only if \( G = K_{d,d} \) and \((v, \beta_0, \mu)\)-design is a design with the parameters \((2d, d, 1)\).
**Proof:** Let $D = (V, B)$ be $(v, \beta_0, \mu)$-design over a regular graph $G$ of degree $d$ and suppose that $r = \mu$. Then, by equation (3.2.1), we have $\beta_0 = v - d$. Let $S$ be a maximum independent set in $G$, whence $|S| = v - d$.

Now, we claim that $v - d = d$. For, if $v - d > d$ then degree of $u$ in $V - S$ is greater than $d$ since every vertex in $S$ is adjacent to every vertex in $V - S$, which contradicts the valency of $G$. On the other hand, if $v - d < d$, then $G$ contains only one maximum independent set, namely $S$ itself, which implies that a vertex in $V - S$ does not belong to any maximum independent set, a contradiction to the fact that $D$ is a $(v, \beta_0, \mu)$-design. This implies that $v = 2d$ and $\beta_0 = d$. This, in turn, implies that $G = K_{d,d}$ and hence $\mu = 1$.

Converse is obvious. $\square$

**Proposition 3.2.3.** Let $G$ be any graph and $\{u_1, u_2, \ldots, u_k\}$ be a set of independent vertices in $G$ and let $H$ be the graph obtained by removal of vertices in $\cup_{i=1}^k N[u_i]$ from $G$. If $S$ is a maximal independent set in $H$, then $S \cup \{u_1, u_2, \ldots, u_k\}$ is a maximal independent set in $G$.

**Proof:** Let $H$ be the graph as defined in the statement of the proposition. Then, $V(H) = V(G) - \cup_{i=1}^k N[u_i]$ and let $S$ be a maximal independent set in $H$. Then, every vertex in $V(H) - S$ is adjacent to at least one vertex in $S$. Now, let $v \in V(G) - S \cup \{u_1, u_2, \ldots, u_k\} = (V(H) - S) \cup (\cup_{i=1}^k N(u_i))$. If
\( v \in V(H) - S \), then \( v \) is adjacent to at least one vertex in \( V(H) - S \) and is \( v \in \bigcup_{i=1}^{k} N(u_i) \). Hence, \( v \) is adjacent to some vertex \( u_j \) in \( \{u_1, u_2, \ldots, u_k\} \).

Thus, in either case, \( v \) is adjacent to at least one vertex in \( V(G) - S \cup \{u_1, u_2, \ldots, u_k\} \). Hence, \( S \) is a maximal independent set in \( G \).

\[ \square \]

**Corollary 3.2.4.** If \( \{u\} \) is any vertex in \( G \) and \( S \) is a maximum independent set in \( G - u \) then \( S \cup \{u\} \) is a maximum independent set in \( G \).

### 3.3 \((v, \beta_0, \mu)\)-designs over \( K_n \times K_n \)

Let \( G = K_n \times K_n \) be a graph obtained by the cartesian product of the complete graph on \( n \geq 2 \) vertices with itself. Clearly, \( G \) has \( n^2 \) vertices and is a regular graph of degree \( 2(n-1) \). As the spectrum of \( K_n \) is

\[
\begin{pmatrix}
  n-1 & -1 \\
  1 & n-1
\end{pmatrix}
\]

we can get the spectrum of \( K_n \times K_n \) by using the Theorem 3.1.9 as

\[
\begin{pmatrix}
  2(n-1) & n-2 & -2 \\
  1 & 2(n-1) & (n-1)^2
\end{pmatrix}
\]
and hence we get $\lambda_1 = 2(n - 1)$ and $\lambda_n = -2$. Clearly, if we arrange the vertices of $G = K_n \times K_n$ in the form of an $n \times n$ array, then the vertices of $G$ which are on the diagonal form an independent set in $G$ and thus $\beta_0(G) \geq n$ holds. On the other hand, by using the Theorem 3.1.7, we have $\beta_0(G) \leq \frac{-n^2(-2)}{2(n-1)(-2)} = n$ and, therefore, $\beta_0(G) = n$ holds. Recently Walikar et al., designed a polynomial time algorithm to generate the total number of maximum independent sets in $K_m \times K_n$ for $m, n \geq 2$ and thereby found that, $M(K_n \times K_n) = nM(K_{n-1} \times K_{n-1})$, where $M(K_n \times K_n)$ denotes the number of maximum independent sets in $K_n \times K_n$. Also one can observe that every vertex in $K_n \times K_n$ occurs in exactly maximum independent sets and hence using the equation $(v - d - 1)\mu = r(\beta_o - 1)$ we get $\mu = \frac{M(K_n \times K_n)}{n(n-1)}$. Thus, we have a $(v, \beta_o, \mu)$-design with the parameters $(n^2, n!, n, 0, (n-2)!)$ where $v = n^2$, $b = n!$, $r = (n-1)!$, $\beta_o = n$, $\mu = (n-2)!$ and whose blocks are the maximum independent sets of the graph.

**Theorem 3.3.1.** There exist $(v, \beta_o, \mu)$-design over $K_n \times K_n$ with $v = n^2$, $\beta_o = n$, and $\mu = (n - 2)!$.

**Remark 3.3.2.** It is noted in [11] that $(v, k, \lambda, \mu)$-designs over strongly regular graphs $G$ are PBIBDs with two association schemes and hence the $(n^2, n, (n - 2)!)$-design obtained above is a PBIBD with two association schemes.
schemes because $G = K_n \times K_n$ is a strongly regular graph as evident by its spectrum.

### 3.4 $(v, \beta, \mu)$-design over the triangular graph $T(n)$

Let $X = \{u_1, u_2, \ldots, u_n\}$ be any set with $n \geq 4$. The regular graph $T(n)$ is the graph whose vertex set $V$ consists of all 2-subsets of $X$ and two vertices $\{u_i, u_j\}$ and $\{u_r, u_s\}$ are adjacent in $T(n)$ if and only if $|\{u_i, u_j\} \cap \{u_r, u_s\}| = 1$; in fact, $T(n)$ is the line graph of the complete graph $K_n$. The maximum independent set in $T(n)$ is nothing but the maximum matching in $K_n$. The number of edges in a maximum matching is called the edge independence number of $G$, denoted $\beta_1(G)$. Thus, by the definition of $T(n)$, we have $\beta_0(T(n)) = \beta_1(K_n)$. If we label the vertices of $K_n$ as $u_1, u_2, \ldots, u_n$, then the maximum matching $M$ in $K_n$ is \{uiuj\} $\cup \hat{M}$, where $\hat{M}$ is the maximum matching in $K_n - u_i - u_j$. Then, the set of all maximum matchings of $K_n$ can be obtained recursively by taking all values of $i$ and $j$ and $i \neq j$.

Thus, if $D = (V, B)$ is $(v, \beta, \mu)$-design over $T(n)$, then the number of blocks are nothing but the number of maximum matchings in $K_n$. As $D$ is a $(v, \beta, \mu)$-design over $T(n)$, the replication number $r$ in $D$ is the number of maximum matchings in $K_n$ containing a given edge $u_iu_j$ of $K_n$. The
number of maximum matchings containing a given edge \( u_iu_j \) of \( K_n \) is the
number of maximum matchings in \( K_n - u_i - u_j \). If we denote the number of
maximum matchings in \( K_n \) by \( m(K_n) \), then we have \( r = m(K_{n-2}) \). Also,
one can easily see that \( \mu = m(K_{n-4}) \). Thus, we have a \((v, \beta_0, \mu)\)-design
over \( T(n) \), whose parameters are \( v = \frac{n(n-1)}{2} \), \( b = m(K_n) \), \( r = m(K_{n-2}) \),
\( \beta_0 = \left\lfloor \frac{n}{2} \right\rfloor \), \( \mu = m(K_{n-4}) \). But the exact values of \( m(K_n) \) and \( \mu = m(K_{n-4}) \)
can be derived from the relation (3.2.2). That is, \( vr = b\beta_0 \), which gives

\[
m(K_n) = (n - 1)m(K_{n-2}). \tag{3.4.1}
\]

The equation (3.4.1) can also be derived from the definition of a matching
polynomial (3.1.2) and the equation (3.1.3) of the Corollary 3.1.4. By
using equation (3.4.1), one can get the exact values of parameters of the
\((v, \beta_0, \mu)\)-design, which we present in the form of a theorem below.

Thus, \((v, \beta_0, \mu)\)-design over \( T(n) \) is a design with the parameters

\[
\left( \frac{n(n-1)}{2}, \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (n-2i+1), \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor-1} (n-2i-1), \left\lfloor \frac{n}{2} \right\rfloor, 0, \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor-2} (n-2i-5) \right)
\]
Theorem 3.4.1. For a triangular graph \( T(n) \) the design with parameters

\[
\left( \frac{n(n-1)}{2}, \left\lfloor \frac{n}{2} \right\rfloor, \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor-2} (n-2i-5) \right)
\]

is a \((v, \beta_0, \mu)\)-design.

3.5 \((v, \beta_0, \mu)\)-designs over \( L_2(n) \) graph

The square lattice graph \( L_2(n) \) \((n \geq 2)\) is the graph with the vertex set \( V = X \times X \), where \( X = \{u_1, u_2, \ldots, u_n\} \) and two vertices \((u_i, u_j)\) and \((u_r, u_s)\) are adjacent in \( L_2(n) \) if and only if \( i = r, j \neq s \) or \( j = s, i \neq r \). Clearly \( L_2(n) \) is the line graph of \( K_{n,n} \); that is, \( L_2(n) = L(K_{n,n}) \). Thus, by the definition of a line graph, a maximum independent set in \( L_2(n) \) is a maximum matching in \( K_{n,n} \). Thus, in a similar fashion as in the case of \((v, \beta_0, \mu)\)-design over \( T(n) \), the number of blocks of \((v, \beta_0, \mu)\)-design over \( L_2(n) \) is the number of maximum matchings \( m(K_{n,n}) \) in \( K_{n,n} \). The blocks of this \((v, \beta_0, \mu)\)-design over \( L_2(n) \) are also obtained recursively as above. Hence, the parameters of the \((v, \beta_0, \mu)\)-design over \( L_2(n) \) are

\[
(n^2, m(K_{n,n}), m(K_{n-1,n-1}), n, 0, m(K_{n-2,n-2}))
\]
Again, using equation (3.2.2) (or using the definition of a matching polynomial and Corollary 3.1.4), we get $m(K_{n,n}) = n!$. Therefore, the parameters of $(v, \beta_0, \mu)$-design over $L_2(n)$ are:

$$(n^2, n!, (n - 2)!, n, 0, (n - 4)!)$$

**Theorem 3.5.1.** There exist a $(v, \beta_0, \mu)$-design over $L_2(n)$ with parameters $(n^2, n, (n - 4)!)$.

### 3.6 $(v, \beta_0, \mu)$-design over the Petersen graph

The Petersen graph is a graph whose vertex set $V$ is the set of all 2-element subsets of a set $X = \{1, 2, 3, 4, 5\}$ and where two vertices are adjacent if and only if their intersection is empty. Figure 3.3, shows the structure of a Petersen graph. We denote 2-element subsets $\{a, b\}$ as simply $ab$.

The Petersen graph is known to possess many important properties of graphs such as being distance transitive, distance regular, being a Moore graph of diameter two, strongly regular graph and many more. Also, it serves as a counterexample to many conjectures. The whole book entitled “Petersen Graph” was written by Sheehan and Holton [10] is devoted to
Figure 3.3: Petersen graph

study this graph. Now, we will look at the \((v, \beta_0, \mu)\)-design over the Petersen graph.

Clearly, from the Figure 3.3, the set \(\{12, 15, 13, 14\}\) is an independent set in \(G\) and hence \(\beta_0(G) \geq 4\) and by using Theorem 3.1.7 we deduce that \(\beta_0(G) \leq \frac{-10(-2)}{3(-2)} = 4\), where \(\lambda_1(G) = 3\), \(\lambda_{10}(G) = -2\) and \(G\) is of order 10 and thus, \(\beta_0(G) = 4\) holds.

Thus, \(D = (V, B)\) is a \((v, \beta_0, \mu)\)-design over the Petersen graph \(G\) having the parameters \(v = 10, b = 5, r = 2, \beta_0 = 4, \mu = 1, V = \binom{X}{2}\) and the blocks in \(B\) consisting of the sets \(B_j = \{\{i, j\} / i \in X - \{j\}\}\), for \(i \neq j\) and \(i, j \in X\).

**Theorem 3.6.1.** There exist a \((v, \beta_0, \mu)\)-design over the Petersen graph
with the parameter \( v = 10, \beta_0 = 4 \) and \( \mu = 1 \).

### 3.7 \((v, \beta_0, \mu)\)-design over the Shrikhande graph

In 1959, S.S. Shrikhande proved that the strongly regular graph \( L_2(n) \) is characterized by its parameters \((n^2, 2n - 2, n - 2, 2)\) except for the case \( n = 4 \) and he showed that for \( n = 4 \) there is a unique exception, which is now popularly known as the Shrikhande graph. In [13], Ryan M. Pedersen has mentioned various ways of constructing the Shrikhande graph. We consider the Shrikhande graph from the Cayley graph concept as described below:

Let \( H \) be a finite group. A Cayley subset \( S \) of \( H \) is a generating set of \( H \) with the property that \( s \in S \) whenever \( s^{-1} \in S \). A Cayley graph of a group \( H \) with respect to \( S \), where \( S \) is a Cayley subset of \( H \), denoted by \( Cay(H, S) \) is the graph with vertex set \( H \), where \( x \in H \) is adjacent to \( y \in H \) whenever \( xy^{-1} \in S \). Now consider a group \( H = Z_4 \times Z_4 \) with \( S = \{\pm(0,1), \pm(1,1), \pm(1,0)\} \). Then the graph \( Cay(H, S) \) in this case is nothing but the Shrikhande graph and is shown in Figure 3.4.
The spectrum of the Shrikhande graph $G$ shown in Figure 3.4. is

\[
\begin{pmatrix}
6 & 2 & -2 \\
1 & 6 & 9 \\
\end{pmatrix}
\]

and hence by Theorem 3.1.7., $\beta_o(G) \leq 16 \frac{-(2)}{6-(2)} = \frac{32}{8} = 4$. But from the Figure 3.4, one can observe that the set $\{00, 21, 02, 23\}$ is an independent set and hence $\beta_o(G) \geq 4$ which, in turn, implies that $\beta_o(G) = 4$. By using Proposition 3.2.3 we see that each vertex $xy$ in the Shrikhande graph $G$, the subgraph $G_1 = (V(G) - N[xy])$ contains three maximal independent sets of cardinality three and hence there are exactly three maximum independent sets of size four containing a given vertex in $G$. This fact gives us $16 \times 3 = 48$ maximum independent sets in $G$. Since each of these occurs
four times, the total number of distinct maximum independent sets of size four in \( G \) are 12 in number. Following list gives the distinct maximum independent sets in \( G \).

\[
\{00, 20, 12, 32\}, \{00, 31, 13, 22\}, \{00, 21, 02, 23\}, \{01, 20, 22, 03\}, \\
\{01, 10, 23, 32\}, \{01, 21, 33, 13\}, \{30, 21, 12, 03\}, \{30, 11, 32, 13\}, \\
\{30, 10, 02, 22\}, \{31, 10, 12, 33\}, \{31, 11, 23, 03\}, \{20, 11, 33, 02\}.
\]

These sets form a (16, 4, 1)-design which is nothing but a (16, 12, 3, 4, 0, 1)-PBIBD with two association schemes.

**Theorem 3.7.1.** For Shrikhande graph the design with parameter \( v = 16 \), \( \beta_0 = 4 \), \( \mu = 1 \) is a \((v, \beta_0, \mu)\)-design.

### 3.8 Non-existence of \((v, \beta_0, \mu)\)-designs over Chang graphs

L.C. Chang in [4] proved that the triangular graph \( T(n) \), with the exception of \( T(8) \), is uniquely determined by the parameters

\[
\left( \frac{n(n - 1)}{2}, 2n - 4, n - 2, 4 \right)
\]

Furthermore, they showed that there are precisely four strongly regular graphs, up to isomorphism, with parameters \((28, 12, 6, 4)\), namely \( T(8) \),
and three other graphs (known as Chang graphs); the three exceptions are also known from the work of Connor, Hoffman, and S. S. Shrikhande. In [15], Seidel proved that these three Chang graphs (as well as Shrikhande graph discussed in section 9) are obtained from $T(8)$ ($L_2(4)$) through the process of so-called Seidel switching.

The following construction of the three Chang graphs are from Seidel switching.

Let $V$ be 2-subsets of the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and we denote an arbitrary 2-subset $\{a, b\}$ of $V$ as $ab$ only.

Define,

$V_1 = \{04, 15, 26, 37\}$

$V_2 = \{01, 12, 23, 34, 45, 56, 67, 07\}$ and

$V_3 = \{01, 12, 23, 34, 04, 56, 67, 75\}$

Note that the above three sets corresponds to three subgraphs of $K_8$ namely $4K_2, C_8$ and $C_5 \cup C_3$.

For $i \in \{1, 2, 3\}$, let

$E_i = \{xy \in V : x, y \in V_i \text{ or } x, y \notin V_i \text{ and } |x \cap y| = 1\} \cup \{xy \in V : x \in V_i , y \notin V_i \text{ and } |x \cap y| = 0\}$.

The graphs $T_i(8) = (V, E_i)$, $i = 1, 2, 3$ are strongly regular graphs with
parameters $(28, 12, 6, 4)$ and are called Chang graphs and all of them have
the same spectra as that of $T(8)$; that is,

$$
\begin{pmatrix}
12 & 4 & -2 \\
1 & 7 & 20 \\
\end{pmatrix}
$$

Hence, by Theorem 3.1.7 $\beta_o(T_i(8)) \leq \frac{-28(-2)}{12-(-2)} = -\frac{56}{14} = 4$.

On the other hand, the sets $\{04, 15, 26, 37\}$, $\{01, 23, 45, 67\}$ and $\{01, 05, 16, 56\}$
are independent sets in $T_1(8)$, $T_2(8)$ and $T_3(8)$ respectively. Hence, $\beta_o(T_i(8)) \geq 4$ and thus we conclude that $\beta_o(T_i(8)) = 4$, for $i = 1, 2, 3$.

Now, we are ready to show the non-existence any $(v, \beta_o, \mu)$-design over
$T_i(8)$, for $i = 1, 2, 3$. By the definition of $(v, \beta_o, \mu)$-design, the paramet-
ric values $v$, $\beta_o$ and $\mu$ are constants and hence to show the non-existence
of $(v, \beta_o, \mu)$-design, it is sufficient to show that $\mu$ is not a constant with
respect to blocks of $(v, \beta_o, \mu)$-design.

### 3.8.1 Non-existence of $(v, \beta_o, \mu)$-design over $T_1(8)$

Consider two non-adjacent vertices 01 and 04 of the graph $T_1(8)$ and the
sets

$N(01) = \{02, 03, 05, 06, 07, 12, 13, 14, 16, 17, 26, 37\}$ and
$N(04) = \{12, 13, 16, 17, 23, 25, 27, 35, 36, 56, 57, 67\}$.

The subgraph $G_1$ induced by the set $V(T_1(8)) - N[01] \cup N[04]$ is as shown in Figure 3.5.

By Proposition 3.2.3 the set $\{01, 04, 15, 45\}$ is the only maximum independent set containing 01 and 04, since the set $\{14, 45\}$ is the only maximum independent set in $G_1$.

On the other hand, consider two non-adjacent vertices 04 and 15 in $T_1(8)$, where $N(15) = \{02, 03, 06, 07, 23, 24, 27, 34, 36, 46, 47, 67\}$ and as above the subgraph $G_2$ induced by $V(T_1(8)) - (N[04] \cup N[11])$ is shown in Figure 3.6.

Again, by using Proposition 3.2.3 the maximum independent sets in $T_1(8)$ containing 04 and 15 are $\{04, 15, 01, 45\}$, $\{04, 15, 05, 14\}$, $\{04, 15, 26, 37\}$.

Thus, every pair of nonadjacent vertices in $T_1(8)$ is not occurring in a fixed
number \( \mu \) of the blocks of \((v, \beta_0, \mu)\)-design and hence there does not exist \((v, \beta_0, \mu)\)-design over \(T_1(8)\).

### 3.8.2 Non-existence of \((v, \beta_0, \mu)\)-design over \(T_2(8)\)

As in the case of \(T_1(8)\), we consider two pairs of nonadjacent vertices 01, 02 and 01, 23 in \(T_2(8)\) and

\[
\begin{align*}
N(01) &= \{12, 07, 24, 25, 26, 27, 35, 36, 37, 46, 47, 57\}, \\
N(02) &= \{03, 04, 05, 06, 24, 25, 26, 27, 34, 45, 56, 67\} \text{ and} \\
N(23) &= \{12, 34, 04, 05, 06, 14, 15, 16, 17, 46, 47, 57\}.
\end{align*}
\]

Let \(H_1\) and \(H_2\) be the subgraphs of \(T_2(8)\) induced by the sets \(V(T_2(8)) - N[01] \cup N[02]\) and \(V(T_2(8)) - (N[01] \cup N[23])\) as shown in Figure 3.7.
Again, by using Proposition 3.2.3 the set \{01, 02, 13, 23\} is the only maximum independent set in \(T_2(8)\) containing 01 and 02 whereas \{01, 23, 02, 13\} and \{01, 23, 67, 45\} are the maximum independent sets containing 01 and 23; this violates the constancy condition on \(\mu\) in the definition of a design. Therefore, there does not exist \((v, \beta_0, \mu)\)-design over \(T_2(8)\).

3.8.3 Non-existence of \((v, \beta_0, \mu)\)-design over \(T_3(8)\)

The same argument may be used to show the non-existence of \((v, \beta_0, \mu)\)-design over \(T_3(8)\) as in the case of 3.8.1 and 3.8.2 by using the set \(V_3 = \{01, 12, 23, 34, 04, 56, 67, 57\}\).

**Theorem 3.8.1.** There does not exists \((v, \beta_0, \mu)\)-design for any of the Chang graphs \(T_1(8)\), \(T_2(8)\) and \(T_3(8)\)
3.9 \((v, \beta_o, \mu)\)-design over Clebsch graph

The Clebsch graph \(G\) has the vertex set consisting of all even cardinality subsets of \(\{1, 2, 3, 4, 5\}\) any two of which are adjacent if and only if their symmetric difference has cardinality four. It is a strongly regular graph with the parameters \((16, 5, 0, 2)\). In Figure 3.8, we have used the numbers 0 to 15 to label its vertices instead of subsets of \(\{1, 2, 3, 4, 5\}\) with even cardinalities.

In the previous chapter we have obtain PBIBDs with two association schemes whose blocks are minimum dominating sets of a graph. The following propositions are established in Chapter 2, in connection with the

83
minimum dominating sets of Clebsch graph. These will be used in the proof of next results.

**Proposition 3.9.1.** The domination number of the Clebsch graph is four.

**Proposition 3.9.2.** If $D$ is a minimum dominating set in the Clebsch graph $G$, then $D$ induces either $K_4$ or $C_4$.

It is well known that for any vertex $u$ in the Clebsch graph $G$, the set $V(G) - N[u]$ induces a Petersen graph $P$. Thus, by Proposition 3.2.3, any maximum independent set in $G$ is $S \cup \{u\}$, where $S$ is a maximum independent set in $P$. Thus, by Corollary 3.2.4, $\beta_0(G) = 5$ because independence number of the Petersen graph is four.

To obtain the independence number $\beta_0(G)$ of the Clebsch graph, one may also use the Proposition 3.1.8, since the number of negative eigenvalues of $G$ is 5 and thus $\beta_0(G) \leq 5$. But the set $\{1, 13, 6, 10, 4\}$ from Figure 3.8 is an independent set and hence $\beta_0(G) \geq 5$ which gives the desired result. The next lemma gives location of maximum independent sets in the Clebsch graph.

**Lemma 3.9.3.** Every maximum independent set in the Clebsch graph $G$ is an open neighborhood of some vertex in $G$. 84
**Proof:** By the structure of the Clebseh graph $G$, for any vertex $u$, the vertex set $V(G)$ can be partitioned into two sets $N[u]$ and $V - N[u]$ which induces $K_{1,5}$ with the central vertex $u$ and the Petersen graph $P$ respectively. We shall prove that any maximum independent set of five vertices is an open neighborhood of some vertex in $G$. On the contrary, assume that there is an independent set of cardinality five which is not an open neighborhood of any vertex in $G$. Let $S = \{u_1, u_2, u_3, u_4, u_5\}$ be one such set in $G$. Let $u$ be any arbitrary vertex in $G$. Then, by the choice of $S$, $S$ is neither a subset of $N(u)$ nor a subset of $V(P)$, since $\beta_o(P) = 4$. Therefore, $S \cap N(u) \neq \emptyset$ and $S \cap V(P) \neq \emptyset$. To prove the result, we consider four cases.

**Case 1:** $|S \cap N(u)| = 1$.

Without loss of generality, let $S \cap N(u) = \{u_1\}$. Then $S \cap V(P) = \{u_2, u_3, u_4, u_5\}$. This implies that $S \cap V(P)$ is a maximum independent set in $P$. Hence, by the structure of $P$, the set $(V(P) - S \cap V(P))$ induces a subgraph isomorphic to $3K_2$ and $u_1$ is adjacent to four vertices of $3K_2$ because the degree of $u_1$ is five and one of the vertices adjacent to $u_1$ is $u$ and $u_1$ is not adjacent to any of the other vertices in $S$. This results in a triangle in $G$, a contradiction.
Case 2: \(|S \cap N(u)| = 2\).

As in case 1, let \(S \cap N(u) = \{u_1, u_2\}\) and \(S \cap V(P) = \{u_3, u_4, u_5\}\). By the structure of the Clebsch graph, the vertices \(\{u_1, u_2\}\) are adjacent to one more common vertex (one is \(u\) itself) say \(v\) in \(P\) other than \(u_3, u_4, u_5\).

But, by Proposition 3.9.2, the vertices \(u, u_1, u_2, v\) form a dominating set in \(G\). Therefore, \(u_3, u_4, u_5\) are adjacent to \(v\), which implies that \(S\) is an open neighborhood of \(v\), a contradiction to our assumption.

Case 3: \(|S \cap N(u)| = 3\).

Let \(N(u) = \{u_1, u_2, u_3, v_1, v_2\}\) and \(u_4, u_5\) be in \(P\). By the structure of the Clebsch graph, \(u_4, u_5\) are adjacent to one common neighbor in \(N(u)\) and one in \(P\). Let the common neighbor of \(u_4\) and \(u_5\) be \(v_1\) (or \(v_2\)) and some \(v\) in \(P\). Thus, the vertices \(v_1, u_4, u_5, v\) induce a 4-cycle in \(G\) and hence a dominating set in \(G\). Thus, \(u_1, u_2, u_3\) are adjacent to \(v\), which shows that \(S\) is an open neighborhood of \(v\), a contradiction to our assumption.

Case 4: \(|S \cap N(u)| = 4\).

Let \(N(u) = \{u_1, u_2, u_3, v_4, v_1\}\) and the vertex \(u_5\) is in \(P\). Again by the structure of the Clebsch graph, \(u_5\) is adjacent to two vertices in \(N(u)\) and one of them must be one of \(u_i\), for \(i = 1, 2, 3, 4\) a contradiction to the assumption that \(S\) is independent in \(G\). This competes the proof. \(\square\)
Thus, the open neighborhood of every vertex in the Clebsch graph is a block in the \((v, \beta_o, \mu)\)-design whose parameters are \((16, 5, 2)\). From Figure 3.8, the blocks of \((16, 5, 2)\)-design are,

\[
\begin{align*}
\{1, 4, 6, 10, 13\}, & \{0, 2, 7, 11, 14\}, \{1, 3, 8, 10, 12\}, \{2, 4, 9, 11, 13\}, \\
\{0, 3, 5, 12, 14\}, & \{4, 7, 8, 10, 11\}, \{0, 8, 9, 11, 12\}, \{1, 5, 9, 12, 13\}, \\
\{2, 5, 6, 13, 14\}, & \{3, 6, 7, 10, 14\}, \{0, 2, 5, 9, 15\}, \{1, 3, 5, 6, 15\}, \\
\{2, 4, 6, 7, 15\}, & \{0, 3, 7, 8, 15\}, \{1, 4, 8, 9, 15\}, \{10, 11, 12, 13, 14\}.
\end{align*}
\]

3.10 \((v, \beta_o, \mu)\)-design over Schläfli graph

The Schläfli graph is a strongly regular graph \(G = (V, E)\) with the parameters \((27, 10, 1, 5)\). The subgraph of the Schläfli graph formed by the non-neighbors of any vertex is isomorphic to the Clebsch graph. Let \(u\) be any vertex in \(G\) and by the parameters \((27, 10, 1, 5)\) of \(G\), the subgraph induced by \(N(u)\) is just \(5K_2\) and \(H = < V(G) - N[u] >\) is the Clebsch graph. Figure 3.9 depicts the Schläfli graph, drawn here using the construction of generalized quadrangle \(GQ(2, 4)\) that appears in [17].
where the numbers in the brackets are the labels of vertices of the Clebsch graph obtained by removal closed neighborhood of vertex 16 of a Schlafli graph, which are adjacent to neighbors of the vertex 16. For instance, 17 which is a neighbor of 16 and is adjacent to 0, 15, 5, 3, 2, 7, 14, 6 of Clebsch graph.

By the parameters of a Schlafli graph $G$ one can easily get the spectrum of $G$ is

$$
\begin{pmatrix}
10 & 1 & -5 \\
1 & 20 & 6
\end{pmatrix}
$$

Hence, by Theorem 3.1.8, $\beta_o(G) \leq 6$. Also, the set $S \cup \{u\}$ is an indepen-
dent set in $G$, where $S$ is a maximum independent set in the Clebsch graph $H$. Thus, $\beta_0(G) = 6$. From the previous section, we note that there are sixteen maximum independent sets in the Clebsch graph $H$ and hence every vertex in the Schlafli graph occurs in these sixteen maximum independent sets. Then, by using the relation $vr = b\beta_0$ we get $b = 72$. Also, by using equation (3.2.1), $(v - d - 1)\mu = r(\beta_0 - 1)$, which yields $\mu = 5$. Thus, the PBIBD with parameters $(27, 72, 16, 6, 0, 5)$ is a $(v, \beta_0(G), \mu)$-design over $G$.

**Theorem 3.10.1.** There exist $(v, \beta_0, \mu)$-design over a Schlafli graph with parameters $v = 27$, $\beta_0 = 6$, $\mu = 5$.

### 3.11 Conclusion and scope

In this chapter, we consider $(v, \beta_0, \mu)$-design over particular class of regular graphs, namely strongly regular graphs; where in $(v, \beta_0, \mu)$-designs are not partially balanced incomplete block designs for instance, $(v, \beta_0, \mu)$-designs over $K_n \times K_n$, for $m \neq n$. The study of sort undertaken in this chapter will give new way of constructing PBIBD’s through the maximum independent sets in a regular graph. Such type of work has been already done by Walikar et al. [16] through minimum dominating sets of cycles, paths and cubic graphs on ten vertices. One may also consider the designs whose blocks...
are subsets of a vertex set of regular graph with given property such as vertex cover, edge-independent sets and many other properties associated with edge set and vertex set of a graph. Exploration of designs of this sort may provide considerable insight into the construction of designs and even may lead to the construction of some special types of codes in coding theory.
REFERENCES


