CHAPTER 5
A NONPARAMETRIC TEST FOR SEVERAL SAMPLE LOCATION PROBLEM BASED ON U-STATISTICS

5.1 Introduction

Let \( X_{i1}, \ldots, X_{in_i}, \ i = 1, \ldots, c \) be \( c \) independent samples with the \( i \)-th sample coming from a population having an absolutely continuous distribution function \( F_i(x) \). We consider here the problem of testing the hypothesis

\[
H_0: F_1(x) = F_2(x) = \ldots = F_c(x) = F(x) \quad \text{for all } x, \quad (5.1.1)
\]

where the functional form of \( F \) is not known, except that it is absolutely continuous. The alternative hypothesis of interest is

\[
H_1: F_i(x) = F(x - \theta_i), \ i = 1, \ldots, c, \quad (5.1.2)
\]

where \( \theta_i \)'s are some real numbers which are not all equal. There are several tests available in the literature for this problem. Among these are Bhapkar's \( V \)-test (1961), Deshpande's \( L \)-test (1965), Sugiura's \( V_{rs} \)-tests (1965) and Shanubhogue and Gore's \( S_m \)-test (1985). All these tests are based on the generalized \( U \)-statistics. The \( S_m \)-test uses subsamples of size \( m \) drawn from each of the \( c \) samples.

In this chapter a new test is proposed, which is also based on \( c \)-sample \( U \)-statistics constructed from the subsamples of size two from each of the \( c \)-samples. This test is given in Section 5.2. The asymptotic distributions of the test statistic under the null hypothesis \( H_0 \) and under the Pitman type location alternatives are obtained in Sections 5.3 and 5.4 respectively. In Section 5.5 the
test shown to be consistent. In the last section the Pitman asymptotic relative efficiencies (ARE) of this test are obtained with respect to the four tests mentioned above and also with respect to the Kruskal-Wallis H-test. These efficiencies are computed for \( c = 2(1) \) 6, 8 and 10 in the case of double exponential, logistic and normal distributions. The proposed test is found to be uniformly better than the \( S_m \)-test in the case of these three distributions. Its performance as compared to the other tests is discussed in this section. (This chapter is based on Biradar and Chikkagoudar (1997)).

### 5.2. The Proposed Test

From each sample, we consider all possible subsamples of size 2 and compute their minima and maxima. Based on these, we shall construct a set of U-statistics as given below. Let

\[
h_i(x_{i1}, x_{i2}; \ldots; x_{i,1}, x_{i,2}; \ldots; x_{c,1}, x_{c,2})
\]

\[
= 1 \quad \text{if} \quad \min(x_{i1}, x_{i2}) > \max (x_{j1}, x_{j2}), \quad \text{for all } j = 1, \ldots, c; \text{ and } j \neq i
\]

\[
= -1 \quad \text{if} \quad \max (x_{i1}, x_{i2}) < \min (x_{j1}, x_{j2}), \quad \text{for all } j = 1, \ldots, c; \text{ } j \neq i
\]

\[
= 0 \quad \text{otherwise}. \quad (5.2.1)
\]

for \( i = 1, 2, \ldots, c \). Using these kernels, we have a set of U-statistics defined as

\[
U_i = \left[ \prod_{i=1}^{c} \binom{n}{2} \right]^{-1} \sum_{1 \leq a_1 < a_2 \leq n} \ldots \sum_{1 \leq b_1 < b_2 \leq n} h_i(X_{ia_1}, X_{ia_2}; \ldots; X_{ib_1}, X_{ib_2}) \quad (5.2.2)
\]

Using these U-statistics the proposed test statistic is

\[
Q = \frac{N}{c^2} \frac{K(c)}{K(1)} \left[ \sum_{i=1}^{c} p_i U_i^2 - \left( \sum_{i=1}^{c} p_i U_i \right)^2 \right] \quad (5.2.3)
\]
where

\[ K(c) = \frac{8 \left(10c - 7\right) \left(4c - 2\right) - 2 \left(2c^2 + 1\right) \left(4c - 3\right)}{(c - 1)^2 \left(2c^2 - 1\right)^2 (4c - 1) (4c - 3) \left(4c - 2\right) (2c - 1)} \]

\[ p_i = \frac{n_i}{N} \]

The test \( Q \) rejects \( H_0 \) at a significance level \( \alpha \), if \( Q \) exceeds a pre-determined number \( Q_\alpha \). The asymptotic distribution of \( Q \), under \( H_0 \), is shown to be Chi-square with \( (c-1) \) degrees of freedom.

**5.3. Asymptotic Distribution of \( Q \) Under \( H_0 \)**

Bhapkar (1961) gave the following generalization of Hoeffding's (1948) theorem on one-sample U-statistics to the case of \( c \)-sample U-statistics. The asymptotic normality of generalized U-statistics has been discussed among others, by Bhapkar (1961) and Lehmann (1963).

**Theorem 5.3.1.** Let \( X_{ij}, j = 1, 2, \ldots, n_i \) be iid random variables with df \( F \), \( i = 1, 2, \ldots, c \). Further, let \( \sum_{i=1}^{c} n_i = N \) and

\[ U_r = \left[ \prod_{i=1}^{r} \left( \frac{n_i}{m_i} \right)^{m_i} \right] \sum_{\alpha} \sum_{\beta} \ldots \sum_{\delta} h_r(X_{i_{\alpha_1}}, \ldots, X_{i_{m_\alpha_1}}, \ldots; X_{i_{\beta_1}}, \ldots, X_{i_{m_\beta_1}}, \ldots; X_{i_{\delta_1}}, \ldots, X_{i_{m_\delta_1}}) \quad (5.3.1) \]

\( r = 1, 2, \ldots, g \), where each \( h_r \) is a function symmetric in each set of its arguments and \( \sum_{\alpha} \ldots \sum_{\delta} \) denotes the summation over all combinations \((\alpha_1, \ldots, \alpha_{m_\alpha}), \ldots, (\beta_1, \ldots, \beta_{m_\beta}), \ldots, (\delta_1, \ldots, \delta_{m_\delta})\) \( m_{lr} \) integers chosen from \((1, 2, \ldots, n_i)\) and so on for \( \beta \)'s \ldots

and \( \delta \)'s. Assume that \( E(h_r) = \eta_r \) and \( E(h_r)^2 < \infty \). Then
(i) $E(U_r) = \eta_r$,

(ii) $\text{Cov}(U_r, U_s) = \sum_{d_i, d_j = 0}^{\infty} \prod_{i=1}^{c} \left( \frac{m_i}{d_i} \right) \left( \frac{n_i - m_i}{m_i - d_i} \right) \xi^{(r,s)}$,

where $m_i(r, s) = \min(m_{ir}, m_{is})$ and

$$
\xi^{(r,s)} = E [h_r (X_{11}, \ldots, X_{1d_1}, X_{1d_1+1}, \ldots, X_{1m_{11}}, \ldots; X_{r}, \ldots, X_{c_{dr}};
X_{cd_r+1}, \ldots, X_{cm_{cr}}), h_s (X_{11}, \ldots, X_{1d_1}, X_{1d_1+1}, \ldots, X_{1m_{11}}, + m_{1s} - d_j; \ldots;
X_{c_j}, \ldots, X_{cd_j}, X_{cm_{cr}+1}, \ldots, X_{cm_{cr}+m_{cs} - d_j}) - \eta_r \eta_s],
$$

(iii) $\sqrt{N} (U - \bar{\eta})$ is, in the limit as $N \to \infty$ in such a way that $n_i = Np_i$, the $p_i$'s being fixed numbers such that $\sum_{i=1}^{g} p_i = 1$, normally distributed with zero mean and asymptotic covariance matrix $\Sigma = (\sigma_{rs})$ given by

$$
\sigma_{rs} = \sum_{i=1}^{c} m_i m_n \xi^{(r,s)}_{0,\ldots,0,1,0,\ldots,0} \quad (1 \text{ at the } i \text{ - th place})
$$

For applying Theorem 5.3.1 to our problem, we note from (5.2.2) that $U_i, i = 1, 2, \ldots, c$ are generalized U-statistics with $g = c$ and $m_{i1} = m_{i2} = \ldots = m_{ic} = 2$. It follows that as $N \to \infty$ in such a way that $n_i/N = p_i, i = 1, 2, \ldots, c$.
remain fixed, \( \sqrt{N} \mathbf{U} \), where \( \mathbf{U} = (U_1, \ldots, U_c) \), has multivariate normal distribution with mean vector zero and covariance matrix \( \Sigma = (\sigma_{ij}) \), \( i, j = 1, 2, \ldots, c \), where

\[
\sigma_{ij} = \lim_{N\to\infty} N \text{Cov}(U_i, U_j)
= \sum_{k=1}^{2c} p_k \xi_{i,j}^{(k)}, \quad (5.3.4)
\]

with

\[
\xi_{i,j}^{(k)} = E[h_i(X_{i1}, X_{12}; \ldots; X_{k1}, X_{k2}; \ldots; X_{c1}, X_{c2}) h_j(X_{i1}', X_{12}'; \ldots; X_{k1}', X_{k2}'; \ldots; X_{c1}', X_{c2}')]\]

Under \( H_0 \)

\[
E(\mathbf{U}) = E[h_i(X_{i1}, X_{12}; \ldots; X_{c1}, X_{c2})]
= P[\text{Min} (X_{j1}, X_{j2}) > \text{Max} (X_{i1}, X_{i2}), \text{for all} j = 1, 2, \ldots, c \text{ and } j \neq i]
- P[\text{Max} (X_{j1}, X_{j2}) < \text{Min} (X_{i1}, X_{i2}), \text{for all} j = 1, 2, \ldots, c \text{ and } j \neq i]
= 0.
\]

Also under \( H_0 \)

\[
\xi_{i,i}^{(k)} = E[h_i(X_{i1}, X_{12}; \ldots; X_{i1}, X_{i2}; \ldots; X_{c1}, X_{c2})]
= E[\pi_1(X_{i1}) - \pi_2(X_{i1})]^2, \quad (5.3.5)
\]

where

\[
\pi_1(x_{i1}) = P[\text{Max} (X_{j1}, X_{j2}) < \text{min} (X_{i1}, X_{i2}), \text{for all} j = 1, \ldots, c; j \neq i]
= \frac{1}{2c-1} \left[ F^{2c-1}(x_{i1}) + F^{2c-2}(x_{i1})[I - F(x_{i1})] \right], \quad (5.3.6)
\]

and

\[
\pi_2(x_{i1}) = P[\text{Min} (X_{j1}, X_{j2}) > \text{Max} (X_{i1}, X_{i2}), \text{for all} j = 1, \ldots, c; j \neq i]
\]

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Using (5.3.6) and (5.3.7), we obtain

\[ \xi_{n,j}^{(i)} = \frac{(c-1)^2}{4} K(c), \]  

(5.3.8)

where \( K(c) \) is as in (5.2.3). Similarly it can be shown that

\[ \xi_{n,j}^{(k)} = \frac{K(c)}{4}, \]

(5.3.9)

and

\[ \xi_{n,j}^{(i)} = \frac{-(c-1)K(c)}{4}. \]  

(5.3.10)

From (5.3.4) and (5.3.8) to (5.3.10), we obtain the elements of the covariance matrix \( \Sigma \) as

\[ \sigma_{ii} = K(c) \left[ \sum_{k=1}^{c} \frac{1}{P_k} + \frac{c^2 - 2c}{p_i} \right] \]

(5.3.11)

and for \( i \neq j \)

\[ \sigma_{ij} = K(c) \left[ \sum_{k=1}^{c} \frac{1}{P_k} \frac{c}{p_i} - \frac{c}{p_j} \right] \]

(5.3.12)

Using (5.3.11) and (5.3.12), \( \Sigma \) can be written as

\[ (K(c))^{-1} \Sigma = aJ_{c,c} + c^2D - cq J_{1,c} - cJ_{c,1} q', \]

(5.3.13)

where

\[ D = \text{diagonal} \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_c} \right), \quad q' = \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_c} \right), \quad a = \sum_{k=1}^{c} \frac{1}{P_k} \]

and \( J_{c,c} \) is a \( c \times c \) matrix with all its elements equal to one.
It can be seen that \( J_{1,c} \Sigma = 0 \). Hence \( \Sigma \) is singular. Also it can be shown that

\[
\sum_{i=r}^c U_i = 0 \quad \text{and rank of } \Sigma \text{ is } c-1.
\]

Hence we consider the asymptotic distribution of \( \sqrt{N} U_0 \), where \( \Sigma_0 = (U_1, \ldots, U_{c-1}) \), which is multivariate normal with mean vector zero and covariance matrix \( \Sigma_0 = (\sigma_{ij})_{c-1 \times c-1} \) for \( i, j = 1, 2, \ldots, c-1 \). Now \( \Sigma_0 \) can be written as

\[
(K(c))^{-1} \Sigma_0 = a J_{c-1,c-1} + c^2 D_0 - c q_0 J_{c-1,1} q_0',
\]

where \( D_0 = \text{diagonal } \left( \frac{1}{p_1}, \ldots, \frac{1}{p_{c-1}} \right) \), \( q_0' = \left( \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_{c-1}} \right) \), \( a = \sum_{k=1}^{c-1} \frac{1}{p_k} \).

Therefore under \( H_0 \), as \( N \to \infty \), \( Q = N U_0' \sum_{i=0}^{c-1} U_i \) has a \( \chi^2 \)-distribution with \( (c-1) \) degrees of freedom. After simplification, the statistic \( Q \) reduces to (5.2.3).

Thus we have proved the following theorem.

**Theorem 5.3.2.** If \( F_1 = \ldots = F_c \) and \( n_i = N p_i \), where \( p_i \) are fixed numbers such that \( \sum_{i=1}^c p_i = 1 \), then the statistic \( Q \) has a limiting \( \chi^2 \) distribution with \( (c-1) \) degrees of freedom.

### 5.4. Consistency of The Q-Test

Here, we state the following theorem due to Bhapkar (1961).

**Theorem 5.4.1.** Let \( \eta_i = f_i(F_1, F_2, \ldots, F_c) \), \( i = 1, \ldots, g \) be real valued functions such that \( f_i(F_1, F_2, \ldots, F) = \eta_{i0} \) for all \( F, F_2, \ldots, F \) in a class \( \mathcal{C}_0 \), where \( \mathcal{C}_0 \) is the class of distributions under the null hypothesis. Let \( T_{n_1, \ldots, n_c}^{(i)} = t_i(X_{11}, \ldots, X_{1n_1}, \ldots, X_{c1}, \ldots, X_{cn_c}) \), \( i = 1, 2, \ldots, g \) be a sequence of real valued statistics such that \( T_{n_1, \ldots, n_c}^{(i)} \) tends to \( \eta_i \) in probability as \( \min(n_1, n_2, \ldots, n_c) \to \infty \).
Suppose that at least one \( f_i(F_1, F_2, \ldots, F_c) \neq \eta_{i0} \) for all \((F_1, F_2, \ldots, F_c)\) in a class \(c_1\), where \(c_1\) is the class of all distributions under the alternative hypothesis.

Further, let

\[
W_{n_1, \ldots, n_c} = w(T_{n_1, \ldots, n_c}^{(1)}; \ldots; T_{n_1, \ldots, n_c}^{(c)})
\]

be a nonnegative function which is zero, if and only if \(T_{n_1, \ldots, n_c}^{(i)} = \eta_{i0}\) for all \(i = 1, 2, \ldots, c\). Then the sequence of tests which rejects \(H_0\) when

\[
W_{n_1, \ldots, n_c} > d_{n_1, \ldots, n_c}
\]

is consistent for testing \(H_0: F_1(x) = \ldots = F_c(x)\) if \(Q > Q_\alpha\) is consistent for all \(F_i(x) = F(x - \theta_i), i = 1, \ldots, c\), provided \(\eta_i\) is different from 0 (\(= E(U_j | H_0)\)) for at least one \(i\), where

\[
\eta_i \equiv P\{\min(X_{j1}, X_{j2}) > \max(X_{j1}, X_{j2}) \text{ for all } j \neq i\} - P\{\max(X_{j1}, X_{j2}) < \min(X_{j1}, X_{j2}) \text{ for all } j \neq i\},
\]

where \(X\)'s are independent random variables with continuous cdf's \(F_1, F_2, \ldots, F_c\) respectively.

5.5 Asymptotic Distribution of \(Q\) Under Location Alternatives

In this section we obtain the limiting distributions of \(U_i\) and \(Q\) under the following sequence of alternative hypotheses

\[
H_n: F_i(x) = F(x - \theta_i / n^{1/2}), \quad \theta_i \in R
\]

(5.5.1)
where all $\theta_i$'s are not equal. Here $n$ is given by the relation $n_i = n s_i$, where $s_i$ are fixed integers. The letter $n$ will be used to index a sequence of situations in which $H_n$ is the true hypothesis.

**Theorem 5.5.1.** (a) Under $H_n$, $\sqrt{N_{ij}}$, $i = 1, \ldots, c$, have jointly in the limit as $n \to \infty$, a multivariate normal distribution with means

$$\eta_i^{(n)} = 4 \left( \sum_{i=1}^{c} s_i \right)^{1/2} \sum_{i=1}^{c} (\theta_i - \bar{\theta}) \Gamma^*,$$

where

$$\Gamma^* = \int \left\{ \left[ (F(x))^{2c-3} - F(x) \right] + (1 - F(x))^{2c-3} F(x) \right\} f(x) dx$$

and the elements of the covariance matrix given by (5.3.11) and (5.3.12) under the following two conditions.

(i) $F$ is absolutely continuous with derivative $f$ and

(ii) there exists a function $g$ such that

$$\int g(x) dF(x) < \infty,$$

and

$$\frac{1}{h} [F(x+h) - F(x)] \leq g(x) \quad \text{for small } h.$$

(b) Under $H_n$ and the above two conditions, $Q$ has in the limit as $n \to \infty$, a noncentral $\chi^2$-distribution with $(c-1)$ degrees of freedom and the noncentrality parameter given by

$$\lambda_n = \frac{16 \Gamma^2}{K(c)} \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2,$$
where $K(c)$ and $T^*$ are as defined in (5.2.3) and (5.5.2) respectively, and

$$\bar{\theta} = \frac{\sum_{i=1}^{c} s_i \theta_i}{\sum_{i=1}^{c} s_i}. \quad \text{(5.5.6)}$$

**Proof of part (a).** Under $H_n$

$$E(U_i | H_n) = 2 \left( \prod_{j=1}^{c} \frac{F(x - \theta_j / \sqrt{n})}{(1 - F(x - \theta_j / \sqrt{n}))} \right) \frac{f^2(x)}{f^2(x)}$$

$$- 2 \left( \prod_{j=1}^{c} \frac{F(x - \theta_j / \sqrt{n})}{(1 - F(x - \theta_j / \sqrt{n}))} \right) \frac{f^2(x)}{f^2(x)} \frac{1}{\sqrt{n}} \quad \text{(5.5.4)}$$

By Lemma 4.1 of Sugiura (1965), (5.5.4) can be expressed as

$$E[U_i | H_n] = 2 \left( \prod_{j=1}^{c} \frac{F(x)}{(1 - F(x))} \right)$$

$$+ 4n \sum_{j=1}^{c} \left( \theta_i - \theta_j \right) \int_{-\infty}^{\infty} \frac{F(x)}{(1 - F(x))} \frac{f^2(x)}{f^2(x)} dx$$

$$- 2 \left( \prod_{j=1}^{c} \frac{F(x)}{(1 - F(x))} \right) \frac{f^2(x)}{f^2(x)} \frac{1}{\sqrt{n}} + o(\sqrt{n}) \quad \text{(5.5.5)}$$

Since $\eta_i = E(N^{1/2} U_i)$, \quad \text{(5.5.6)}

and $\sqrt{N/n} = (\Sigma s_i)^{1/2}$ (5.5.6) reduces to (5.5.2).

Proceeding exactly on similar lines and using conditions, (i) and (ii) of the
theorem and Lemma 4.1 of Suguira (1965), we see that,
\[ \lim_{N \to \infty} N \text{Cov} (U_j, U_j | H_n) = \lim_{N \to \infty} N \text{Cov} (U_j, U_j | H_0), \]

Hence part (a) of the theorem follows.

Proof of part (b). It follows from part (a) of the theorem that \( N^{1/2} U_j \) is asymptotically multivariate normal with mean vector \( 4(\sum_{i=1}^{c} \delta_i)^{1/2} \Gamma^* \delta \), where \( \delta' = (\delta_1, \ldots, \delta_c) \), with \( \delta_i = c\theta_i - \sum_{j=1}^{c} \theta_j \) and \( \Gamma^* \) as defined in (5.5.2), and the covariance matrix \( \Sigma = (\sigma_{ij}) \), where \( \sigma_{ij} \)'s are given by (5.3.11) and (5.3.12).

Hence \( Q \) is distributed as noncentral \( \chi^2 \) distribution with \((c-1)\) degrees of freedom and the noncentrality parameter
\[ \lambda_Q = 16 \left( \sum_{i=1}^{c} \delta_i \right) \left( \sum_{i=0}^{c} \delta_i \right) \Gamma^{*2}, \]

with \( \delta_0 = (\delta_1, \ldots, \delta_{c-1}) \). This simplifies to (5.5.3) and the theorem follows.

5.6. Asymptotic Relative Efficiency

We know from Andrew (1954) and Hannquin (1956), that the Pitman asymptotic relative efficiency (ARE), of one test with respect to another, is equal to the ratio of the non-centrality parameters of the two test statistics, provided that they are asymptotically distributed as non-central \( \chi^2 \)-variates with the same degrees of freedom under the given sequence of alternative hypotheses.

We shall compare the Q-test proposed here with the Kruskal-Wallis H-test (1952), Bhapkar's V-test (1961), Deshpande's L-test (1965), Sugiuira's \( V_{rs} \)-test (1965) and Shanubhogue and Gore's \( S_m \)-test (1985). It is shown in these
papers that under $H_n$ these test statistics, in the limit, have noncentral $\chi^2$
distributions with $(c-1)$ degrees of freedom and with the corresponding
noncentrality parameters

$$\lambda_H = 12 \left( \int f^2(x) dx \right)^2 \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2,$$  \hspace{1cm}(5.6.1)$$

$$\lambda_V = c^2 (2c - 1) \lambda_H^2 \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2,$$  \hspace{1cm}(5.6.2)$$

$$\lambda_L = \frac{(c-1)^2 (2c-1)}{2} \left( \frac{2c-2}{c-1} \right)^2 \left( \lambda_H + \lambda_c \right)^2 \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2,$$  \hspace{1cm}(5.6.3)$$

$$\lambda_{v_n} = \frac{1}{K(r,s)} \lambda_H^2 \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2,$$  \hspace{1cm}(5.6.4)$$

and

$$\lambda_{s_n} = c^2 m^2 (2cm - 1) \Gamma^2 \sum_{i=1}^{c} s_i (\theta_i - \bar{\theta})^2.$$  \hspace{1cm}(5.6.5)$$

respectively. Here

$$\lambda_H = \int (1 - F(x))^{c-2} f^2(x) dx, \quad \lambda_c = \int (F(x))^{c-2} f^2(x) dx,$$

$$\lambda_n = \int [r(F(x))^{r-1} + s(1 - F(x))^{s-1}] f^2(x) dx, \quad \Gamma = \int (1 - F(x))^{cm-2} f^2(x) dx \quad \text{and}$$

$$K(r,s) = \frac{r^2}{(2r+1)(r+1)^2} + \frac{s^2}{(2s+1)(s+1)^2} + \frac{2}{(r+1)(s+1)} - 2B(r+1,s+1).$$

Efficiencies of the Q-test as compared to these tests are obtained in the case of
double exponential, logistic and normal distributions. The numerical values
of the efficiencies for these distributions are given in Tables 5.1, 5.2 and
5.3 respectively. In the case of normal distribution the numerical values of \( \Gamma^* \), \( \Gamma \), \( \lambda_{rs} \), \( \lambda_1 \) and \( \lambda_c \) are obtained using the method given in Ruben (1954). In the case of the \( S_m \)-test the statistics \( S_2 \) and \( S_4 \) are considered.

From these tables it can be seen that the Q-test is uniformly better than the \( S_m \)-test with \( m = 2 \) and 4 in the case of all the three distributions considered here. It is found that \( e(Q, S_m) \) increases with \( c \) and \( m \). The Q-test is also better than the V-test for these distributions except for \( c = 2 \) in the case of logistic and normal distributions, it is also seen that \( e(Q, V) \) increases with \( c \).

**Table 5.1**

**ARE of Q-Test for double exponential distribution**

<table>
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<th>c</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
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<tbody>
<tr>
<td>e(Q,H)</td>
<td>1.072</td>
<td>1.038</td>
<td>0.916</td>
<td>0.783</td>
<td>0.671</td>
<td>0.513</td>
<td>0.414</td>
</tr>
<tr>
<td>e(Q,V)</td>
<td>1.072</td>
<td>1.107</td>
<td>1.154</td>
<td>1.188</td>
<td>1.218</td>
<td>1.276</td>
<td>1.328</td>
</tr>
<tr>
<td>e(Q,L)</td>
<td>1.072</td>
<td>1.038</td>
<td>0.975</td>
<td>0.915</td>
<td>0.873</td>
<td>0.833</td>
<td>0.820</td>
</tr>
<tr>
<td>e(Q,V_{0.6})</td>
<td>2.292</td>
<td>2.218</td>
<td>1.959</td>
<td>1.674</td>
<td>1.433</td>
<td>1.096</td>
<td>0.884</td>
</tr>
<tr>
<td>e(Q,S_2)</td>
<td>1.351</td>
<td>1.885</td>
<td>2.281</td>
<td>2.514</td>
<td>2.648</td>
<td>2.792</td>
<td>2.872</td>
</tr>
<tr>
<td>E(Q,S_4)</td>
<td>2.669</td>
<td>4.098</td>
<td>4.989</td>
<td>5.437</td>
<td>5.660</td>
<td>5.724</td>
<td>5.974</td>
</tr>
</tbody>
</table>

As is pointed out by Sugiura (1965), in the class of his \( V_{rs} \)-tests, \( V_{01} \), \( V_{10} \), \( V_{11} \) and \( V_{22} \) are equivalent to the Kruskal-Wallis H-test in the case of symmetric distributions. \( e(Q,H) \) values are shown in the tables. From Tables 3, 4 and 5 given by Sugiura (1965), \( e(Q,V_{rs}) \) values are found to be greater than one for many combinations of \( r \), \( s \), and \( c \) values. From these tables it can also
be found that the pairs of the best and the worst members of the $V_{rs}$ family are respectively, $(V_{44}, V_{0,10})$ for the normal distribution and $(V_{11}, V_{0.6})$ for double exponential as well as logistic distributions. So the Q-test has been compared with these pairs of tests. By similar computations $e(Q,H)$ is found to be greater than one for $c > 4$ in the case of uniform (over $(0,1)$) and exponential distributions. For the uniform distributions $e(Q,V)$ is also greater than one for $c > 4$. For both of these distributions $e(Q, V_{rs})$ is found to be greater than one for some combinations of $r$, $s$ and $c$.

Table 5.2

**ARE of Q-Test for logistic distribution**

<table>
<thead>
<tr>
<th>$c$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(Q,H)$</td>
<td>0.988</td>
<td>0.996</td>
<td>0.992</td>
<td>0.954</td>
<td>0.899</td>
<td>0.783</td>
<td>0.684</td>
</tr>
<tr>
<td>$e(Q,V)$</td>
<td>0.988</td>
<td>1.062</td>
<td>1.181</td>
<td>1.272</td>
<td>1.335</td>
<td>1.409</td>
<td>1.452</td>
</tr>
<tr>
<td>$e(Q,L)$</td>
<td>0.988</td>
<td>0.996</td>
<td>0.998</td>
<td>0.989</td>
<td>0.957</td>
<td>0.920</td>
<td>0.896</td>
</tr>
<tr>
<td>$e(Q,V_{0.6})$</td>
<td>1.647</td>
<td>1.659</td>
<td>1.654</td>
<td>1.591</td>
<td>1.489</td>
<td>1.305</td>
<td>1.140</td>
</tr>
<tr>
<td>$e(Q,S_{2})$</td>
<td>1.176</td>
<td>1.478</td>
<td>1.786</td>
<td>2.026</td>
<td>2.201</td>
<td>2.433</td>
<td>2.578</td>
</tr>
<tr>
<td>$e(Q,S_{4})$</td>
<td>1.779</td>
<td>2.439</td>
<td>3.083</td>
<td>3.597</td>
<td>3.984</td>
<td>4.512</td>
<td>4.851</td>
</tr>
</tbody>
</table>
Table 5.3

ARE of Q-Test for normal distribution

<table>
<thead>
<tr>
<th>c</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>e(Q,H)</td>
<td>0.948</td>
<td>0.970</td>
<td>1.009</td>
<td>1.016</td>
<td>0.999</td>
<td>0.938</td>
<td>0.871</td>
</tr>
<tr>
<td>e(Q,V)</td>
<td>0.948</td>
<td>1.035</td>
<td>1.169</td>
<td>1.275</td>
<td>1.350</td>
<td>1.443</td>
<td>1.498</td>
</tr>
<tr>
<td>e(Q,L)</td>
<td>0.948</td>
<td>0.970</td>
<td>0.987</td>
<td>0.982</td>
<td>0.968</td>
<td>0.943</td>
<td>0.925</td>
</tr>
<tr>
<td>e(Q,V_{4,4})</td>
<td>0.917</td>
<td>0.937</td>
<td>0.976</td>
<td>0.982</td>
<td>0.965</td>
<td>0.906</td>
<td>0.842</td>
</tr>
<tr>
<td>e(Q,V_{0,10})</td>
<td>1.716</td>
<td>1.756</td>
<td>1.826</td>
<td>1.839</td>
<td>1.808</td>
<td>1.697</td>
<td>1.176</td>
</tr>
<tr>
<td>e(Q,S_2)</td>
<td>1.099</td>
<td>1.311</td>
<td>1.553</td>
<td>1.747</td>
<td>1.890</td>
<td>2.084</td>
<td>2.209</td>
</tr>
<tr>
<td>e(Q,S_4)</td>
<td>1.460</td>
<td>1.836</td>
<td>2.243</td>
<td>2.575</td>
<td>2.830</td>
<td>3.189</td>
<td>3.432</td>
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</table>