Chapter 5

ROMAN DOMINATION

5.1 Preamble

In Scientific American, Dec 1999, pp. 136 – 138, there was an article by Ian Stewart titled “Defend the Roman Empire” [15]. The summary of the article is as follows: The Roman Empire in 300 A.D had eight prominent places (vertices) Britain, Iberia, Gaul, Rome, North Africa, Egypt, Asia Minor and Constantinople.

![Figure 1: The Roman Empire around 300 A.C.](image)

The lines (edges) show the path between the places. Each location has to be stationed with the legions in order to safeguard them. A location is considered unsecured if no legions are stationed there and secured otherwise. An unsecured location can be secured by bringing a legion from the adjacent
location. But Emperor Constantine the Great, in the fourth century A.D decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured. Thus, two legions must be stationed at a location before one of the legions can be sent to an adjacent location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. Now the question is: "Where should, the Roman armies be placed so that the least number of armies can protect the whole Empire?"

Motivated by this article Michael A Henning and Stephen T Hedetniemi explored a new strategy of defending the Roman Empire. They believed that the Roman Empire had the potential of saving the Emperor Constantine the Great, substantial costs of maintaining legions, while still defending the Roman Empire.

In graph theoretic terminology, let $G=(V,E)$ be a graph and let $f$ be a function $f: V \rightarrow \{0, 1, 2\}$. A vertex $u$ with $f(u)=0$ is said to be undefended with respect to $f$ if it is not adjacent to a vertex with positive weight. The function $f$ is a weak Roman dominating function (WRDF) if each vertex $u$ with $f(u)=0$ is adjacent to a vertex $v$ with $f(v)>0$ such that the function $f': V \rightarrow \{0, 1, 2\}$, defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u,v\}$, has no undefended vertex. The weight of $f$ is $w(f) = \sum_{v \in V} f(v)$. 

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The weak Roman domination number, denoted $\gamma_w(G)$, is the minimum weight of a WRDF in $G$.

Meanwhile Cockayne defined a Roman dominating function (RDF) on a graph $G(V,E)$ to be a function $f: V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. For a real valued function $f: V \rightarrow R$ the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$.

The Roman Domination number (RDN) denoted $\gamma_R(G)$ is the minimum weight of an RDF in $G$; that is, $\gamma_R(G) = \min\{w(f): f \text{ is a WRDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called as $\gamma_R(G)$-function.

Stated in other words, a Roman dominating function is a coloring of the vertices of a graph with the colors $\{0,1,2\}$ such that every vertex colored '0' is adjacent to at least one vertex colored '2'.

5.2 Preliminary results and properties

In this section, we list the basic results and trivial properties without proof from [5], which are useful for the further sections. Let $G$ be a graph and $V(G)$ be its vertex set. We partition $V$ into $V_0$, $V_1$, $V_2$ that indicates the vertices labeled 0, 1 and 2 respectively. Further we take $|V_0| = n_0$, $|V_1| = n_1$, $|V_2| = n_2$. 
Proposition 5.2.1: For any graph \( G \), \( \gamma (G) \leq \gamma_R (G) \leq 2 \gamma (G) \). Further the graphs for which the equality of the upper bound holds is said to be Roman Graphs. Example: Star, Complete graph

Proposition 5.2.2: \( \gamma_R (G) = n_1 + 2n_2 \)

Proposition 5.2.3: \( \gamma (G) = \gamma_R (G) \) if and only if \( G = \bar{K}_n \)

Proposition 5.2.4: \( \gamma_R (G) = 2 \) if \( G \) is either star or a complete graph.

Proposition 5.2.5: \( \gamma_R (P_n) = \left\lceil \frac{2n}{3} \right\rceil = \gamma_R (C_n) \)

Proposition 5.2.6: The maximum degree of the subgraph induced by \( \langle V_1 \rangle \) is atmost one.

Proposition 5.2.7: No edge of \( G \) joins \( V_1 \) and \( V_2 \).

Proposition 5.2.8: Each vertex of \( V_0 \) is adjacent to at most two vertices of \( V_1 \).

Proposition 5.2.9: \( V_2 \) is a \( \gamma \) - set of \( \langle V_0 \cup V_2 \rangle \)

Proposition 5.2.10: For each \( v \in V_2 \), there exists at least two private neighbors in \( \langle V_0 \cup V_2 \rangle \).

Proposition 5.2.11: For the classes of paths \( P_n \) and cycles \( C_n \), we have

\[
\gamma_R (P_n) = \gamma_R (C_n) = \left\lceil \frac{2n}{3} \right\rceil
\]

Proposition 5.2.12: Let \( G = K_{m_1, m_2, m_3, \ldots, m_n} \) be the complete \( n \) - partite graph with \( m_1 \leq m_2 \leq \ldots \leq m_n \).

a) If \( m_1 \geq 3 \) then \( \gamma_R (G) = 4 \).
b) If \( m_1 = 2 \) then \( \gamma_R(G) = 3 \).

c) If \( m_1 = 1 \) then \( \gamma_R(G) = 2 \).

**Proposition 5.2.13:** If \( G \) is a graph of order \( n \) which contains a vertex of degree \( n - 1 \), then \( \gamma(G) = 1 \) and \( \gamma_R(G) = 2 \).

**Proposition 5.2.14:** The Roman domination number of a ladder \( G_{2,n} \) is \( \gamma_R(G_{2,n}) = n + 1 \).

**Proposition 5.2.15:** If \( G \) is any isolate free graph of order \( n \), then \( \gamma_R(G) = n \) if and only if \( n \) is even and \( G = \left( \frac{n}{2} \right) K_2 \).

### 5.3 Bounds for Roman domination number

**Theorem 5.3.1:** For any Graph \( G \) of order \( n \) and maximum degree \( \Delta \), the bounds for the Roman domination number is as follows:

\[
\left\lfloor \frac{2n}{\Delta + 1} \right\rfloor \leq \gamma_R(G) \leq n - \Delta + 1
\]

(Note: The lower bound is mentioned in [5] without proof)

**Proof:** For any Roman domination function \( f : V \rightarrow \{0, 1, 2\} \), we have

\[
\gamma_R(G) = n_1 + 2n_2 \quad \text{.................................(1)}
\]

where \( n_1 = |V_1| \), \( n_2 = |V_2| \).

Any element of \( V_2 \) defend utmost \( \Delta + 1 \) vertices, whereas the elements of \( V_1 \) defend just one. Therefore \( n_1 + (\Delta + 1) n_2 \geq n \)
Equivalently $\frac{2n_1}{\Delta+1} + 2n_2 \geq \frac{2n}{\Delta+1}$...............................(II)

For any connected Graph $G$, $\Delta \geq 1$.

Hence $\frac{2n_1}{\Delta+1} + 2n_2 \leq n_1 + 2n_2$...............................(III)

Using (I) and (II) in (III), we get

$$\frac{2n}{\Delta+1} \leq \gamma_R(G)$$

Since the Roman domination number is an integer we could write the above bound as $\left[ \frac{2n}{\Delta+1} \right] \leq \gamma_R(G)$...............................(IV)

Let $u \in V(G)$ such that $\deg(u) = \Delta$.

If $f(u) = 2$, $f(N(u)) = 0$ and $f(V(G) - N[u]) = 1$, then $f$ is clearly a Roman domination function.

Weight $W = \sum_{v \in V} f(u) = 0(\Delta) + 1(n - \Delta - 1) + 2(1)$

$$= n - \Delta + 1$$

Therefore $\gamma_R(G) \leq n - \Delta + 1$...............................(V)

From (IV) and (V), we have $\left[ \frac{2n}{\Delta+1} \right] \leq \gamma_R(G) \leq n - \Delta + 1$

Note: The very popular bounds for the usual Domination in terms of the number of vertices $n$ and the maximum degree $\Delta$ due to Berge, Walikar,
Acharya and Sampath Kumar becomes the particular case of the above theorem i.e., \[ \frac{n}{\Delta + 1} \leq \gamma(G) \leq n - \Delta \]

**Theorem 5.3.2:** Let \( G \) be a graph of order \( n \) and maximum degree \( \Delta \).

If \( \deg(u_i) = \Delta \) for \( i = 1 \) to \( k \) such that \( \bigcap_{i=1}^{k} N[u_i] = \phi \),

then \( \gamma_r(G) \leq n - k(\Delta - 1) \)

**Proof:** Let \( f(u_i) = 2 \ \forall \ i = 1 \) to \( k \) and \( f(N(u_i)) = 0 \). Assign ‘1’ to the remaining vertices of \( G \). Clearly \( f \) is a Roman dominating function and its weight is \( W = \sum_{u \in V} f(u) = 0(\Delta) + 1(n - k\Delta - k) + 2(k) \)

\[ = n - k\Delta + k \]

\[ = n - k(\Delta - 1) \]

Therefore \( \gamma_r(G) \leq n - k(\Delta - 1) \)

**5.4 Trees with small and large RDN**

In this section, we classify all the trees with small Roman domination number i.e., one, two, three four, five and large Roman domination number viz. \( n \) and \( n - 1 \) where ‘\( n \)’ is the order of a tree.

**Proposition 5.4.1:** \( \gamma_r(T) = 1 \) if and only if \( T = K_1 \)

**Proof:** Suppose \( \gamma_r(T) = 1 \), then we first claim that \( V_2 = \phi \). Suppose \( V_2 \neq \phi \), then \( \gamma_r(T) \geq 2 \), which is a contradiction to the fact that \( \gamma_r(T) = 1 \). Therefore
$V_2 = \phi$ and hence $V_0 = \phi$, and naturally $n_1 = 1$ implies that $T = K_1$. The converse is obvious.

**Proposition 5.4.2:** $\gamma_r(T) = 2$ if and only if $T = K_{1,n-1}$

**Proof:** Suppose $\gamma_r(T) = 2$, then we consider the following two cases:

**Case 1:** $V_2$ is empty

If $V_2 = \phi$ then $V_0 = \phi$, Therefore $n_1 = 2$. Since $T$ is a connected graph, we have $T = K_2$.

**Case 2:** $V_2$ is not empty

Suppose $V_2 \neq \phi$, then there exists exactly one vertex in $V_2$, otherwise $\gamma_r(T) \geq 4$, which is a contradiction to the fact that $\gamma_r(T) = 2$. This implies that $V_1$ is empty and all the vertices of $V_0$ are adjacent to a vertex of $V_2$. Hence $T = K_{1,n-1}$.

The converse is obvious.

**Proposition 5.4.3:** $\gamma_a(T) = 3$ if and only if $T = S_{1,m}$

**Proof:** Suppose $\gamma_a(T) = 3$, then we claim that $n_2 \leq 1$. Suppose $n_2 \geq 2$ then $\gamma_a(T) \geq 4$, which is a contradiction to the fact that $\gamma_a(T) = 3$. Now consider the following two cases:

**Case 1:** $V_2$ is empty

If $V_2 = \phi$ then it implies $V_0 = \phi$. Therefore $n_1 = 3$. Since tree is connected, $V_i$ is not independent. More over $\Delta(\langle V_i \rangle) \leq 1$ implies that this case is not possible.
**Case 2:** $V_2$ is not empty

That is $n_2 = 1$, obviously implies that $n_1 = 1$, because $\gamma_R(T) = 3$. Therefore all the vertices of $V_0$ are adjacent a vertex of $V_2$ and only one vertex of $V_0$ is adjacent to the lone vertex of $V_1$. Therefore $T = S_{1,n}$.

The converse is obvious.

**Proposition 5.4.4:** $\gamma_R(T) = 4$ if and only if $T$ is one of the following trees.

![Fig-5.1](image1)

![Fig-5.2](image2)
Proof: It is not difficult to see that, if $T$ is one of the above tree, then $\gamma_s(T) = 4$. Conversely, let $\gamma_s(T) = 4$, then we claim that $n_2 \leq 2$. Suppose $n_2 \geq 3$ then $\gamma_s(T) \geq 6$, which is a contradiction to the fact that $\gamma_s(T) = 4$. Now consider the following three cases:
Case 1: $V_2$ is empty

$V_2 = \emptyset$ naturally implies $V_0 = \emptyset$, Therefore $n_1 = 4$. Since tree is connected, $V_1$ is not independent. Moreover $\Delta (\langle V_1 \rangle) \leq 1$ implies that this case is not possible.

Case 2: $V_2$ contains exactly one vertex

If $n_2 = 1$, then obviously it implies that $n_1 = 2$, because $\gamma_h(T) = 4$. Naturally all the vertices of $V_0$ are adjacent to a vertex of $V_2$ and there arises two subcases.

Subcase 2.1: $V_1$ is independent.

If $V_1$ is independent, then the two vertices of $V_1$ are adjacent to two vertices of $V_0$. This gives the tree $T_1$ as shown in the figure 5.1.

Subcase 2.2: $V_1$ is not independent.

If $V_1$ is not independent, then $\langle V_1 \rangle$ is $K_2$. Therefore one of the vertices of $V_1$ is adjacent to the vertex of $V_0$ and the other is a pendant vertex. This gives the tree $T_2$ of the figure 5.2.

Case 3: $V_2$ contains two vertices

That is if $n_2 = 2$, then $n_1 = 0$, because $\gamma_h(T) = 4$. Therefore all the vertices of $V_0$ are adjacent to two vertices of $V_2$ and at most one vertex of $V_0$ is adjacent to both the vertices of $V_2$. This gives the trees $T_3$ and $T_4$ of the figures 5.3 and 5.4 respectively.

Hence the proposition.
Proposition 5.4.5: $\gamma_r(T) = 5$ if and only if $T$ is one of the following trees.

$T_5$: Fig-5.5

$T_6$: Fig-5.6
Proof: It is not difficult to see that if $T$ is one of the above tree, then $\gamma_R(T) = 5$. Conversely let $\gamma_R(T) = 5$, then we claim that $n_2 \leq 2$. Suppose $n_2 \geq 3$ then $\gamma_R(T) \geq 6$, which is a contradiction to the fact that $\gamma_R(T) = 5$. Now consider the following three cases:

**Case 1:** $V_2$ is empty

That is, if $V_2 = \emptyset$ then $V_0 = \emptyset$, Therefore $n_2 = 5$. Since tree is connected, $V_1$ is not independent. More over $\Delta(V_1) \leq 1$ implies that this case is not possible.

**Case 2:** $V_2$ consists of exactly one vertex

That is, if $n_2 = 1$, then it implies that $n_1 = 3$, because $\gamma_R(T) = 5$. Naturally all the vertices of $V_0$ are adjacent to a vertex of $V_2$ and hence there arises two subcases.
Subcase 1: \( V_i \) is independent.

If \( V_i \) is independent, then the three vertices of \( V_i \) are adjacent to three vertices of \( V_0 \), which can be found in \( T_s \).

Subcase 2: \( V_i \) is not independent.

If \( V_i \) is not independent, then \( (V_i) \) consists of \( K_i \cup K_2 \). Therefore one end of \( K_2 \) of \( (V_i) \) is adjacent to any one of the vertex of \( V_0 \) and the other is a pendant vertex adjacent to any other vertex of \( V_0 \). This is the tree \( T_6 \) as shown in the figure 5.6.

Case 3: \( V_2 \) consists of two vertices

That is, if \( n_2 = 2 \), then \( n_1 = 1 \), because \( \gamma_r(T) = 5 \). Therefore all the vertices of \( V_0 \) are adjacent to two vertices of \( V_2 \) and at most one vertex of \( V_0 \) is adjacent to both the vertices of \( V_2 \). Along with this, there is exactly one vertex of \( V_0 \) that is adjacent to the lone vertex of \( V_1 \). The pictures of these is as depicted in \( T_7 \), \( T_8 \) and \( T_9 \).

Hence the proposition.

Proposition 5.4.6: \( \gamma_r(T) = n \) if and only if \( T = K_1 \) or \( T = K_2 \)

Proof: If \( T = K_1 \) or \( T = K_2 \), then the result is obvious. Conversely let \( \gamma_r(T) = n \). From theorem 5.3.1, we have \( \gamma_r(G) \leq n - \Delta + 1 \). Therefore we get \( \Delta \leq 1 \). Hence \( T = K_1 \) or \( T = K_2 \).
Proposition 5.4.7: \( \gamma_r(T) = n - 1 \) if and only if \( T = P_3, P_4, P_5 \)

Proof: From Theorem 5.3.1, we have \( \gamma_r(G) \leq n - \Delta + 1 \)

Since \( \gamma_r(T) = n - 1 \), the above equation becomes \( n - 1 \leq n - \Delta + 1 \)

This implies \( \Delta \leq 2 \). A tree with maximum degree two is a path.

But \( \gamma_r(P_n) = \left\lceil \frac{2n}{3} \right\rceil \Rightarrow n - 1 = \left\lceil \frac{2n}{3} \right\rceil \)

This is true if and only if \( 3 \leq n \leq 5 \)

Hence \( T = P_3, P_4, P_5 \).

The Converse can be easily verified.

5.5 Roman Domination of some special class of Graphs

During the study of Chemical graphs and its Weiner number, the Yugoslavian Chemist Ivan Gutman introduced the concept of Thorn graphs. This idea was further extended to the broader concept of generalized Thorny graphs by Danail Bonchev and Douglas J Klein of USA. This class of graphs gain importance in Spectral theory as it represents the structural formula of aliphatic and aromatic hydrocarbons.
Thorn Rod

A Thorn rod is a graph $P_{p,t}$ which includes a linear chain (termed as a rod) of $p$ vertices and degree $t$—terminal vertices at each of the two rod ends.

Example: $p = 4$ and $t = 3$ in Fig 5.10 and $p = 5$ and $t = 4$ in Fig 5.11.

Fig - 5.10

Fig - 5.11
Theorem 5.5.1: For any thorny rod $P_{p,t}$,

$$
\gamma_R(P_{p,t}) = \begin{cases}
\left\lfloor \frac{2p}{3} \right\rfloor & \text{for } t = 1 \\
\left\lfloor \frac{2(p+2)}{3} \right\rfloor & \text{for } t = 2 \\
\end{cases}
$$

and for $t \geq 3$

$$
\gamma_R(P_{p,t}) = \begin{cases}
2k+3 & \text{for } p = 3k+2 \\
2(k+1) & \text{otherwise}
\end{cases}
$$

Proof: If $t = 1$, then the thorny rod reduces to a path of $p$ vertices.

Hence $\gamma_R(P_{p,1}) = \gamma_R(P_p) = \left\lfloor \frac{2p}{3} \right\rfloor$.

If $t = 2$, again we have a path of order $p+2$.

Hence $\gamma_R(P_{p,2}) = \gamma_R(P_{p+2}) = \left\lfloor \frac{2(p+2)}{3} \right\rfloor$.

For $t \geq 3$, label the linear chain as $u_1, u_2, \ldots, u_n$, the left siblings as $l_1, l_2, \ldots, l_{t-1}$ and right siblings as $r_1, r_2, \ldots, r_{t-1}$.

Claim 1: $f(u_i) \neq 0$ and $f(u_n) \neq 0$.

If possible let $f(u_i) = 0 = f(u_n)$ then

i) The siblings cannot be assigned ‘0’, because none of them are adjacent to a vertex labeled ‘2’.
ii) The siblings cannot be assigned ‘1’ because each vertex of $V_0$ is adjacent to utmost two vertices of $V_1$.

iii) The siblings cannot be assigned ‘2’ because each vertex of $V_2$ has at least two private neighbors in $\langle V_0 \cup V_2 \rangle$.

Therefore neither $f(u_i)$ nor $f(u_n)$ is zero.

**Claim 2:** $f(u_i) \neq 1$ and $f(u_n) \neq 1$

If possible let $f(u_i) = 1 = f(u_n)$ then

i) The siblings cannot be assigned ‘0’, because none of them are adjacent to a vertex labeled ‘2’.

ii) The siblings cannot be assigned ‘1’ because the maximum degree induced by $\langle V_1 \rangle$ is utmost one.

iii) The siblings cannot be assigned ‘2’ because there is no edge joining the vertices of $V_1$ and $V_2$.

Therefore neither $f(u_i)$ nor $f(u_n)$ can be assigned one.

From the above two claims, we declare that $f(u_i) = f(u_n) = 2$.

Hence $f(u_2) = f(u_n) = f(1) = f(r) = 0$. The remaining part of the graph is a path of $p - 4$ vertices.

Therefore $\gamma_k'(P_{p,1}) = 4 + \left\lfloor \frac{2(p-4)}{3} \right\rfloor$.

Case 1: If $P = 3k$ then $\gamma_k'(P_{p,1}) = 4 + \left\lfloor \frac{2(3k-4)}{3} \right\rfloor = 2(k+1)$
Case 2: If $P = 3k + 1$ then $\gamma_k(P_{p,1}) = 4 + \left\lceil \frac{2(3k + 1 - 4)}{3} \right\rceil = 2(k + 1)

Case 3: If $P = 3k + 2$ then $\gamma_k(P_{p,1}) = 4 + \left\lceil \frac{2(3k + 2 - 4)}{3} \right\rceil = 2k + 3$

Hence the theorem.

Thorn Star

Thorn Stars are the graphs obtained from a k-arm star by attaching $t-1$ terminal vertices to each of the star arms and are denoted as $S_{k,t}$.

Example 1: $S_{4,4}$

Fig. 5.12
Example 2: $S_{3,4}$

Note: If $K = 2$, then thorn star becomes a thorn rod.

**Theorem 5.5.2:** For any thorn star $S_{k,t}$ and $k > 2$, we have

\[
\gamma_{R}(S_{k,t}) = \begin{cases} 
2 & \text{for } t < 2 \\
 k+2 & \text{for } t = 2 \\
 2k & \text{for } t > 2
\end{cases}
\]

**Proof:** If $t = 1$, then $S_{k,1}$ is a star $K_{1,k}$. Therefore $\gamma_{R}(S_{k,1}) = 2$.

If $t = 2$ then $S_{k,2}$ is a healthy spider. A healthy spider $S_{k,2}$ consists of $2k+1$ vertices. $k$ vertices are of degree one, $k$ vertices of degree two and a vertex of
degree ‘$k$’. Label the vertex of degree $k$ as ‘$u$’, the vertices of degree two as $v_1, v_2, \ldots, v_k$ and their corresponding pendant vertices as $w_1, w_2, \ldots, w_k$.

**Case 1:** If $f(u) = 0$, then $\exists i$ such that $f(v_i) = 2$ and $f(w_i) = 0$.

For any other $j \neq i$, $f(v_j) = 0 \Rightarrow f(w_j) = 2$

$f(v_j) = 1 \Rightarrow f(w_j) = 1$

$f(v_j) = 2 \Rightarrow f(w_j) = 0$.

Therefore $\sum_{u \in V} f(u) = 2k$

**Case 2:** If $f(u) = 1$, then

$f(v_i) \neq 2$ because there is no edge connecting the vertices of $V_1$ and $V_2$.

Further $f(w_i) \neq 2$ because each vertex of $V_2$ should have at least two private neighbors in $\langle V_0 \cup V_2 \rangle$.

Therefore $f(v_i) = 1 = f(w_i)$.

Hence $\sum_{u \in V} f(u) = 2k + 1$.

**Case 3:** $f(u) = 2 \Rightarrow f(v_i) = 0 \Rightarrow f(w_i) = 1$.

Hence $\sum_{u \in V} f(u) = k + 2$

From the above three cases $\gamma_k(S_{k,2}) = \min \left\{ \sum_{u \in V} f(u) \right\} = k + 2$.

If $t > 2$ then $S_{k,t}$ is of order $1 + tk$. One vertex is of degree $k$, $k$ vertices are of degree $t$ and the remaining $k(t-1)$ are pendant vertices. Label the vertex of
degree \( k \) as ‘\( w_0 \)’, the vertices of degree \( t \) as \( u_1, u_2, \ldots u_k \). The pendant vertices of \( u_i \) is labeled as \( u_{i,j} \) for \( j = 1 \) to \( t - 1 \).

**Claim:** \( f(u_i) \neq 1 \)

Suppose \( f(u_i) = 1 \) then \( f(u_{i,j}) \neq 0 \) because no adjacent vertex of it is labeled ‘2’. \( f(u_{i,j}) \neq 2 \) as there is no edge connecting the vertices of \( V_1 \) and \( V_2 \). Further \( f(u_{i,j}) \neq 1 \) because \( \Delta(V_1) \leq 1 \). Therefore \( f(u_i) \neq 1 \). Hence \( f(u_i) = 0 \) or 2.

**Case 1:** If \( f(u_i) = 0 \) then \( f(u_{i,j}) = 1 \) and \( f(w_0) = 2 \). Hence \( \sum_{u \in V} f(u) = 2 + k(t - 1) \).

**Case 2:** If \( f(u_i) = 2 \) then \( f(u_{i,j}) = 0 = f(w_0) \) and \( \sum_{u \in V} f(u) = 2k \).

**Case 3:** If \( f(u_i) \) is combination of 0’s and 2’s then it is easily verifiable that \( \sum_{u \in V} f(u) > 2k \).

From the above three cases \( \gamma_R(S_k, t) = \min \left\{ \sum_{u \in V} f(u) \right\} = 2K \) for \( t > 2 \).

Hence

\[
\gamma_R(S_k, t) = \begin{cases} 
2 & \text{for } t < 2 \\
k + 2 & \text{for } t = 2 \\
2k & \text{for } t > 2 
\end{cases}
\]
**Thorn Ring**

A $t$-thorny ring has a simple cycle as the parent, and $t - 2$ thorns at each cycle vertex.

Example 1: A thorn ring with $n = 3$ and $t = 3$

![Fig-5.14]

Example 2: A thorn ring with $n = 4$ and $t = 5$

![Fig-5.15]
Theorem 5.5.3: \( \gamma_R^* (C_n^+) = \begin{cases} 
4k & \text{if } n = 3k \\
2(2k + 1) & \text{if } n = 3k + 1 \\
4k + 3 & \text{if } n = 3k + 2 
\end{cases} \)

Proof: \( C_n^+ \) consists of \( 2n \) vertices. 'n' vertices on cycle are of degree three and remaining \( n \) vertices are pendant vertices. Label the vertices on cycle as \( u_1, u_2, \ldots, u_n \) in order and its corresponding pendant vertices as \( v_1, v_2, \ldots, v_n \).

The possible assignments for vertices are \( f(u_i) = 0 \) or \( 1 \) or \( 2 \).

Similarly \( f(v_i) = 0 \) or \( 1 \) or \( 2 \).

Thereby for any pair \( (u_i, v_j) \), there are nine possible assignments.

They are (0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).

i) The pairs (1,2) and (2,1) will not exist as there is no edge that joins \( V_1 \) and \( V_2 \).

ii) \( f(v_i) \neq 2 \), because for each \( v \in V_2 \), there exists at least two private neighbors in \( (V_0 \cup V_2) \).

iii) \( f(v_i) = 0 \Rightarrow f(u_i) = 2 \). Hence (0,0) and (1,0) is not possible.

Hence the nine possibilities has reduced to three, they are (0,1), (1,1) and (2,0). In other words \( f(u_i) = 0 \Rightarrow f(v_i) = 1 \), \( f(u_i) = 1 \Rightarrow f(v_i) = 1 \), \( f(u_i) = 2 \Rightarrow f(v_i) = 0 \). This implies that the maximum number of zero's on cycle will reduce the total weight of the Roman Domination function.
Maximum number of zero's that could be inducted on the cycle of 'n' vertices is \[ \left\lfloor \frac{2n}{3} \right\rfloor \] such that the cycle retains the property of \( \gamma_R \) function.

**Case 1:** If \( n = 3k \) then the maximum of \( \left\lfloor \frac{2(3k)}{3} \right\rfloor = 2k \) vertices on the cycle can be assigned zeros and its corresponding pendant vertices are assigned one. The remaining 'k' vertices are assigned two and its pendant vertices are assigned zero. Hence \( \gamma_R(C_{3k}^*) = 0(2k) + 2(k) + 1(2k) + 0(k) = 4k \).

**Case 2:** If \( n = 3k+1 \) then the maximum of \( \left\lfloor \frac{2(3k+1)}{3} \right\rfloor = 2k \) vertices on the cycle can be assigned zeros and its corresponding vertices are assigned one. The remaining \( k+1 \) vertices on cycle are assigned two and its corresponding pendant vertices are assigned zero.

Hence \( \gamma_R(C_{3k+1}^*) = 0(2k) + 2(k+1) + 1(2k) + 0(k+1) = 2(2k+1) \).

**Case 3:** If \( n = 3k+2 \) then the maximum of \( \left\lfloor \frac{2(3k+2)}{3} \right\rfloor = 2k+1 \) vertices on the cycle can be assigned zeros and its corresponding pendant vertices are assigned one. The remaining \( k+1 \) vertices on cycle are assigned two and its corresponding pendant vertices are assigned zero.

Hence \( \gamma_R(C_{3k+2}^*) = 0(2k+1) + 2(k+1) + 1(2k+1) + 0(k+1) = 4k+3 \).

Therefore \( \gamma_R(C_n^*) = \begin{cases} 4k & \text{if } n = 3k \\ 2(2k+1) & \text{if } n = 3k+1 \\ 4k+3 & \text{if } n = 3k+2 \end{cases} \)
Theorem 5.5.4: $\gamma_R(C_n^\gamma) = 2n$ for $t \geq 4$

Proof: Thorny ring $C_n^\gamma$ has $n(t-1)$ vertices. 'n' vertices on the cycle are each of degree $t$ and there are $n(t-2)$ pendant vertices. Label the vertices on the cycle as $u_1, u_2, \ldots, u_n$ and the siblings of $u_i$ are labeled as $v_{i,j}$ for $j = 1$ to $t-2$.

Note that
i) $f(v_{i,j}) = 0$ only if $f(u_i) = 2$.

ii) $f(v_{i,j}) \neq 2$ because for each $v \in V_2$, there exists at least two private neighbors in $(V_1 \cup V_2)$.

iii) If $f(u_i) = 2$ then $f(v_{i,j}) \neq 1$, because there is no edge joining $V_1$ and $V_2$.

iv) If $f(u_i) = 1$ then $f(v_{i,j}) \neq 0, f(v_{i,j}) \neq 2$. Hence $f(v_{i,j}) = 1$ $\forall j$. But $(V_1)$ has maximum degree 1, therefore $f(u_i) \neq 1$.

The above statements reveal that there are only two possible ways of assignment. a) $f(u_i) = 0, f(v_{i,j}) = 1$

b) $f(u_i) = 2, f(v_{i,j}) = 0$.

Hence the minimum weight of these two gives the roman domination number.

i.e. $\gamma_R(C_n^\gamma) = 2n$ for $t \geq 4$.

Now we discuss another set of special graphs that are most useful in Computer Applications.
Acyclic connected graph is a tree. If a tree consists of a vertex of degree two and the remaining vertices is of either degree one or three then it is said to be a binary tree. The vertex of degree two is called as a root. It is also called as parent and the two neighbors are called as children of level one. These vertices act as parent if they have two more neighbors each and those vertices are called as children of level two and so on. Therefore there are vertices which act both as parent as well as child. The vertices which do not have children are called as siblings. In other words, the pendant vertices of a binary tree are called as siblings. Consider a binary tree of \( l \) levels namely \( L_0, L_1, L_2, \ldots, L_{l-1} \). If there exists \( 2^{l-1} \) siblings in the level \( L_{l-1} \) of a binary tree, then the tree is called as complete binary tree. In other words, the binary tree is said to be complete binary tree if and only if the siblings are found only in final level.

Example:

![Diagram of a binary tree]

Fig – 5.16
Lemma 5.5.5: The complete binary tree \((CBT)\), for \(l \geq 3\) is not a \(\gamma_R\) function, if any of the vertex of the level \(L_{i-2}\) is assigned '1'.

Proof: Suppose a vertex of the level \(L_{i-2}\) is assigned '1', then the siblings cannot receive '0' as it violates the basic definition of \(\gamma_R\) function. They cannot receive '1' because \(\Delta(V_i) \leq 1\). Further they also cannot receive '2' because an edge between \(V_1\) and \(V_2\) is not allowed. In general '1' cannot be assigned to any of the vertex of the level \(L_{i-2}\) in order to make the \((CBT)\), a \(\gamma_R\) function.

Lemma 5.5.6: In a complete binary tree \((CBT)_l\), \(\gamma_R(CBT)_l \leq \frac{2^{i+2}}{7}\).

Proof: Consider a complete binary tree \((CBT)_l\), of \(l\) levels, namely \(L_0, L_1, L_2, \ldots \ldots L_{j-1}\). Assign '2' to all the vertices of the level \(L_{i-2}, L_{i-5}, L_{i-8}, \ldots\) and '0' to the remaining vertices. This is clearly a Roman domination function with the weight

\[
2 \left[2^{i-2} + 2^{i-5} + 2^{i-8} + \ldots \ldots \right] = \frac{4}{7} \cdot 2^i = \frac{2^{i+2}}{7}.
\]

Hence \(\gamma_R(CBT)_l \leq \frac{2^{i+2}}{7}\).

Lemma 5.5.7: A complete binary tree \((CBT)_l\), of level \(l\) can't be a \(\gamma_R\) function, if the vertices of the level \(L_{i-2}\) are assigned zero.
Proof: If all the vertices of the level $L_{i-2}$ are assigned zero, then invariably any one of the three neighbors should be assigned ‘2’. In applying the principle of minimal assignment, invariably siblings get the value ‘1’ and the vertices of the level $L_{i-3}$ will get ‘2’ throughout. The weight of this part of the complete binary tree, i.e., the weight of the levels $L_{i-1}$, $L_{i-2}$ and $L_{i-3}$ amounts to

$$1(2^{i-1}) + 0(2^{i-2}) + 2(2^{i-3}) = \frac{3}{4}2^i.$$ 

This implies that $\gamma_R(CBT_i) \geq \frac{3}{4}2^i$.

But from Lemma 5.5.6, we have $\gamma_R(CBT_i) \leq \frac{2^{i+2}}{7}$, which is a contradiction. Therefore a complete binary tree $(CBT)_l$ of level $l$ can't be a $\gamma_R$ - function, if the vertices of the level $L_{i-2}$ are assigned zero.

Theorem 5.5.8: The Roman domination number of a complete binary tree is as follows:

$$\gamma_R(CBT)_l = \begin{cases} 
\frac{4}{7}(2^l - 1) & \text{if } l = 3k \\
\frac{2}{7}(2^{l+1} - 1) & \text{if } l = 3k - 1 \\
\frac{8}{7}(2^{l-1} + 1) & \text{if } l = 3k - 2 
\end{cases}$$

Proof: Name the $l$ levels of a complete binary tree as $L_0, L_1, L_2, \ldots, L_{l-1}$. By the previous lemma it is evident that the vertices of the level $L_{i-2}$ should be assigned only ‘2’. Hence the vertices of the levels $L_{i-3}$ and $L_{i-1}$ will receive ‘0’. Deleting these levels we again get the complete binary tree of $l-3$ levels. Applying the same procedure, we assign ‘2’ to the vertices of the level
and '0' to the vertices of the level \( L_{i-4} \) and \( L_{i-6} \). Continuing in this manner, we finally end up assigning '2' to the levels \( L_{i-2}, L_{i-5}, L_{i-8}, \ldots \) and '0' to the vertices of the intermediate levels.

**Case 1: \( l = 3k \)**

It is easily verifiable that the vertices of the level \( L_1 \) will receive '2' and obviously the root can be assigned '0'. Therefore the total weight is

\[
2 \left[ 2^{l-2} + 2^{l-5} + 2^{l-8} + \ldots + 2^0 \right]
\]

\[
= 2 \left[ 2^{l-2} + 2^{l-5} + 2^{l-8} + \ldots \right] \frac{l}{3} \text{ terms}
\]

\[
= \frac{4}{7} \left( 2^l - 1 \right)
\]

**Case 2: \( l = 3k - 1 \)**

If the vertices of the levels \( L_{i-2}, L_{i-5}, L_{i-8}, \ldots \) are assigned '2' then obviously the root will also receive '2'. Therefore the total weight is

\[
2 \left[ 2^{l-2} + 2^{l-5} + 2^{l-8} + \ldots + 2^0 \right]
\]

\[
= 2 \left[ 2^{l-2} + 2^{l-5} + 2^{l-8} + \ldots \right] \frac{l}{3} \text{ terms}
\]

\[
= \frac{2}{7} \left( 2^{l+1} - 1 \right)
\]

**Case 3: \( l = 3k - 2 \)**

If the vertices of the levels \( L_{i-2}, L_{i-5}, L_{i-8}, \ldots \) are assigned '2' then the vertices of the intermediate levels are assigned '0'. It is easy to verify that the
only remaining root must be assigned ‘1’. Therefore the total weight is

\[ 2 \left( 2^{t-2} + 2^{t-5} + 2^{t-8} + \ldots + 2^{t+1} \right) + 1 \]

\[ = 2 \left( 2^{t-2} + 2^{t-5} + 2^{t-8} + \ldots + 2^{t+1} \right) \frac{1}{3} \text{ terms} + 1 \]

\[ = \frac{8}{7} \left( 2^{t-1} - 1 \right) + 1 \]

5.6 Graphs with given roman domination number ‘k’

A triplet \( G(n, m, k) \) is said to be realizable, if there exists a graph \( G \) of order \( n \), size \( m \) and the roman domination number \( \gamma_R(G) = k \).

**Proposition 5.6.1:** It is possible to construct a tree of any given Roman Domination number.

**Proof:** Consider a graph \( P_n \) (a path of order \( n \)). Define a function \( f: P_n \to \mathbb{N} \) such that \( f(n) = \left\lceil \frac{2n}{3} \right\rceil \). Clearly the function defines the Roman domination number of the Path. Moreover \( f \) is onto. Therefore for every given Roman Domination number, we could construct a Path and hence a tree.

**Theorem 5.6.2:** Let \( G \) be a connected graph of order \( n \), size \( m \) and \( \gamma_R(G) = k \), then \( m \leq \left\lceil \frac{n(n-k+1)}{2} \right\rceil \).
Proof: Consider the inequality of the theorem 5.3.1 i.e.,
\[ \gamma_R(G) \leq n - \Delta + 1. \]
For the graphs with roman domination number \( \gamma_R(G) = k \), we have \( k \leq n - \Delta + 1 \) or
\[ \Delta \leq n - k + 1 \] 
.................................(I)

From Handshaking property, we have \( \sum d(v_i) = 2m \) ......................(II)
But for any graph, we have \( \sum d(v_i) \leq n\Delta \)

\[ \leq n (n - k + 1) \quad [\text{from (I)}] \]
\[ 2m \leq n (n - k + 1) \quad [\text{from (II)}] \]
\[ m \leq \frac{n (n - k + 1)}{2} \]

Since edge is an integer, we could write the above inequality as
\[ m \leq \left[ \frac{n (n - k + 1)}{2} \right] \]

In other words, a triple \( G(n, m, k) \) is realizable for any connected graph \( G \),
if the following bound satisfies \( n - 1 \leq m \leq \left[ \frac{n(n - k + 1)}{2} \right] \).

Now instead of triplet \( G(n, m, k) \) let us analyse a pair \( G(n, k) \) and
\( G(m, k) \) separately.

a) “Can we construct a graph with given roman domination number
\( \gamma_R(G) = k \) and given order \( n \)?”
b) "Can we construct a graph with given roman domination number
\( \gamma_R(G) = k \) and given size \( m \)?"

In particular, if \( k = 2 \), then we get a class of graphs with \( \gamma_R(G) = 2 \).

Example: Star, Complete graph

It is easily verifiable that, we can construct a graph with given roman domination number \( \gamma_R(G) = k \) of any given order \( n \geq 2 \) and similarly, we can construct a graph with given roman domination number \( \gamma_R(G) = k \) and any given size \( m \). But the realizability of the connected graph \( G(n, m, 2) \) can be obtained by taking \( k = 2 \) in the relation \( n - 1 \leq m \leq \frac{n(n-k+1)}{2} \).

Therefore the bounds for the number of edges in a connected graph with \( \gamma_R(G) = k \) is \( n - 1 \leq m \leq \frac{n(n-1)}{2} \). Now we proceed to show the possibility of construction of the graphs of roman domination number \( \gamma_R(G) = k \), for each value of 'm' in the above bound.

**Construction of the connected graphs with roman domination number 2**

Consider the complete graph \( K_n \) for the fixed order \( n \). Here after we refer it as \( G_0 \). This has \( n \) vertices and \( \frac{n(n-1)}{2} \) edges.

Label the vertices as \( v_1, v_2, v_3, \ldots, v_n \) and edges as \( e_1, e_2, e_3, \ldots, e_{n-1}, e_n, e_{n+1}, \ldots, e_{\frac{n(n-1)}{2}} \) such that
\( e_i = (v, v_n) \) for all \( i = 1 \) to \( n-1 \). Hence the edges \( e_1, e_2, \ldots, e_{n-1} \) are incident to \( v_n \), while the edges \( e_n, e_{n+1}, \ldots, e_{\frac{n(n-1)}{2}} \) are not incident to \( v_n \) and they are \( \frac{n^2 - 3n + 2}{2} \) in number.

Define a function \( f : V \rightarrow \{0, 1, 2\} \) such that \( f(V) = 2 \) and \( f(V_i) = 0 \) \( \forall i < n \). Clearly \( f \) is a Roman domination function with \( G_0 \) being the graph with \( \gamma_R(G) = 2 \).

Define the graph \( G_{j+1} = G_j - e_{n+j} \) for \( j = 0 \) to \( \frac{n(n-3)}{2} \). As \( j \) increases, the edge not adjacent to \( v_n \) is removed and hence the roman domination number of the graph \( G_{j+1} \) remains to be 2. The final value of \( j = \frac{n(n-3)}{2} \) results in the graph \( G_{\frac{n^2 - 3n + 2}{2}} \), which is a star. Therefore, we can construct a graph with \( \gamma_R(G) = 2 \) for each value of \( m \) in \( \left( (n-1), \frac{n(n-1)}{2} \right) \) and there are \( \frac{n^2 - 3n + 4}{2} \) number of such values of \( m \).

**Construction of the connected graphs with roman domination number 4**

Consider the graph \( G_0 \) of fixed order \( n \) such that \( \delta(G_0) = n - 3 = \Delta(G) \).

Therefore \( G_0 \) have \( n \) vertices and \( \frac{n(n-3)}{2} \) edges.
Label the vertices as $v_1, v_2, v_3, \ldots, v_n$ and edges as $e_1, e_2, e_3, \ldots, e_{n-1}, e_n, e_{n+1}, \ldots, e_{n(n-3)/2}$ such that

$e_i = (v_i, v_{i+1})$, $e_{n-2} = (v_1, v_{n-2})$, $e_{n-1} = (v_3, v_n)$ for all $i = 1$ to $n-3$. In simple

$v_1$ and $v_n$ are not adjacent and $v_i$ is adjacent to all other vertices except $v_{i-1}$ and $v_{i+1}$ for all $i = 2$ to $n-1$.

Define a function $f: V \rightarrow \{0, 1, 2\}$ such that $f(V_{n-1}) = 2$, $f(V_i) = 0 \ \forall \ i = 1$ to $n-3$ and $f(V_{n-2}) = 1 = f(V_n)$. Clearly $f$ is a

Roman domination function with $G_0$ being a graph with $\gamma_R(G) = 4$.

Define the graph $G_{j+1} = G_j - e_{n+j}$ for $j = 0$ to $n(n-5)/2$. As $j$

increases, the redundant edges is getting removed and hence the graph $G_{j+1}$

remains to be a graph with $\gamma_R(G) = 4$. The final value of $j = n(n-5)/2$

results in the graph $G_{n^2-5n+2}$, which is a tree. Therefore, we can construct a

graph with $\gamma_R(G) = 4$, for each value of $m$ in $\left( n-1, \frac{n(n-3)}{2} \right)$ and

there are $\frac{n^2 - 5n + 4}{2}$ number of such values of $m$.

5.7 Open problems

1. Characterize the graphs that attains the bounds in 5.3.1

2. Construct all possible graphs with $\gamma_R(G) = 3$ that satisfies 4.7.3
References

[1] Alon N, Spencer J H,  
*The Probabilistic Method*,  

[2] Aris, Paolo, Konrad, Kathleen, David and Peter,  
Server Placements, Roman Domination & other Dominating Set variants  
*Preprint*.

Dominating sets in chordal graphs,  
*Research report CS-80-34*, University of waterloo, 1980.

[4] Booth K S and Johnson J H,  
Dominating sets in Chordal graphs,  

[5] Cockayne E J, Paul A Dreyer, Stephen Hedetniemi, Sandra Hedetniemi,  
Roman Domination in graphs,  

The Algorithmic complexity of Roman domination,  
*Preprint*.

Protection of a graph,  
*Preprint*.

[8] Danail Bonchev, Douglas J Klein,  
On the Wiener number of Thorn trees, stars, rings and rods,  
*CCACAA 75*(2) 613 – 620 (2002)

[9] Dreyer P A,  
Defending the Roman Empire,  

[10] Farber M,  
Independent domination in chordal graphs,  

[11] Frank Harary,
Graph Theory,
Addison-Wesley, Reading Mass (1972).

[12] Haynes, Hedetniemi S T, Slater P J,
Fundamentals of Domination in Graphs,

[13] Hedetniemi S T,
Roman domination in graphs II,
Slides and notes from 9th Quad. Int. Con. On Graph theory, June 2000.

[14] Michael A Henning,
A Characterization of Roman trees,

Defending the Roman Empire – A new strategy,

[16] Ian Stewart,
Defend the Roman Empire!
Scientific American, (Dec 1999).

[17] Ivan Gutman,

[18] Narsingh Deo,
Graph theory with Applications to Engg. And comp. Sc.,
Prentice-Hall of India (1999)

[19] ReVelle C S and Rosing K E,
Defends imperium romanum: a classical problem in military strategy,

[20] Walikar H B and Acharya B D,
Domination critical graphs,

[21] Walikar H B and Sudershan Reddy L,
Roman domination in Special class of graphs,
Preprint