In this chapter we have obtained a new general class of generating function for the generalized modified Laguerre polynomials $L^{(\alpha)}_n(x)$ by group theoretic method. Also, we introduce the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials with the help of two linear partial differential operators. Furthermore, we recover the result of Majumdar [45] and notice that the result of Das and Chatterjea [23] is the particular case of our result.
2.1 Introduction

In a theoretical connection with the unification of generating functions has great importance in the study of special functions. With the steps forward in this directions has been made by some researchers [13, 14, 18, 59]. Also, the special functions has great deal with applications in pure and applied mathematics. They are appears in different frameworks. They are often used in combinatorial analysis [54], and even in statistics [41]. Moreover, Laguerre polynomials have been applied in many other contexts, such as the Blissard problem (see [59]), the representation of Lucas polynomials of the first and second kinds [9, 11], the recurrence relations for a class of Freud-type polynomials [8], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [42]. Consequently they were also used [10] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way so called Robert formulas [55]. In their study Darus and Ibrahim [22] used deformed calculus to define generalized Laguerre polynomials and other special functions. Moreover, they gave the explicit representation formulas for the deformed Laguerre-type derivative of a composite function and illustration with applications. While Mukherjee [50] extend the bilateral generating function involving Jacobi polynomials derived by Chongdar [19] is well presented by group-theoretic method. Also, he has been proved the existence of quasi bilinear generating function implies the existence of a more general generating function. In their paper [1],
2.2 Generating Functions of Laguerre Polynomials

Alam and Chongdar obtained some results on bilateral and trilateral generating functions of modified Laguerre polynomials. Furthermore, they made some comments on the results of Laguerre polynomials obtained by Das and Chatterjea [23]. On the similar way Banerji and Mohsen [6] established a result on generating relation involving modified Bessel polynomials.

This chapter is organized in four sections. In the first section, we gave the introduction to the problem. While in section two, we develop the new general class of modified Laguerre and Jacobi polynomials. Also, there we have introduce bilateral generating function. In the third section of this chapter, we gave there some applications in connection with results obtained so far. In the last section of this chapter, we conclude the results therein.

2.2 Generating Functions of Laguerre Polynomials

In this section we develop the new general class of generating functions for modified Laguerre polynomials. Also, we introduce the bilateral generating function for modified Laguerre and Jacobi polynomials.

In the useful way of Weisner’s group-theoretic method, Das and Chatterjea [23] have claimed that the following operator $R_1$ is of double interpretations to both the index ($n$) and the parameter ($\alpha$) of the Laguerre polynomial. In a step forward with extending the ideas involve in the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$, we introduce the bilateral generating function for the generalized modified Laguerre and
Jacobi polynomials by means of theorem 2.2.1 and theorem 2.2.2. The Laguerre polynomials, as introduced by the later author [52], are defined as,

\[ L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} F_1(-n; 1 + \alpha; x), \quad Re(\alpha) > -1. \]

The following theorem determines the new generating relation from the given bilateral function.

**Theorem 2.2.1.** If there exists a generating function of the form

\[ G(x, u, w) = \sum_{n=0}^{\infty} a_n w^n L_n^{(\alpha)}(x) P_m^{(n, \beta)}(u), \quad (2.2.1) \]

then

\[ \exp(-wx)(1 - wt)^{(1 + \beta + m)}(1 + w)^\alpha G \left( x(1 + w), \frac{u + wt}{1-wt}, \frac{w}{1-wt} \right) = \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1 + n + \alpha + m)_q}{p!q!} L_{(n+p)}^{(\alpha-p)}(x) P_m^{(n+q, \beta)}(u)t^q. \]

**Proof.** Let us carry forward with the following linear partial differential operators, which has been referred from [1, 50].

\[ R_1 = xy^{-1}z \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} - xy^{-1}z, \quad (2.2.3) \]

and

\[ R_2 = (1 + u)t \frac{\partial}{\partial u} + t^2 \frac{\partial}{\partial t} + (1 + \beta + m)t. \]

So that

\[ R_1[y^\alpha z^n L_n^{(\alpha)}(x)] = (1 + n) L_{(n+1)}^{(\alpha-1)}(x)y^{(\alpha-1)}z^{(n+1)}, \quad (2.2.5) \]

and

\[ R_2[t^n P_m^{(n, \beta)}(u)] = (1 + n + \beta + m) P_m^{(n+1, \beta)}(u)t^{(n+1)}. \]

\[ (2.2.6) \]
Also, we have
\[ \exp(wR_1)f(x, y, z) = \exp\left(-\frac{wxz}{y}\right)f(x + wxy^{-1}z, y + wz, z), \quad (2.2.7) \]
and
\[ \exp(wR_2)f(u, t) = (1 - wt)^{-(1+\beta+m)}f\left(\frac{u + wt}{1 - wt}, \frac{t}{1 - wt}\right). \quad (2.2.8) \]

Now, we consider generating function (2.2.1) and replacing there \( w \) by \( wtz \); and then multiplying both sides by \( y^\alpha \), we get
\[ y^\alpha G(x, u, wtz) = y^\alpha \sum_{n=0}^{\infty} a_n(wtz)^n L_n^{(\alpha)}(x) P_m^{(\beta)}(u). \quad (2.2.9) \]

Operating \( \exp(wR_1) \), \( \exp(wR_2) \) on both sides of (2.2.9), we have
\[ \exp(wR_1)\exp(wR_2) \left[ y^\alpha G(x, u, wtz) \right] = \exp(wR_1)\exp(wR_2) \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) y^\alpha P_m^{(\beta)}(u)(wtz)^n. \quad (2.2.10) \]

With the help of (2.2.7) and (2.2.8) the left hand side of (2.2.10) can be simplified as
\[ \exp\left(-\frac{wxz}{y}\right)(1 - wt)^{-(1+\beta+m)}(y + wz)^\alpha G\left(x + wxy^{-1}z, \frac{u + wt}{1 - wt}, \frac{wtz}{1 - wt}\right). \quad (2.2.11) \]

Also, the right hand side of (2.2.10) with the help of (2.2.5) and (2.2.6) is simplified as
\[ \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1 + n)_p}{p!} L_n^{(\alpha-p)}(x) y^{\alpha-p} \frac{(1 + n + \beta + m)_q}{q!} P_m^{(n+q,\beta)}(u) z^{n+p} t^{n+q}. \quad (2.2.12) \]
Therefore, the simplified form of (2.2.10) is

\[
\exp \left( \frac{-wxz}{y} \right) (1 - wt)^{-(1+\beta+m)}(y + wz)^{\alpha} G \left( x + wxy^{-1}z, \frac{w+wt}{1-wt}, \frac{wtz}{1-wt} \right)
\]

\[
= \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1 + n)_p(1 + n + \beta + m)_q}{p!q!} L_{n+p}^{(\alpha-p)}(x) P_{m}^{(n+q,\beta)}(u)
\]

\[
\times y^{-p} z^{n+p} t^{n+q}.
\]

(2.2.13)

Finally substituting \( z/y = 1 \) in (2.2.13), we obtain bilateral generating function (2.2.14) for generalized modified Laguerre and Jacobi polynomials.

\[
\exp \left( -wx \right) (1 - wt)^{-(1+\beta+m)}(1 + w)^{\alpha} G \left( x + wx, \frac{w+wt}{1-wt}, \frac{w}{1-wt} \right) = \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1 + n)_p(1 + n + \beta + m)_q}{p!q!} L_{n+p}^{(\alpha-p)}(x) P_{m}^{(n+q,\beta)}(u) t^{q}.
\]

(2.2.14)

This completes the proof of the theorem. \( \square \)

**Theorem 2.2.2.** If there exists bilateral generating relation of the form

\[
G(x, v, w) = \sum_{n=0}^{\infty} a_n w^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v),
\]

then

\[
\left( \frac{1 + w}{1 + 2w} \right)^{\alpha} \exp (-wv) G \left( \frac{x + 2w}{1 + 2w}, v + wv, w \right) = \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1 + n)_q}{q!} P_{n+p}^{(\alpha,\beta-p)}(x)L_{n+q}^{(\alpha-q)}(v).
\]

(2.2.16)

**Proof.** Now we replace the variables \( x, y \) and \( z \) in the operator \( R_1 \) by \( v, s \) and \( t \) respectively. With this replacement we can rewrite the operator \( R_1 \) as;

\[
R_1 = vs^{-1} t \frac{\partial}{\partial v} + t \frac{\partial}{\partial s} - vs^{-1} t.
\]
2.2 Generating Functions of Laguerre Polynomials

So that

\[ R_1\left(s^n t^n L_n^{(\alpha)}(v)\right) = (1 + n)L_{(n+1)}^{(\alpha-1)}(v)s^{(\alpha-1)}t^{(n+1)}. \quad (2.2.17) \]

Let us define the operator \( R_3 \)

\[ R_3 = (1 - x^2)y^{-1}z\frac{\partial}{\partial x} - z(x - 1)\frac{\partial}{\partial y} - (1 + x)y^{-1}z^2\frac{\partial}{\partial z} \quad (2.2.18) \]

(One may refer [50] for more details about the operator \( R_3 \).) Operating \( R_3 \) on \( y^\beta z^n P_n^{(\alpha, \beta)}(x) \), we get

\[ R_3\left(y^\beta z^n P_n^{(\alpha, \beta)}(x)\right) = -2(1 + n)P_{n+1}^{(\alpha, \beta-1)}(x)y^{\beta-1}z^{n+1}. \quad (2.2.19) \]

Also, we have

\[ \exp(wR_3)f(x, y, z) = \left(\frac{y}{y + 2wz}\right)^{\alpha+1}f\left(\frac{xy + 2wz}{y + 2wz}, \frac{y(y + 2wz)}{y + 2wz}, \frac{yz}{y + 2wz}\right), \quad (2.2.20) \]

and

\[ \exp(wR_1)f(v, s, t) = \exp\left(-\frac{wvt}{s}\right)f(v + wvs^{-1}t, s + wt, t). \quad (2.2.21) \]

Now, we consider (2.2.15) and replacing there \( w \) by \( wtz \); and then multiplying both sides by \( y^\beta s^\alpha \), we get

\[ y^\beta s^\alpha G(x, v, wtz) = y^\beta s^\alpha \sum_{n=0}^{\infty} a_n(wtz)^n P_n^{(\alpha, \beta)}(x)L_n^{(\alpha)}(v). \quad (2.2.22) \]

Operating \( \exp(wR_1), \exp(wR_3) \) on both sides of (2.2.22), we have

\[ \exp(wR_1)\exp(wR_3) \left[y^\beta s^\alpha G(x, v, wtz)\right] \]

\[ = \exp(wR_1)\exp(wR_3) \sum_{n=0}^{\infty} a_n(wtz)^n P_n^{(\alpha, \beta)}(x)L_n^{(\alpha)}(v)y^\beta s^\alpha. \quad (2.2.23) \]
2.2 Generating Functions of Laguerre Polynomials

With the help of (2.2.17) and (2.2.19) the right hand side of (2.2.23) can be simplified as

\[
\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \left( \frac{1+n}{p!} \frac{(1+n)_q}{q!} (-2)^p \right) P_{n+p}^{(\alpha, \beta-p)}(x) L_n^{(\alpha-q)}(v) \times y^{(\beta-p)} s^{(\alpha-q)} z^{(n+p)} t^{(n+q)}.
\]

(2.2.24)

Also, the left hand side of (2.2.23) with the help of (2.2.20) and (2.2.21) is simplified as

\[
y^\alpha (s + wt)^\alpha \exp\left( -\frac{wst}{s} \right) \left( \frac{y + 2wz}{y + 2wz} \right)^{\alpha+1} G\left( \frac{xy + 2wz}{y + 2wz}, v + wvs^{-1}t, \frac{wtyz}{y + 2wz} \right).
\]

(2.2.25)

Therefore, the simplified form of (2.2.23) is

\[
y^{(\alpha+\beta+1)} \left( \frac{s + wt}{y + 2wz} \right)^\alpha \exp\left( -\frac{wst}{s} \right) (y + 2wz)^{-1} G\left( \frac{xy + 2wz}{y + 2wz}, v + wvs^{-1}t, \frac{wtyz}{y + 2wz} \right)
\]

\[
= \sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \left( \frac{1+n}{p!} \frac{(1+n)_q}{q!} (-2)^p \right) P_{n+p}^{(\alpha, \beta-p)}(x) L_n^{(\alpha-q)}(v) \times y^{(\beta-p)} s^{(\alpha-q)} z^{(n+p)} t^{(n+q)}.
\]

(2.2.26)

Finally substituting \( s = y = z = t = 1 \) in (2.2.26), we arrive at the proof of theorem.

With the help of bilateral generating function for set of Laguerre and Jacobi polynomials, we determine two new classes of generating functions in the form of theorem 2.2.1 and theorem 2.2.2. In the following section we gave the applications to newly investigated class of generating function, which are nothing but the particular cases of above results.
2.3 Application

If we put \( m = 0 \), we notice that \( P_o^{(n, \beta)}(u) = 1 \). Hence, from theorem 2.2.1, we deduce that

\[
(1 + w)^\alpha \exp(-wx)G(x + wx, w) = \sum_{n, p=0}^{\infty} a_n w^{n+p} \frac{(1 + n)_p}{p!} L^{(\alpha-p)}_{n+p}(x).
\]

(2.3.1)

1. If we put \( a_n = 1 \), in (2.3.1), we obtain

\[
(1 + w)^\alpha \exp(-wx)L_n^{(\alpha)}(x(1 + w)) = \sum_{p=0}^{\infty} w^p \binom{n + p}{p} L^{(\alpha-p)}_{n+p}(x).
\]

(2.3.2)

This result is as same as obtained by Das and Chatterjea in their paper [23].

2. If we multiply both sides of (2.3.1) by \( r^n \), we get

\[
(1 + w)^\alpha \exp(-wx)G(x + wx, wr) = \sum_{n, p=0}^{\infty} a_n w^{n+p} \frac{(1 + n)_p}{p!} r^n L^{(\alpha-p)}_{n+p}(x),
\]

\[
= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{n} a_{n-p} \binom{n}{p} r^{n-p} L^{(\alpha-p)}_{n-p}(x),
\]

\[
= \sum_{n=0}^{\infty} w^n \sigma_n(x, r),
\]

(2.3.3)

where

\[
\sigma_n(x, r) = \sum_{s=0}^{\infty} \binom{n}{s} a_s r^s L^{(\alpha-n+s)}_{n}(x).
\]
2.4 Conclusion

Incidentally this happens to be the theorem 1 in the paper of Majumdar (p:195 cf.[45]).

In this section of applications, we means that; we recovered the famous result of Majumdar as well as the result of Das and Chatterjea through the theorem 2.2.1.

2.4 Conclusion

In this chapter, we have introduced a new general class of generating functions in the form of (2.2.1), for modified Laguerre and Jacobi’s polynomials. Whereas, a function in (2.2.15) is a bilateral generating function for Laguerre and Jacobi’s polynomials. It may be noted that the result of Das and Chatterjea (one may refer [23]) fall out as particular cases of theorem 2.2.1 for \( m = 0 \) and \( a_n = 1 \). Also, we have shown that the result of Majumdar [45] given by (2.3.3) is the particular case of theorem 2.2.1.

**Note 2.4.1.** This work has been published in the International Journal “Journal of Inequality and Special Functions” (cf.[24])