Chapter VI

Solution of Modified Reynolds Equation with Variable coefficients using Wavelet-Multigrid Scheme

6.1 Introduction

This chapter deals with an application of the method discussed in chapter V to a problem encountered in bio-medical engineering problem. The effects of surface roughness on a squeeze film behavior of poroelastic bearings with Newtonian fluid as lubricant is investigated. Modified and realistic forms of constitutive relations of the viscoelastic matrix are incorporated in the derivation of relevant partial differential equation. The modified Reynolds equation is derived on the basis of stochastic theory, describing roughness structure of the articular cartilage and the constitutive equation of the poroelastic material. Recently developed fast wavelet-multigrid method is used for the solution of modified Reynolds equation, which has many advantages over classical methods. The influences of roughness on poroelastic material provides increase in the squeeze film pressure and hence load carrying capacity; that is sustained by articular cartilage during normal conditions. This helps design biomedical engineers in choosing suitable parameters while manufacturing the artificial knee-joint. To illustrate advantages of the wavelet-multigrid method in CFD and bio-
medical engineering applications, Wavelet-Multigrid solution of Surface Roughness Effects on Squeeze Film Poroelastic Bearings is presented.

Recently much attention has been focused on the study of human joints with increased interest of both engineers and orthopedic surgeons into biomechanics of joints because of their importance in human locomotion. These joints provide low friction and wear properties. Many modes of lubrication in movable joints, for example, weeping lubrication, boosted lubrication squeeze film and elastohydrodynamic lubrication (Torzilli (1978)), have been proposed for the study of negligible friction and wear characteristics of articular cartilage in human joints. Articular cartilage is the load bearing and shock absorbing tissue within all the diarthrodial joints such as hips, knees shoulders etc. These joints have remarkable tribological characteristics that are more superior to man-made bearings. But for various reasons, a long term process of cartilage leads to osteoarthritis, the major biomedical problem which deteriorates the normal functioning of the joint.

The simplest and successful linear biphasic model for articular cartilage has been developed by Mow et al.(1980) . This model includes the small deformation of poroelastic material which corresponds to Biot (1941) theory for soil consolidation. Using this (Biot's) theory, the governing equations for cartilage deformation and motion of interstitial fluid were formulated. Monsour et al. (1973) have modeled the joint as a single layer of homogenous fluid filled, porous permeable, deformable elastic material (articular cartilage). This tissue secretes the viscous and highly non-Newtonian fluid called synovial fluid which mainly consists of hyaluronic acid,
nutrients, glycoprotein etc. This synovial fluid bathes and supplies nutrients to both the cartilage surfaces. Hou et al (1992) have analyzed the squeeze film lubrication of articular cartilage by assuming synovial fluid to be linearly viscous fluid. Detailed analyses about articular cartilage and non-Newtonian characteristics of synovial fluid are given in Torzilli and Mow (1976).

Poroelasticity is a continuum theory for the analysis of porous media with elastic matrix consisting of interconnected fluid filled pores. When fluid permeated into a poroelastic material, the drag force between the fluid and the porous medium may cause deformation in porous matrix. This leads to volumetric changes in the pores. Since the pores are filled with fluid, the presence of the fluid not only acts as a stiffener of the material, but also results in the flow of pore fluid between regions of higher and lower pore pressure. A successful model for cartilage with interstitial fluid has been developed by Mow and his co-workers (1980). This simplest linear version of biphasic mixture includes the small deformation of the porous elastic matrix, which corresponds to Biot’s model for soil consolidation. The above biphasic model for a homogeneous and isotropic articular cartilage has been used in a series of papers to model the fluid flow across the articular surface in geometrically idealized step-loaded synovial joints (Forster and Fisher(1996). Various aspects of articular cartilage and non-Newtonian characteristics of the synovial fluid are presented by Torzilli and Mow. Collins (1982) considered Biot’s theory to model poroelastic matrix for cartilage which is assumed to satisfy generalized form of Darcy’s law for unsteady flow in an elastic porous medium. Later, modified and corrected forms of the
constitutive equations are given in (Jin et al. (1992), Ateshian et al. (1994) and Hlavacek (2000)). Most of the studies on synovial joint mechanics and lubrication have used elastic single-phase models of cartilage and a Newtonian single-phase model for synovial fluid. Recently, Mercer and Barry (1999) gave a numerical method on finite difference approximations for the calculation of deformation, pressure and flow within a finite two-dimensional poroelastic medium by considering slip and no-slip boundaries.

Sayles et al. (1979) experimentally revealed that the cartilage surface is rough and that roughness height distribution is Gaussian in nature. This motivates us to investigate the effect of roughness in cartilage surface. Christensen (1969) developed the stochastic theory to investigate the effect of roughness in hydrodynamic lubrication on the assumption that roughness can be represented as a randomly varying quantity. It is assumed that in hydrodynamic lubrication theory, the roughness heights are small compared to the film thickness. This theory consists of two types of roughness structures namely longitudinal and transverse roughness. The former one has the roughness striations in the form of long ridges and narrows in $x$-direction whereas latter one in $y$-direction.

The present chapter is organized as follows. The simplified modified constitutive relations of poroelastic material are considered in the section 6.2. In section 6.3, the modified Reynolds equation which holds in the film region is derived for two roughness patterns namely longitudinal and transverse roughness. In section 6.4, we discuss briefly the wavelet-multigrid method for the solution of modified
Reynolds equation. Section 6.5 is devoted to the discussion of the results obtained in the previous sections for various parameters. The concluding section summarizes the major results of the present study.

6.2. Formulation of the Problem

Fluid film region:

The physical configuration of the problem is shown in Fig. 6.1(b), which is the simplified form of synovial knee joint (shown in Fig.6.1(a)). The bone ends are covered by articular cartilage to prevent natural abrasion, which is in a sac containing fluid for lubricating the two surfaces. A tough fibrous capsule together with the muscles, ligaments and intra-articular structures etc encloses the normal joint cavity. The inner lining of this capsule, the synovial membrane secretes viscous and highly lubricating fluid called synovial fluid. This fluid bathes both articular surfaces and intra-articular structures. Following Walker and Erkman (1972), as the load bearing area of the synovial knee joint is small, two articular surfaces may be considered to be parallel under high loading conditions and for mathematical simplicity the average of three layers of the cartilages is modeled as a single poroelastic layer. So, the problem considered is that of squeeze film lubrication between two rectangular surfaces with finite dimensions. The lubricant in the joint cavity is assumed to be Newtonian fluid i.e. linearly viscous and incompressible fluid.
The upper rigid rough impervious spherical indenter approaches the lower poroelastic matrix normally with a constant velocity $\frac{dH}{dt}$. The film thickness between two plates is characterized by

$$H = h(x, y) + h_s(x, y, \xi)$$

(6.2.1)

where $h(x, y) = h_0 + \xi$, is the nominal smooth part of the geometry, $h_0$ is the minimum thickness, $R$ is the radius of the indenter in $x$ and $y$ directions, $h_s$ is the height of the surface asperities measured from the nominal level which is randomly varying quantity of zero mean and $\xi$ is the index parameter determining a definite roughness structure. The lubricant in the film region is assumed to be Newtonian i.e. linearly viscous and incompressible fluid. All the articulations of knee joint under fluid film lubrication involve cartilage-viscous fluid-cartilage interactions. The lubricant in the film region is assumed to be Newtonian fluid i.e. linearly viscous. So, the problem considered would be that of three dimensional squeeze film lubrication between the upper rigid spherical indenter and the lower smooth poroelastic matrix.

With usual assumptions of lubrication, the governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (6.2.2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}, \quad (6.2.3)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\mu} \frac{\partial p}{\partial y} \quad (6.2.4)$$
where \( u, v \) and \( w \) are the velocity components in \( x, y \) and \( z \) directions respectively, \( p \) is the pressure and \( \mu \) is the viscosity of the fluid.

**Poroelastic Region**

Following Mow et al. (1980), the poroelastic material is assumed to be made of a linearly elastic solid phase (deformable cartilage matrix) and an inviscid fluid phase in which these two phases are intrinsically incompressible. For the cartilage matrix deformation and viscous fluid contained in its pores, we write modified coupled equations of motion as (Ateshian et al. (1994-98))

Matrix:

\[
\rho_s \frac{\partial^2 \mathbf{U}}{\partial t^2} = \text{div} \sigma_s - (\phi')^2 \frac{\mu}{k} \left( \frac{\partial \mathbf{U}}{\partial t} - \mathbf{V} \right)
\]  

Fluid:

\[
\rho_f \frac{\partial \mathbf{V}}{\partial t} = \text{div} \sigma_f + (\phi')^2 \frac{\mu}{k} \left( \frac{\partial \mathbf{U}}{\partial t} - \mathbf{V} \right)
\]  

and

\[
\nabla \cdot \left( \phi' \frac{\partial \mathbf{U}}{\partial t} + \phi' \mathbf{V} \right) = 0
\]  

where \( \phi' \) is the porosity and \( \phi' = 1 - \phi' \) is the solidity of the poroelastic material. \( \rho_s \) and \( \rho_f \) denote the densities of solid matrix and fluid respectively, \( \mathbf{U} \) is the corresponding displacement vector, \( k \) is the permeability of the cartilaginous matrix to fluid. The left hand terms denote the local forces (mass times acceleration), which are counterbalanced by the right hand terms namely the surface forces, \( \text{div} \sigma \), and the porous medium driving forces (Darcy's law) respectively. These two component equations may be simply viewed as generalized forms of Darcy's law for unsteady
flow in a deformable porous medium in terms of relative velocity \( \frac{\partial U}{\partial t} - V \) between the moving cartilage and the fluid contained in its pores. Also, equations (6.2.6) and (6.2.7) denote force balances for the linear elastic solid and viscous fluid components of the cartilage respectively. The classical stress tensor \( \sigma \) for a continuous homogeneous medium may be expressed for the matrix (cartilage) and fluid (synovial). Thus the constitutive relations for the solid and fluid phases respectively are

\[
\sigma_s = -\phi' P I + 2N e + A e I, \\
\sigma_f = -\phi' P I + E' e I,
\]

in terms of the elastic parameters \( N, E' \) and \( A \) of the cartilage and the hydrostatic pressure \( P \) and \( I \) the identity tensor, \( e \) the cartilage dilation. The inertial terms in (6.2.6) and (6.2.7) are neglected because in the balance of momentum equation the fluid-fluid viscous stress is negligible compared with the drag between the fluid and solid matrix [10]. After neglecting inertia terms in (3.6) & (3.7) and using this result into equation (3.8), we get,

\[
\nabla^2 e - \frac{\mu}{k(A + E' + 2N)} \frac{\partial e}{\partial t} = 0,
\]

(6.2.11)

where \( e = \nabla \cdot U \). The typical stress-strain curve obtained from confined compression tests an articular cartilage under loading stresses is essentially a linear relationship beyond the initial stress level (Dowson and Jin (1992)). Following Collins (1982),
their results may be characterized by a simple linear equation in terms of the corresponding average bulk modulus $K$ and pressure $P$

$$e = e_0 + \frac{P}{K}. \quad (6.2.12)$$

Using this empirical relation (6.2.12) into eqn (6.2.11) we get the equation describing pressure in the poroelastic region,

$$\nabla^2 P = 0. \quad (6.2.13)$$

**Boundary Conditions**

The relevant boundary conditions for the velocity field ($0 < z < H$) are

$$u(x, y, 0) = u(x, y, H) = v(x, y, 0) = v(x, y, H) = 0, \quad (6.2.14)$$

$$w(x, y, 0) = -w^*, \quad w(x, y, H) = -\frac{dH}{dt}, \quad (6.2.15)$$

where $w^*$ represents the normal component of the relative velocity of the fluid at the cartilage surface. Conditions (6.2.14) are no-slip velocity conditions.

**6.3. Solution Procedure**

Integrating equation (6.2.13) with respect to $z$ over the porous layer thickness ($-\delta$, 0) and using the solid backing boundary condition $\frac{\partial P}{\partial z} = 0$ at $z = -\delta$, we get

$$\left. \frac{\partial P}{\partial z} \right|_{z=0} = -\int_{-\delta}^{0} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \, dz. \quad (6.3.16)$$

where $\delta$ is the thickness of the poroelastic layer. Using the Morgan-Cameron approximation (1957) valid for the poroelastic layer to be small and incorporating the pressure continuity condition ($p = p^*$) at the porous interface $z=0$, we get
\[
\begin{align*}
\frac{\partial P}{\partial z} \bigg|_{z=0} &= -\delta \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right). \\
(6.3.17)
\end{align*}
\]

After neglecting inertia terms, equation (6.3.7) may be arranged in terms of relative velocity in the form

\[
\begin{align*}
\left( V - \frac{\partial U}{\partial t} \right) &= -\frac{k}{\mu \left( \phi' \right)^2} \nabla P \left( \phi' - \frac{E'}{K} \right). \\
(6.3.18)
\end{align*}
\]

Elimination of \( e \) through (6.3.12) and (6.3.18) gives

\[
\begin{align*}
\left( V - \frac{\partial U}{\partial t} \right) &= -\frac{k}{\mu \left( \phi' \right)^2} \nabla P \left( \phi' - \frac{E'}{K} \right). \\
(6.3.19)
\end{align*}
\]

The normal component of the relative fluid velocity at the cartilage surface is

\[
\begin{align*}
\left. w_n \right|_{\text{cartilage}} &= -\left( V - \frac{\partial U}{\partial t} \right) \bigg|_{\text{cartilage}} = -\frac{k}{\mu \left( \phi' \right)^2} \left( \frac{E'}{K} - \phi' \right) \frac{\partial P}{\partial z} \bigg|_{z=0}. \\
(6.3.20)
\end{align*}
\]

Using equation (6.3.17) in equation (6.3.20), we get

\[
\begin{align*}
\left. w_n \right|_{\text{cartilage}} &= -\frac{k \delta}{\mu \left( \phi' \right)^2} \left( \frac{E'}{K} - \phi' \right) \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right). \\
(6.3.21)
\end{align*}
\]

Equations (6.2.3) and (6.2.4) can be integrated for \( u \) and \( v \) with respect to \( z \) using boundary conditions (6.2.14). Substituting \( u \) and \( v \) in the continuity equation (6.2.2) and integrating across the film thickness from \( z = 0 \) to \( z = H \) with respect to \( z \) using boundary conditions (6.2.15), we obtain modified Reynolds equation.
\[
\frac{\partial}{\partial x} \left[ H^3 - \frac{24\delta k}{(\phi')^3} \left( \frac{E'}{K} - \phi' \right) \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial y} \left[ H^3 - \frac{24\delta k}{(\phi')^3} \left( \frac{E'}{K} - \phi' \right) \frac{\partial p}{\partial y} \right] = -12\mu \frac{dH}{dt}, \tag{6.3.22}
\]

for the pressure distribution in the joint cavity. For including roughness features, taking the stochastic average of (6.3.22) with respect to \( h_s \), we get

\[
\frac{\partial}{\partial x} \left[ E \left[ H^3 - \frac{24\delta k}{(\phi')^3} \left( \frac{E'}{K} - \phi' \right) \frac{\partial p}{\partial x} \right] \right] + \frac{\partial}{\partial y} \left[ E \left[ H^3 - \frac{24\delta k}{(\phi')^3} \left( \frac{E'}{K} - \phi' \right) \frac{\partial p}{\partial y} \right] \right] = -12\mu \frac{dE(H)}{dt}, \tag{6.3.23}
\]

where expectancy operator \( E(*) \) is defined by

\[
E(*) = \int f(h_s) dh_s, \tag{6.3.24}
\]

\( f \) is the probability density function of the stochastic film thickness \( h_s \). In most of the problems, surfaces show a roughness height distribution which is Gaussian in nature. Therefore, polynomial form which approximates the Gaussian is chosen in the analysis. Such a probability density function is (Christensen (1969))

\[
f(h_s) = \begin{cases} 
\frac{35}{32c^7}(c^2 - h_s^2)^5, & -c < h_s < c \\
0, & \text{elsewhere}
\end{cases}
\]

where \( c = 3\sigma_i \) and \( \sigma_i \) is the standard deviation.
Longitudinal roughness

In the context of rough surfaces, there are two types of roughness patterns, which are of special interest. The one-dimensional longitudinal structure where the roughness has the form of long narrow ridges and valleys running in the $x$-direction and the one-dimensional transverse structure where roughness striations are running in the $y$-direction in the form of narrow and valleys. However, the present study is restricted to one-dimensional longitudinal roughness since the one roughness structure can be obtained from other by just rotation of coordinate axes. For the one-dimensional longitudinal roughness pattern, the film thickness assumes the form

$$H = h(t) + h(x, \xi),$$

then, Reynolds equation (6.3.23) takes the form

$$\begin{align*}
\frac{\partial}{\partial x} & \left[ E(H') \left( \frac{24k}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial E(p)}{\partial x} \right] + \\
\frac{\partial}{\partial y} & \left[ \frac{1}{E(H' + H^2)} \left( \frac{24k}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial E(p)}{\partial y} \right] = -12\mu \frac{dE(H)}{dt} \tag{6.3.25}
\end{align*}$$

Transverse roughness

In this structure, the roughness striations are running in the $y$-direction in the form of narrow ridges and valleys. In this case, the film thickness takes the form

$$H = h + h_y(y, \xi)$$

and modified Reynolds equation becomes

$$\begin{align*}
\frac{\partial}{\partial x} & \left[ \frac{1}{E(H' + H^2)} \left( \frac{24k}{(\phi')^2} \frac{\partial E(p)}{\partial x} \right) \right] + \\
\frac{\partial}{\partial y} & \left[ E(H') \left( \frac{24k}{(\phi')^2} \frac{\partial E(p)}{\partial y} \right) \right] = -12\mu \frac{dE(H)}{dt} \tag{6.3.26}
\end{align*}$$
By confining to longitudinal roughness parameter for the distribution function given by (6.3.24), we have

\[ E(H) = h. \] (6.3.27)

\[ E(H^3) = h^3 + \frac{hc^2}{3}. \] (6.3.28)

\[ E(I/H^3) = \frac{35}{32c^7} \left[ 3(\hbar^2 - c^2)(c^2 - h^2) \log\left( \frac{h + c}{h - c} \right) + 2c h (15\hbar^2 - 13c^2) \right] \] (6.3.29)

Therefore, Reynolds equation (6.3.25) for longitudinal roughness takes the form

\[
\frac{\partial}{\partial x} \left( E(H^3) - \frac{24k\delta}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial E(p)}{\partial x} + \frac{\partial}{\partial y} \left( \frac{1}{E(H^3)} - \frac{24k\delta}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial E(p)}{\partial y} = -12\mu \frac{dh}{dt}. \] (6.3.30)

Introducing the following non-dimensional parameters and variables

\[
\bar{x} = \frac{x}{a}, \quad \bar{y} = \frac{y}{a}, \quad \bar{H} = \frac{H}{h_0} = 1 + \frac{\bar{x}^2 + \bar{y}^2}{2R_h}, \quad R_l = \frac{R}{a}, \quad \bar{h}_l = \frac{h}{h_0}, \quad \bar{h}_r = \frac{h_r}{h_0},
\]

\[
\bar{k} = \frac{k_\delta}{h_0^n}, \quad \bar{\rho} = \frac{E(p)h_0^2}{\mu h_0^2 \frac{dh}{dt}}, \quad C = \frac{c}{h_0},
\]

in equation (6.3.30), we get,

\[
\frac{\partial}{\partial \bar{x}} \left( E(\bar{H}^3) - \frac{24\bar{k}}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left( \frac{1}{E(\bar{H}^3)} - \frac{24\bar{k}}{(\phi')^2} \left( \frac{E'}{K} - \phi' \right) \right) \frac{\partial \bar{p}}{\partial \bar{y}} = -12. \] (6.3.31)

where
\[
E(\vec{H}^2) = \vec{H}_0^2 + \frac{\vec{H}^2}{3}
\]

\[
E(\vec{H}^{-1}) = \frac{35}{32C^3} \left( 3(5\vec{H}^2 - C^2)(\vec{H}^2 - \vec{H}^2) \log \left( \frac{\vec{H} + C}{\vec{H} - C} \right) + 2C\vec{H}(15\vec{H}^2 - 13C^2) \right)
\]

and for pressure distribution, the boundary conditions are

\[
\bar{p} = 0 \text{ at } \bar{x} = \pm 1,
\]

\[
\bar{p} = 0 \text{ at } \bar{y} = \pm 1.
\]  

(6.3.32)

6.4. Wavelet-multigrid Method

Since, the equation (6.3.30) is an elliptic equation with variable coefficients, it is too difficult to solve analytically; we solve it numerically using recently developed wavelet-multigrid method (Bujurke et al.(2007)). Using the first order finite difference scheme for derivatives in (6.3.30), we write

\[
A_0\vec{p}_{i+1,j} + A_1\vec{p}_{i-1,j} + A_2\vec{p}_{i,j+1} + A_3\vec{p}_{i,j-1} + A_4\vec{p}_{i,j} = A_5
\]  

(6.4.33)

where,

\[
A_0 = \frac{a_{\text{int},j}}{(\Delta x)^2}, \quad A_1 = \frac{a_{\text{int},j}}{(\Delta y)^2}, \quad A_2 = \frac{\beta_{\text{int},j}}{(\Delta y)^2}, \quad A_3 = \frac{\beta_{\text{int},j}}{(\Delta y)^2}, \\
A_4 = -(A_0 + A_1 + A_2 + A_3), \quad A_5 = -12,
\]

\[
a_{ij} = E(\vec{H}_{ij}^2) - \frac{24k}{(\psi_j')^2} \left( \frac{E_j'}{K} - \phi_j' \right), \quad \beta_{ij} = E(\vec{H}_{ij}^2) - \frac{24k}{(\psi_j')^2} \left( \frac{E_j'}{K} - \phi_j' \right).
\]

Let the matrix formulation of (6.4.33) be

\[
Ax = y
\]  

(6.4.34)

By considering D-4 family of wavelets, the wavelet-multigrid method is applied to the matrix equation (6.4.34) as illustrated and explained in chapter V.
To begin with, FWT is performed on $A$ and $y$ of equation (6.4.34) recursively till the coarsest level is reached at level $-J$. Then $\hat{A} \hat{x} = \hat{y}$, to obtain $\hat{x}$, is solved at the coarsest level using Gauss-Jordan method. Finally, IWT performed on $\hat{x}$ to obtain $\hat{x}_0$ level -1, which gives the distribution of fluid film pressure $\bar{p}$. In the calculation, for all numerical experiments D-4 wavelets are employed. However, one has the freedom and flexibility to choose any wavelet basis. To test the accuracy, the problem is solved at resolutions $2^4$ and $2^8$. It is observed that, although, there is a marginal increase in the accuracy of the solution; better accuracy can be achieved by increasing the resolution and / or the order of the wavelet family. It is also observed that, the amount of computational effort is considerably less than that of classical multigrid method.

Once the fluid film pressure is obtained by using wavelet-multigrid method, the load carrying capacity can be evaluated. The non-dimensional load carrying capacity $\bar{W}$ per unit area of the joint surface is:

$$\bar{W} = \frac{1}{1 \times 1} \int_{-1}^{1} \int_{-1}^{1} \bar{p}(\bar{x}, \bar{y}) \, d\bar{x} \, d\bar{y}.$$  

(6.4.35)

6.5. Results and Discussion

A simplified mathematical model has been developed for the study of effect of surface roughness on the poroelastic bearing by modeling articulate cartilage as poroelastic matrix and synovial fluid as a linearly viscous fluid. The problem considered is that of the squeeze film lubrication between the rough spherical indenter
and smooth poroelastic material. The modified Reynolds equation incorporates the constitutive equations of poroelastic matrix. This Reynolds equation is solved using wavelet-multigrid method. The values of $E', K$ and $\bar{k}$ are taken from Torzilli (1978), which are associated with healthy human cartilage while normal functioning. The parameters $\bar{k}$ and $\phi'$ are chosen in such way that the load carrying capacity that can be sustained by the joints should be more. The pressure $\bar{p}$ and $\bar{w}$ are functions of non-dimensional parameters $C, \bar{k}$ and $\bar{H}$ and constant parameters $\phi'$ and $\frac{E'}{K}$. As the radius of the upper rigid indenter increases, it becomes relatively flat and uniform, which leads to increase in area of the film region. This wide film area avoids exit of lateral fluid and is responsible for retaining the large amount of fluid which enhances the pressure and load carrying capacity. For large radius, the spherical indenter becomes relatively flat and thus the study reduces to the squeeze film lubrication between two parallel plates.

The distribution of pressure $\bar{p}$ with rectangular coordinates $\bar{x}$ and $\bar{y}$, is given in Fig.6.2 (a-b). For $C = 0.3$, the roughness effects are more pronounced when compared with $C = 0.1$. In Fig.6.3, it is observed that, the effect of roughness is to increase the pressure distribution for increasing values of $C$. The roughness increases the pressure in the film region compared to smooth case. This is because, the presence of surface asperities reduces the fluid flow and therefore, large fluid is deposited in the film region which enhances the pressure built up. The variation of load $\bar{w}$ that can be
sustained by a joint, with roughness parameter \( C \) for different elastic parameter \( \frac{E'}{K} \), is shown in Fig.6.4. The load carrying capacity increases with increasing \( \frac{E'}{K} \) for different roughness parameters \( C \). Presence of hyaluronic acid, molecular and other molecular weight substances in the synovial fluid results in the formation of a thick dense substance on the cartilage surface during squeezing action. This leads to increase in the pressure distribution and in turn increase in the load carrying capacity. This trend is preserved for all values of \( \frac{E'}{K} \).

6.6. Conclusions

Wavelet-multigrid method is found to be accurate for the solution of Reynolds equation, which incorporates surface roughness and poroelastic nature of articular cartilage. The method is proved to be more efficient than the classical multigrid method. Comparing the rates of convergence of the two methods, it is found that Wavelet-multigrid method converges faster compared with Multigrid solution. It provides the ability to carry out calculations, focusing only in regions where it is needed. It can as well be generalized to non-uniform grids, which is the unique feature of this approach. In classical scheme we find that as the grid size decreases the condition number increases a serious disadvantage whereas in the present wavelet-Multigrid scheme the conditioning of the matrix is incorporated in the procedure itself without requiring separate analysis for it. It is observed that, the effect of roughness is to increase the pressure built up (and hence load carrying capacity) compared to
classical case. The governing equations describing complex structure of cartilage and synovial fluid are complicated because of non-linearity, and also, joints have wide range of articulating features. However, the proposed model does predict some of the salient features of bearing characteristics, which would enable bio-medical engineers in selecting suitable design parameters.
Fig. 6.1. (a). A schematic diagram of a synovial knee joint.

Fig. 6.1(b). Schematic diagram of simplified synovial knee joint.
Fig. 6.2(a). Fluid film pressure distribution for $C = 0.3$ with $\bar{k} = 7.65 \times 10^{-3}$, $\phi^f = 0.8$ and $\frac{E'}{K} = 1.0$
Fig. 6.2(b). Fluid film pressure distribution for $C = 0.1$ with $\bar{k} = 7.65 \times 10^{-5}$, $\phi' = 0.8$ and $\frac{E'}{K} = 1.0$
Fig. 6.3. Variation of pressure $\bar{p}$ with $\bar{x}$ for different values of $C$ with $k = 7.65 \times 10^{-5}$, $\phi_f = 0.8$ and $\frac{E'}{K} = 1.0$
Fig. 6.4. Variation of load $\bar{W}$ with $C$ for different values of $E'/K$ with $k = 7.65 \times 10^{-5}$, $\phi' = 0.8$. 