6.1. Introduction

In this chapter a nonparametric test for the c-sample scale problem is proposed. Unlike in the case of other tests for this problem, the c distribution functions from which samples are drawn are assumed to have a common quantile. As in Chapter 5, let $X_{i1}, \ldots, X_{in_i}$, $i = 1, 2, \ldots, c$ be independent random samples from c populations with absolutely continuous distribution functions $F_i$, $i = 1, 2, \ldots, c$. It is assumed that these distribution functions have zero as the common quantile of order $\alpha$ ($0 < \alpha < 1$), that is $F_i(0) = \alpha$ for $i = 1, 2, \ldots, c$. It is also assumed that $F_i$'s are identical in all respects, except possibly their scale parameters. The problem is of testing the null hypothesis.

$$H_0: F_1(x) = \ldots = F_c(x) = F(x) \text{ say; } F_i(0) = \alpha, \quad (6.1.1)$$

$i = 1, 2, \ldots, c$ against the scale alternative

$$H_1: F_i(x) = F(\theta_i, x), \quad \theta_i > 0, \quad (6.1.2)$$

where all $\theta_i$'s are not equal; $F_i(0) = \alpha$, $i = 1, 2, \ldots, c$.

In spite of quite significant growth of literature on nonparametric two-sample scale problem, the work on the c-sample scale problem seems to be comparatively inadequate. Puri (1965) has generalized two-sample tests of Mood (1954), Ansari and Bradely (1960) and Klotz (1962) to the c-sample scale problem and has also studied their asymptotic relative efficiencies (ARE's). These tests make use of the assumption that the common quantile is of
order $\alpha = 1/2$, that is, the distributions have the same median. Other c-sample scale tests are those proposed by Chatterjee (1966), Sugiura (1965) and Deshpande (1970); these tests are based on U-statistics. Deshpande's (1970) class of tests contain the tests of Sugiura (1965) as a special case. Later, Gore and Shanubhogue (1985) proposed a test based on subsample extrema. However, none of these tests is suitable when the common quantile is different from the median. Deshpande and Kusum (1984) proposed a test based on U-statistics for the non-parametric two-sample scale problem, when the assumption of common median is replaced by that of common quantile of order $\alpha$. The situations where two populations may have a common quantile of order other than 1/2 arise in practice, have been explained in Deshpande and Kusum (1984). The similar situations will naturally arise in the case of c (> 2) samples also.

The proposed test statistic for testing $H_0$ against $H_1$ is given in Section 6.2. The asymptotic distribution of the test statistic under the null hypothesis $H_0$ is obtained in Sections 6.3. In Section 6.4, the test is shown to be consistent. In Section 6.5. The asymptotic distribution of the test statistic under the Pitman-type scale alternatives is obtained. In the last section the proposed test is compared with c-sample version of Mood's test given by Puri (1965), using the criterion of the Pitman asymptotic relative efficiency (ARE), when the underlying distributions are double exponential and uniform.
6.2. The Proposed Test

Let us define
\[ h_i^*(x_1, \ldots, x_c) \]
\[
\begin{cases}
  1 & \text{if } 0 < x_j < x_i \text{ or } x_i < x_j < 0 \text{ for all } j \neq i = 1, 2, \ldots, c \\
  -1 & \text{if } 0 < x_i < x_j \text{ or } x_j < x_i < 0 \text{ for all } j \neq i = 1, 2, \ldots, c \\
  0, & \text{otherwise,}
\end{cases}
\]

for \( i = 1, 2, \ldots, c \). Then the \( c \) - sample U - statistic corresponding to the kernel \( h_i^* \) is given by
\[
U_i = \frac{1}{n_1n_2\ldots n_c} \sum_{\alpha_i=1}^{n_i} \ldots \sum_{\alpha_c=1}^{n_c} h_i^*(X_{1\alpha_i}, X_{2\alpha_i}, \ldots, X_{c\alpha_i}).
\]  

For testing \( H_0 \) against \( H_1 \), with \( F_i(0) = \alpha \), for \( i = 1, 2, \ldots, c \), the proposed test based U-statistic is
\[
T = Nc^{-2}B^{-1}(c, \alpha) \left[ \sum_{i=1}^{c} P_i U_i^2 - (\sum_{i=1}^{c} P_i U_i)^2 \right],
\]  

where \( N = \sum_{i=1}^{c} n_i \), \( P_i = n_i/N \) and
\[
B(c, \alpha) = \frac{2 \left( \left( \frac{2c-2}{c-2} \right) - 1 \right) \left( (1-\alpha)^{2c-1} + \alpha^{2c-1} \right)}{(c-1)^2 (2c-1) \left( \frac{2c-2}{c-2} \right)}. 
\]

The test \( T \) rejects \( H_0 \) at a significance level \( \alpha \) if \( T \) exceeds the upper \( \alpha \)-th quantile of the null distribution of \( T \). When \( H_0 \) is true asymptotic distribution of \( T \) is shown to be chi-square with \( (c-1) \) degrees of freedom.
6.3. Asymptotic Distribution of T Under $H_0$

Obviously $U_j$'s defined by (6.2.2) are $U$-statistics generalized to the c-sample case. Thus, form Theorem 5.2.1, $\sqrt{N} U$, where $U = (U_1, \ldots, U_c)$, has multivariate normal distribution with mean vector zero and covariance matrix $\Sigma = (\sigma_{ij})$, $i, j = 1, 2, \ldots, c$, where

$$
\sigma_{ij} = \lim_{N \to \infty} N \text{ cov} (U_i, U_j) = \sum_{k=1}^{c} \frac{1}{P_k} \xi_{i,j}^{(k)},
$$

(6.3.1)

with

$$
\xi_{i,j}^{(k)} = E[h_i^*(x_1, \ldots, x_k, \ldots, x_c) h_j^*(x_1, \ldots, x_k, \ldots, x_c)].
$$

It may be noted that under $H_0$

$$
E (U_i) = E (h_i^*(x_1, \ldots, x_c)) = P [0 < X_j < x_i, j = 1, \ldots, c; j \neq i] + P [X_j < 0, j = 1, \ldots, c; j \neq i] - P [0 < X_i < X_j, j = 1, \ldots, c; j \neq i] - P [X_j < 0, j = 1, \ldots, c; j \neq i] = 0.
$$

Now, under $H_0$

$$
\xi_{i,j}^{(0)} = E[h_i^*(X_1, \ldots, X_i, \ldots, X_c) h_j^*(X_1, \ldots, X_i, \ldots, X_c)].
$$

= $E[\Psi_1'(X_i)I(X_i > 0) + \Psi_2'(X_i)I(X_i < 0) - \Psi_3'(X_i)I(X_i < 0) - \Psi_4'(X_i)I(X_i > 0)].$

(6.3.2)

where

$$
I (X \in A) = 1, \ X \in A
$$

= 0, \ X \notin A,

and

$$
\Psi_1(x_i) = P [0 < X_j < x_i \text{ for all } j = 1, \ldots, c; j \neq i]
$$

= $[F(x_i) - \alpha]^{c-1},$

(6.3.3)
\[ \Psi_2(x_i) = \mathbb{P}[x_i < X_j < 0, \text{ for all } j = 1, \ldots, c; j \neq i] \]
\[ = [\alpha - F(x_i)]^{c-1} \quad (6.3.4) \]
\[ \Psi_3(x_i) = \mathbb{P}[x_i < X_j < 0, \text{ for all } j = 1, \ldots, c; j \neq i] \]
\[ = [1 - F(x_i)]^{c-1} \quad (6.3.5) \]
\[ \Psi_4(x_i) = \mathbb{P}[X_j < x_i < 0, \text{ for all } j = 1, \ldots, c; j \neq i] \]
\[ = (F(x_i))^{c-1} \quad (6.3.6) \]

Using expressions (6.3.3) to (6.3.6) in (6.3.2), we have

\[ \xi^{(k)}_{x_i} = \int_0^\alpha [F(x_i) - \alpha]^{2c-2} dF(x_i) + \int_0^\alpha \left[ \alpha - F(x_i) \right]^{2c-2} dF(x_i) \]
\[ + \int_0^\alpha [1 - F(x_i)]^{2c-2} dF(x_i) + \int_0^\alpha \left[ F(x_i) \right]^{2c-2} dF(x_i) \]
\[ - 2 \int_0^\alpha \left[ F(x_i) - \alpha \right]^{c-1} [1 - F(x_i)]^{c-1} dF(x_i) \]
\[ - 2 \int_0^\alpha \left[ \alpha - F(x_i) \right]^{c-1} [F(x_i)]^{c-1} dF(x_i) \]
\[ = \frac{2(1 - \alpha)^{2c-1}}{2c-1} + \frac{2\alpha^{2c-1}}{2c-1} - 2(1 - \alpha)^{2c-1} B(c, c) - 2\alpha^{2c-1} B(c, c) \]
\[ = (c-1)^2 B(c, \alpha), \quad (6.3.7) \]

where \( B(c, \alpha) \) is given by (6.2.4).

\[ \xi^{(k)}_{x_i} = \mathbb{E}[h_1(X_1, \ldots, X_k, X_\ell) h_2'(X_1', \ldots, X_k', X_\ell')] \]
\[ = \mathbb{E}[\Psi_2(X_k)I(X_k > 0) + \Psi_2(X_k)I(X_k < 0) + \Psi_3(X_k)I(X_k > 0) + \Psi_3(X_k)I(X_k < 0) \]
\[ + \Psi_4(X_k)I(X_k < 0) - 2\Psi_1(X_k)\Psi_3(X_k)I(X_k > 0) - 2\Psi_2(X_k)\Psi_4(X_k)I(X_k < 0)] \]
\[ \quad (6.3.8) \]

Here

\[ \Psi_1(x_k) = \mathbb{P}[0 < x_k < X_1, \text{ for all } l \neq i \neq k \text{ and } 0 < x_k < X_1] \]
\[
= \int_{x_i} \left[ F(x_i) - \alpha \right]^{c-2} \, dF(x_i)
\]
\[
= (c-1)^{-1} \left\{ (1-\alpha)^{c-1} - (F(x_k) - \alpha)^{c-1} \right\}
\]
\[
\Psi_2(x_k) = P[X_i < X_k < 0 \text{ for all } 1 \neq i \neq k \text{ and } X_i < x_k < 0]
\]
\[
= \int_{x_k}^{x_i} \left[ \alpha - F(x_i) \right]^{c-2} \, dF(x_i)
\]
\[
= (c-1)^{-1} \left\{ \alpha^{c-1} - (\alpha - F(x_k))^{c-1} \right\},
\]
similarly, we get
\[
\Psi_3(x_k) = P[0 < X_i < X_k \text{ for all } 1 \neq i \neq k \text{ and } 0 < X_i < x_k]
\]
\[
= (c-1)^{-1} \left\{ (1-\alpha)^{c-1} - (1 - F(x_k))^{c-1} \right\} \text{ and}
\]
\[
\Psi_4(x_k) = P[X_i < X_k < 0 \text{ for all } 1 \neq i \neq k \text{ and } x_k < X_i < 0]
\]
\[
= (c-1)^{-1} \left\{ \alpha^{c-1} - (F(x_k))^{c-1} \right\}.
\]
Now
\[
E[\Psi_1^2(X_k)I(X_k > 0)] = \frac{1}{(c-1)^2} \int_0^1 (1-\alpha)^{2c-2} \, dF(x_k)
\]
\[
+ \frac{1}{(c-1)^2} \int_0^1 (F(x_k) - \alpha)^{2c-2} \, dF(x_k)
\]
\[
- \frac{2(1-\alpha)^{c-1}}{(c-1)^2} \int_0^1 (F(x_k) - \alpha)^{c-1} \, dF(x_k)
\]
\[
= \frac{2(1-\alpha)^{2c-1}}{c(2c-1)}, \quad (6.3.9)
\]
Similarly we obtain
\[
E[\Psi_2^2(X_k)I(X_k < 0)] = \frac{2\alpha^{2c-1}}{c(2c-1)} \quad (6.3.10)
\]
\[
E[\Psi_3^2(X_k)I(X_k > 0)] = \frac{2(1-\alpha)^{2c-1}}{c(2c-1)} \quad (6.3.11)
\]
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\[ E[\Psi_4'(X_k)I(X_k < 0)] = \frac{2\alpha^{2c-1}}{c(2c-1)} \]  
\[ E[\Psi_1(X_k)\Psi_2(X_k)I(X_k > 0)] = \frac{(1-\alpha)^{2c-1}}{(c-1)^2} - \frac{2(1-\alpha)^{2c-1}}{c(c-1)^2} + \frac{(1-\alpha)^{2c-1}}{(c-1)^3}B(c, c) \]  
\[ E[\Psi_2'(X_k)\Psi_4(X_k)I(X_k < 0)] = \frac{\alpha^{2c-1}}{(c-1)^2} - \frac{2\alpha^{2c-1}}{c(c-1)^2} + \frac{\alpha^{2c-1}}{(c-1)^3}B(c, c) \]

and

Using expressions (6.3.9) to (6.3.14) in (6.3.8), we get

\[ \xi_{i,k}^{(k)} = B(c, \alpha) \]  
\[ \xi_{i,j}^{(k)} = B(c, \alpha) \]  
\[ \xi_{i,j}^{(i)} = \xi_{i,j}^{(j)} = - (c-1)B(c, \alpha) \]  

Substituting (6.3.7), (6.3.15), (6.3.16) and (6.3.17) in (6.3.1), we obtain

\[ \sigma_{ii} = B(c, \alpha) \left[ \sum_{k=1}^{c} \frac{1}{p_k} + \frac{c^2 - 2c}{p_i} \right] \]  
\[ \sigma_{ij} = B(c, \alpha) \left[ \sum_{k=1}^{c} \frac{1}{p_k} - \frac{c}{p_i} - \frac{c}{p_j} \right] \ (i \neq j). \]

From (6.3.18), we write the covariance matrix \( \Sigma \) as

\[ (B(c, \alpha))^{-1} \Sigma = aJ_{c,c} + c^2D - cqJ_{1,c} - cJ_{c,1}q', \]  

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where

\[
a = \sum_{k=1}^{c} \frac{1}{p_k}, \quad D = \text{diagonal} \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_c} \right) \quad q' = \left( \frac{1}{p_1}, \ldots, \frac{1}{p_c} \right) \quad \text{and} \quad J_{c,c} = (1)_{c\times c}.
\]

The covariance matrix \( \Sigma \) is singular since \( \sum_{i=1}^{c} \sigma_{ij} = 0 \) for every \( j \) and the rank of \( \Sigma \) is \((c-1)\). Hence we consider the asymptotic distribution of \( N^{1/2} U_0 \), where \( U_1 = (U_1, U_2, \ldots, U_c) \), which is multivariate normal random vector with mean vector zero and covariance matrix \( \Sigma_0 \), where \( \Sigma_0 \) can be written as.

\[
(B(c, \alpha))^{-1} \Sigma_0 = aJ_{c-1, c-1} + \sigma^2 D_0 - c q_0 J_{1, c-1} - c J_{c-1,1} q_0',
\]  

(6.3.20)

where

\[
D_0 = \text{diagonal} \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_{c-1}} \right) \quad q_0' = \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_{c-1}} \right).
\]

Therefore, under \( H_0 \) as \( N \to \infty \), \( T = N U_0' \Sigma_0^{-1} U_0 \) has a \( \chi^2 \)-distribution with \((c-1)\) degrees of freedom. After simplification the statistic \( T \) reduces to (6.2.3).

Thus, we have the following theorem.

Theorem 6.3.1. If \( F_1 = F_2 = \ldots = F_c \) and \( n_i = N p_i \), where \( p_i \) are fixed numbers such that \( \sum_{i=1}^{c} p_i = 1 \), then the statistic \( T \) has a limiting \( \chi^2 \)-distribution with \((c-1)\) degrees of freedom.

6.4 Consistency of \( T \)-Test

Using Theorem 5.4.1, we can see that the test which rejects \( H_0: F_i(x) = \ldots = F_c(x) \), if \( T > t_\alpha \) is consistent for all \( F_i(x) = F(\theta_i, x) \), \( i = 1, \ldots, c \), provided \( \eta_i \) is different from 0 (= \( E(U_i | H_0) \)) for at least one \( i \), where
\[ T_i = \sum_{j=1}^{c} \left[ F_j(x) - \alpha_i \right] dF_j(x) + \sum_{j=1}^{c} \left[ \alpha_i - F_j(x) \right] dF_j(x) \]

\[ - \sum_{j=1}^{c} \int_{j}^{\infty} \left[ 1 - F_j(x) \right] dF_j(x) - \int_{0}^{\alpha} \sum_{j=1}^{c} \left[ F_j(x) \right] dF_j(x). \] (6.4.1)

### 6.5 Asymptotic Distribution of T Under Scale Alternatives

In this section we obtain the limiting distributions of \( U_i \)'s and \( T \) under the sequence of scale alternatives,

\[ H_n : F_i(x) = F(x \left( 1 + \frac{G_i}{n^{1/2}} \right)), \quad i = 1, 2, \ldots, c; \quad n = 1, 2, \ldots \] (6.5.1)

Here 'n' is given by the relation \( n_i = n_{ij} \) where \( s_j \)'s are some fixed positive integers, \( \sigma_i > 0 \) for each \( i \) and \( \sigma_i \)'s are not all equal.

**Theorem 6.5.1.** (a) Under \( H_n \), \( N^{1/2} U_i, \ i = 1, 2, \ldots, c \) have jointly in the limit as \( n \to \infty \), a multivariate normal distribution with means

\[ \eta_{i} = \left( \sum_{j=1}^{c} \mu_j \right)^{-1/2} \sum_{j=1}^{c} (\sigma_j - \sigma_i) \mu(\alpha), \] (6.5.2)

where

\[ \mu(\alpha) = \int_{0}^{\alpha} x \left[ \left( F(x) - \alpha \right)^{\alpha/2} + \left( 1 - F(x) \right)^{\alpha/2} \right] f^2(x) dx \]

\[ - \int_{-\infty}^{0} x \left[ \left( \alpha - F(x) \right)^{\alpha/2} + F^{\alpha/2}(x) \right] f^2(x) dx \] (6.5.3)

and the elements of the covariance matrix are given by (6.3.18) under the following two conditions:

(i) \( F \) is absolutely continuous with derivative \( f \) and

(ii) there exists a function \( g \) such that

\[ \int_{-\infty}^{\infty} x g(x) dF(x) < \infty \]

and

\[ \left| \frac{1}{h} \left[ F(x + h) - F(x) \right] \right| \leq g(x) \text{ for small } h. \]
(b) Under $H_n$ and the above two conditions, $T$ has, in the limit as $n \to \infty$, a noncentral $\chi^2$ distribution with $(c-1)$ degrees of freedom and the noncentrality parameter given by

$$\lambda_T = B^{-1}(c, \alpha) \sum_{i=1}^{c} s_i (\sigma_i - \bar{\sigma})^2 \mu^2(\alpha), \quad (6.5.4)$$

where $B(c, \alpha)$ and $\mu(\alpha)$ are given by (6.2.4) and (6.5.3) respectively and

$$\bar{\sigma} = \frac{\sum_{i=1}^{c} s_i \sigma_i}{\sum_{i=1}^{c} s_i}$$

Proof of part (a). Under $H_n$

$$E(U_i \mid H_n) = \int_0^\infty \prod_{j \neq i} [F_j(x) - \alpha] dF_i(x) + \int_0^\infty \prod_{j \neq i} [\alpha - F_j(x)] dF_i(x)$$

$$- \int_0^\infty \prod_{j \neq i} [1 - F_j(x)] dF_i(x) - \int_0^\infty \prod_{j \neq i} [F_j(x)] dF_i(x)$$

$$= \int_0^\infty \prod_{j \neq i} [F(x(1 + \sigma_j / \sqrt{n})) - \alpha] dF(x(1 + \sigma_j / \sqrt{n}))$$

$$+ \int_0^\infty \prod_{j \neq i} [\alpha - F(x(1 + \sigma_j / \sqrt{n}))] dF(x(1 + \sigma_j / \sqrt{n}))$$

$$- \int_0^\infty \prod_{j \neq i} [1 - F(x(1 + \sigma_j / \sqrt{n}))] dF(x(1 + \sigma_j / \sqrt{n}))$$

$$- \int_0^\infty \prod_{j \neq i} [F(x(1 + \sigma_j / \sqrt{n}))] dF(x(1 + \sigma_j / \sqrt{n})). \quad (6.5.5)$$

Using Lemma 6.10 of Sugiura (1965), the above expression simplifies to

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\[ E[U_i | H_n] = n^{-1/2} \sum_{j=1}^{c} (\sigma_j - \sigma_i) \left\{ \int_{0}^{\alpha} x[F(x) - \alpha]^{c-2} + (1 - F(x))^{c-2} f^2(x) dx \right\} \\
- \int_{0}^{\alpha} x[(\alpha - F(x))^{c-2} + F^{-2}(x)] f^2(x) dx + o(n^{-1/2}) \]  
(6.5.6)

Multiplying (6.5.6) by \(N^{1/2}\) and taking the limit as \(n \to \infty\), we obtain (6.5.2).

Similarly, it can be shown that

\[ \lim_{N \to \infty} N \text{Cov} ((U_i, U_j) | H_n) = \lim_{N \to \infty} N \text{Cov} ((U_i, U_j) | H_0), \]

Hence part (a) of the theorem follows.

Proof of part (b): It follows from part (a) of the lemma that \(N^{1/2} U\), is

asymptotically multivariate normal with mean vector \(\left( \sum_{i=1}^{c} s_i \right)^{1/2} \delta \mu(\alpha)\), where

\[ \delta' = (\delta_1, ..., \delta_c) \text{ with } \delta_i = \sum_{j=1}^{c} \sigma_j - c \sigma_i, \quad \mu(\alpha) \text{ is given by (6.5.3) and covariance matrix } \Sigma = (\sigma_{ij}), \]

with \(\sigma_{ij}\) given by (6.3.18). Hence \(T\) is distributed as noncentral \(\chi^2\)-distribution with \((c - 1)\) degrees of freedom and the noncentrality parameter

\[ \lambda_T = (\sum_{i=1}^{c} s_i) \delta_0^{0} \Sigma_0^{-1} \delta_0 \mu^2(\alpha), \]

with \(\delta_0 = (\delta_1, ..., \delta_{c-1})\). This simplifies to (6.5.4) and the lemma follows.

6.6. Asymptotic Relative Efficiency

Here the asymptotic relative efficiency of the proposed test as compared to Mood’s L(M)-test has been studied. Puri (1965) gave a c-sample analogue of Mood’s test for two-sample scale problem. The corresponding test statistic is as given below.
Let $Z_{nj}^{(i)} = 1$, if the $j$-th smallest observation from the combined sample of size $N = \sum_{i=1}^{c} n_i$ is from the $i$th sample and $Z_{nj}^{(i)} = 0$ otherwise. Then $\mathcal{L}(M)$-test statistic is given by

$$\mathcal{L}(M) = 180 \sum_{i=1}^{c} n_i (M_{N,i} - \bar{E}_N)^2,$$

where

$$n_i M_{N,i} = \sum_{j=1}^{N} E_{N,j} Z_{nj}^{(i)}$$

with

$$E_{N,j} = \left( \frac{j}{N+1} - \frac{1}{2} \right)^2$$

and

$$\bar{E}_N = \frac{1}{N} \sum_{j=1}^{N} E_{N,j}.$$

Puri (1965) has shown that under the alternatives $H_n$, given by (6.5.1), the test statistic $\mathcal{L}(M)$ has in the limit as $n \to \infty$ a non-central $\chi^2$ distribution with $(c-1)$ degrees of freedom and the noncentrality parameter

$$\lambda_M = 180 \sum_{i=1}^{c} s_i (\sigma_i - \overline{\sigma})^2 \left[ \int_{-\infty}^{\infty} x f(x)(2F(x) - 1) dF(x) \right]^2,$$

where

$$\overline{\sigma} = \frac{\sum_{i=1}^{c} s_i \sigma_i}{\sum_{i=1}^{c} s_i}.$$

Puri (1965) has assumed that $F_i(0) = 1/2$. However, it can be easily seen that the noncentrality parameter remains the same as given in (5.2) even if $F_i(0) = \alpha$ ($\neq 1/2$).

It is well known that if both the test statistics have asymptotically noncentral $\chi^2$-distributions with the same degrees of freedom under the given sequence of alternative hypotheses, then the ARE can be computed as the ratio of their noncentrality parameters. The ARE of the $T$-test with respect to the $L(M)$-test is
quite difficult to evaluate in closed form for many of the distributions. However, for the double exponential and uniform distributions with location parameter $\theta$, the ARE’s are simpler to calculate.

For the double exponential distribution with location parameter $\theta$ the ARE expression can be obtained as

$$e(T, \mathcal{L}(M)) = \frac{72(c-1)^2(2c-1)(2c-2)(2c-1)\mu_2^2(\alpha)}{125((2c-1)^2-1)\{1-\alpha\}^2 + \alpha^2},$$ \hspace{1cm}(6.6.3)

where

$$\mu(\alpha) = \begin{cases} 
\phi(\alpha), & \text{if } 0 \leq \alpha \leq 1/2 \\
\phi(1-\alpha), & \text{if } 1/2 < \alpha < 1 
\end{cases} \hspace{1cm}(6.6.4)$$

with

$$\phi(\alpha) = \sum_{k=0}^{c-2} \frac{(-1)^k(c-2)^k}{2^{k+2}(k+2)^2} \left( (1-\alpha)^{c-2-k} - (-1)^c \alpha^{c-2-k} + 2^{k+2}(-\alpha)^{c} + 2^{k+2}\alpha^c \right)$$

$$+ \frac{\ln(2\alpha)}{c(1-c)} \left( 2(1/2-\alpha)^c - (-\alpha)^c - (1-\alpha)^c + 2^{-(c-1)} - 1 \right) + \frac{2^c \alpha^c + 1}{2^c c^2} \hspace{1cm}(6.6.5)$$

Thus $\mu(\alpha)$ is symmetric about $\alpha=1/2$. Since the denominator of $e(T, \mathcal{L}(M))$ is also symmetric about $\alpha=1/2$, it follows that $e(T, \mathcal{L}(M))$ is symmetric about $\alpha=1/2$. It may be noted that when $c=2$, $e(T, \mathcal{L}(M))$ will reduce to the corresponding expression given by Deshpande and Kusum (1984). The values of $e(T, \mathcal{L}(M))$ are computed for $c=2(1)5,7$ and $\alpha=0.01, 0.05, 0.1(0.1) 0.5$ and are given in Table 6.1.
Table 6.1

e(T, \mathcal{L}(M)) for double exponential distribution

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>13.6287</td>
<td>13.6280</td>
<td>12.8589</td>
<td>11.7616</td>
<td>9.5287</td>
</tr>
<tr>
<td>0.05</td>
<td>5.3653</td>
<td>5.3551</td>
<td>5.1076</td>
<td>4.7393</td>
<td>3.9575</td>
</tr>
<tr>
<td>0.10</td>
<td>3.1424</td>
<td>3.1124</td>
<td>3.0014</td>
<td>2.8287</td>
<td>2.4370</td>
</tr>
<tr>
<td>0.20</td>
<td>1.6492</td>
<td>1.5741</td>
<td>1.5347</td>
<td>1.4865</td>
<td>1.3553</td>
</tr>
<tr>
<td>0.30</td>
<td>1.1144</td>
<td>1.0152</td>
<td>0.9712</td>
<td>0.9456</td>
<td>0.8948</td>
</tr>
<tr>
<td>0.40</td>
<td>0.9103</td>
<td>0.8480</td>
<td>0.8040</td>
<td>0.7625</td>
<td>0.6876</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8640</td>
<td>0.8640</td>
<td>0.8916</td>
<td>0.9130</td>
<td>0.9183</td>
</tr>
</tbody>
</table>

It may be noted from Table 6.1 that the ARE $e(T, \mathcal{L}(M))$ of the proposed test relative to $\mathcal{L}(M)$ test for the double exponential distribution, increases for every fixed value of $c$ as $\alpha \neq 0.5$ moves away from 0.5 in either direction; for $c=2$ this increasing process starts from $\alpha=0.5$ itself. Also the ARE increases as $c$ increases when $\alpha = 0.5$ and it decreases as $c$ increases for values of $\alpha$ other than 0.5. It follows from (5.3) to (5.5) that $e(T, \mathcal{L}(M))$ tends to $\infty$ as $\alpha \to 0$ or $\alpha \to 1$, for fixed value of $c$.

In the case of uniform distribution over ($\theta -1/2, \theta +1/2$) the ARE can be easily obtained as

$$e(T, \mathcal{L}(M)) = \frac{(2c-1)(2c-3)(1-\alpha)^c + \alpha^c}{10((2c-3)(1-\alpha)^{2c-1} + \alpha^{2c-1})}$$  \hfill (6.6.6)

It can be easily seen that $e(T, \mathcal{L}(M))$ is symmetric about $\alpha =1/2$. The values of $e(T, \mathcal{L}(M))$ are computed for $c=2(1) 6,8,10$ and $\alpha =0.01, 0.05, 0.1(0.1) 0.5$ and are given in Table 6.2.
Table 6.2

\(e(T, \mathcal{L}(M))\) for the uniform distribution

\[
\begin{array}{l|cccccccc}
\alpha & c & 2 & 3 & 4 & 5 & 6 & 8 & 10 \\
\hline
0.1 & 0.5941 & 0.5940 & 0.7294 & 0.9039 & 1.0934 & 1.4850 & 1.8810 \\
0.05 & 0.5731 & 0.5702 & 0.6970 & 0.8674 & 1.0492 & 1.4250 & 1.8050 \\
0.1 & 0.5527 & 0.5415 & 0.6633 & 0.8218 & 0.9940 & 1.3500 & 1.7100 \\
0.2 & 0.5335 & 0.4946 & 0.5940 & 0.7319 & 0.8843 & 1.2000 & 1.5200 \\
0.3 & 0.5455 & 0.4818 & 0.5497 & 0.6574 & 0.7826 & 1.0524 & 1.3306 \\
0.4 & 0.5794 & 0.5345 & 0.5989 & 0.6838 & 0.7751 & 0.9694 & 1.1746 \\
0.5 & 0.6000 & 0.6000 & 0.7368 & 0.9130 & 1.1044 & 1.5000 & 1.9000 \\
\end{array}
\]

It is observed from Table 6.2 that the ARE of the proposed test relative to \(\mathcal{L}(M)\)-test, in case of uniform distribution, increases as \(c\) increases when \(c\) is greater than 2, for every fixed value of \(\alpha\). T-test performs better than \(\mathcal{L}(M)\)-test, for \(c\) greater than 6, and for all \(\alpha\) (except for \(c=8\) and \(\alpha =0.4\)); when \(c=6\) it performs better for \(\alpha\) near 0 or 0.5.