CHAPTER V

Design of Equiripple FIR Higher-Order Digital Differentiators by Modified MEA and WLS Methods.

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5.1 Introduction

Designing of FIR higher-order DD is a concept that is being pursued in the last decade[1-29]. They have been designed as FIR linear-phase filters that approximate the ideal frequency characteristics, which vary as a power of frequency with frequency. In many signal-processing systems it is often necessary to obtain first and the higher derivatives of signals for analysis and further use [1,2]. In the study of velocity and acceleration in human locomotion [3], for analyzing radar signals [4] and also in sonar echo returns it is required to find first and higher derivatives [5]. The higher-order DD's are very useful for calculation of geometric moment's [6] and for biological signal processing [1].

Conventionally, well known Parks-McClellan program [9] is used for the design of first order DD's. Later Rahenkamp and Vijaya Kumar [10] have modified this program for designing higher-order DD's. First order DD's with wide bandwidths have been proposed by Rabiner and Schafer [7], where the algorithm of McClellan et al. [9] has been used to calculate the weighting coefficients of the DD. Vaidyanathan and Nguyen [11] have introduced the eigenfilter method for designing linear-phase FIR digital filters. The unique advantage of this over the Parks-McClellan algorithm is that it is enough to incorporate both time and frequency domain constraints. The eigenfilter method [11] has been extended to the design of higher-order DD's [12] by formulating an error function in the quadratic form. The error function involves the square of the difference between desired amplitude response of the designed filter at an arbitrary reference frequency as opposed to being equal to ideal amplitude characteristics. The filter coefficients are found by computing the eigenvector corresponding to the smallest eigenvalue of a positive definite symmetric matrix. The
Fourier series method [8] in conjunction with accuracy constraint [13] has been extended to the design of higher-order DD’s. In this technique differentiators are designed by imposing magnitude and derivative constraints at a particular frequency. Consequently these differentiators are accurate only in the neighbourhood of the frequency at which these constraints are imposed. Sunder et al. [14] described least-squares approach for the design of digital differentiators satisfying prescribed specifications. Least-squares approach [15] has been extended for the design of higher-order digital differentiators. This procedure involves the formulation of the absolute mean square error between the practical and ideal differentiators as a quadratic function. The coefficients of the differentiators are obtained by solving a system of linear equations. The explicit inclusion of the ideal amplitude response in the error function leads to a more meaningful formulation than the eigenfilter method and does not necessitate the use of reference frequency. Shian-Tang Tzeng et al. [16] designed FIR higher-order DD using genetic algorithm (GA) approach. By minimizing a quadratic measure of the error in the frequency band, appropriate crossover-, mutation-, and selection-operation are used to get the filter coefficients.

Another approach, namely, WLS technique described by Sunder et al. [19] for the design of DD’s and HT’s is based on an iterative algorithm in which the appropriate frequency-dependent weighting function that yields an equiripple design is determined. According to Sunder et al. this method can be treated as an alternate to Parks-McClellan method and it is optimal in minimax sense. A perusal of literature suggests that higher-order DD’s have not been designed using WLS technique, as far as author is aware.
Design methods, such as eigenfilter, least-squares, GA are not equiripple and have more peak errors near the band edges. Author's ultimate aim is to design equiripple higher-order DD's and as far as possible minimum peak error.

In view of the above considerations the methods for the design of higher-order DD's are being modified from time to time. In this chapter, a new method for the design of equiripple FIR higher-order DD's based on modified MEA is described. Further, modified MEA is extended for the design of frequency-selective higher-order DD's such as lowpass, highpass, bandpass and bandstop differentiators. This new method is not only simple but fast and optimal in minimax sense. In this topic, the author has made an attempt to apply weighted least-squares (WLS) technique for designing equiripple FIR higher-order DD's.

A look at the literature suggests that even order DD's can only be designed by symmetric impulse response [20, case 1 and 2] whereas the odd-order DD's can only be designed by antisymmetric impulse response [20, case 3 and 4]. Bearing this in mind the modified MEA and WLS methods are used for the design of equiripple higher-order DD's. In general, the peak errors so obtained by these methods are very close to those determined by Parks-McClellan method, while they are very much lower when compared with the values obtained by other methods like eigenfilter, least-squares and GA method.

5.2 Problem Formulation

Consider a typical transfer function of a FIR filter of length N, which can be represented as
Depending on the values of $N$ (odd or even) and type of symmetry of the filter impulse response $h(n)$ (symmetric or antisymmetric), there are four possible types of linear phase FIR filters [20]. The amplitude response of these four types of filters can be expressed as

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}.$$  \hfill (5.1)

The amplitude response of these four types of filters can be expressed as

$$A(\omega) = \begin{cases} \sum_{n=0}^{M} b(n) \text{trig}(\omega, n) & \text{for case 1} \\ \sum_{n=1}^{M} b(n) \text{trig}(\omega, n) & \text{for cases 2, 3 and 4} \end{cases}$$ \hfill (5.2)

where $\text{trig}(\omega, n)$ is an appropriate trigonometric function and coefficients $b(n)$ are related to impulse response $h(n)$ of the filter, whereas $M$ is a function of the filter length $N$. Table 5.1 lists the relationship among them for four types of linear phase filters. Defining the column vectors

$$b = \begin{bmatrix} b(0), b(1), \ldots, b(M) \end{bmatrix}^T$$ for case 1

$$b = \begin{bmatrix} b(1), b(2), \ldots, b(M) \end{bmatrix}^T$$ for cases 2, 3 and 4

and

$$C(\omega) = \begin{bmatrix} \text{trig}(\omega, 0), \text{trig}(\omega, 1), \ldots, \text{trig}(\omega, M) \end{bmatrix}^T$$ for case 1

$$C(\omega) = \begin{bmatrix} \text{trig}(\omega, 1), \text{trig}(\omega, 2), \ldots, \text{trig}(\omega, M) \end{bmatrix}^T$$ for cases 2, 3 and 4

equation (5.2) can be rewritten in terms of these vectors as

$$A(\omega) = b^T C(\omega) = C^T(\omega) b$$ \hfill (5.5)

Now, the problem is to find the filter coefficient vector $b$ such that amplitude response $A(\omega)$ approximates desired amplitude response $D(\omega)$ of higher-order DD's as close as possible. An ideal $k^{th}$-order DD has a frequency response [15]

$$H_1(\omega) = D(\omega)e^{j\frac{\pi}{2}}$$ \hfill (5.6)
Table 5.1
Relationship between h(n), N and M, b(n), trig(ω,n)

<table>
<thead>
<tr>
<th>Type of symmetry</th>
<th>Case</th>
<th>M, b(n), trig(ω,n)</th>
</tr>
</thead>
</table>
| h(n): Symmetric  | 1    | M = \( \frac{N-1}{2} \)  
| N: Odd           |      | b(0) = h(\( \frac{N-1}{2} \)), b(n) = 2h(\( \frac{N-1}{2} - n \))  
|                  |      | n = 1, 2, ..., M  
|                  |      | trig(ω,n) = cos(ωn) |
| h(n): Symmetric  | 2    | M = \( \frac{N}{2} \)  
| N: Even          |      | b(n) = 2h(\( \frac{N}{2} - n \))  
|                  |      | n = 1, 2, ..., M  
|                  |      | trig(ω,n) = cos(ω(\( n - \frac{1}{2} \))) |
| h(n): Antisymmetric | 3  | M = \( \frac{N-1}{2} \)  
| N: Odd           |      | b(n) = 2h(\( \frac{N-1}{2} - n \))  
|                  |      | n = 1, 2, ..., M  
|                  |      | trig(ω,n) = sin(ωn) |
| h(n): Antisymmetric | 4  | M = \( \frac{N}{2} \)  
| N: Even          |      | b(n) = 2h(\( \frac{N}{2} - n \))  
|                  |      | n = 1, 2, ..., M  
|                  |      | trig(ω,n) = sin(ω(\( n - \frac{1}{2} \))) |

where \( D(\omega) = \left( \frac{\omega}{2\pi} \right)^k \) for \( 0 \leq \omega \leq \omega_p \leq \pi \), ideal amplitude response and \( \omega_p \) is the highest frequency for which differentiating action is required.

For even order DD's (k is even), it can be seen that \( H_1(\omega) \) is real valued function. Hence, only a FIR filter with symmetrical impulse response (case 1 and 2)
can be used for the design of even-order differentiators. However, case 1 is suitable for designing full-band or nonfull-band DD’s, case 2 is useful only for designing nonfull-band DD’s due to the inherent zero folding frequency constrain: [20]. Similarly, for an odd-order DD (k is odd), the fact that \( H_i(\omega) \) is purely imaginary function, mandates the design of a FIR filter with an antisymmetrical impulse response (case 3 and 4). For odd-order DD’s case 3 is only suitable for designing the nonfull-band DD due to the inherent zero folding frequency constraints, case 4 is suitable for both full-band as well as nonfull-band DD’s.

5.3 Error function minimization

In this section, the optimal coefficients are obtained by minimizing the following squared error:

\[
E_{ue} = \int_{0}^{\omega_p} E_s^2(\omega) d\omega ,
\]

(5.7)

where \( E_s(\omega) = D(\omega) - A(\omega) \),

(5.8)

is the error function. In minimizing \( E_{ue} \), we set \( \frac{\partial E_{ue}}{\partial b} = 0 \) to obtain a system of linear equations \( b_{ks} = E^{-1}f_1 \),

(5.9)

where matrix \( E \) and vector \( f_1 \) are

\[
E = \int_{0}^{\omega_p} C(\omega)C'(\omega) d\omega,
\]

(5.10)

\[
f_1 = \int_{0}^{\omega_p} C(\omega)D(\omega) \omega d\omega.
\]

(5.11)
In our approach, we will use error function equation (5.8) to search extremal frequencies, and least square solution equation (5.9) as the initial guess solution in the iterative modified MEA.

5.4 Modified multiple exchange algorithm

A typical amplitude and error response for an equiripple linear phase FIR higher-order DD is shown in Figs. 5.1(a) and (b).

![Amplitude response](image1)

**Fig. 5.1(a)** A typical amplitude response of higher-order DD.

![Error response](image2)

**Fig. 5.1(b)** Typical error response.

Let us assume that there are $N_p$ extrema in passband $[0, \omega_p]$ of a higher-order DD. There is no alternation theorem to specify extreme number $N_p$ such that
optimality is guaranteed [21]. However, a modified MEA can be developed to design higher-order DD. By selecting the extremal frequencies $\omega_i$ (i =1, 2, ..., $N_p$) in its passband [17,18] as

$$\omega_1 < \omega_2 < \omega_3 < \omega_4 < ... < \omega_{N_p}, \quad (5.12)$$

we want to find a set of filter coefficients that satisfy the condition expressed as

$$A(\omega) = D(\omega_i) - (-1)^i \delta, \quad (5.13)$$

where $\delta$ is error.

Now, our goal is to find a solution vector $b$ that satisfies equation (5.13). Substituting equation (5.2) in equation (5.13), we have the following matrix form

$$
\begin{bmatrix}
\text{trig}(\omega_1,1) & \text{trig}(\omega_2,2) & \ldots & \text{trig}(\omega_1,M) \\
\text{trig}(\omega_2,1) & \text{trig}(\omega_2,2) & \ldots & \text{trig}(\omega_2,M) \\
\vdots & \vdots & \ddots & \vdots \\
\text{trig}(\omega_{N_p},1) & \text{trig}(\omega_{N_p},2) & \ldots & \text{trig}(\omega_{N_p},M)
\end{bmatrix} \begin{bmatrix} b(1) \\ b(2) \\ \vdots \\ b(M) \end{bmatrix} = \begin{bmatrix} (\omega_1 \frac{2\pi}{k}) \\ (\omega_2 \frac{2\pi}{k}) \\ \vdots \\ (\omega_{N_p} \frac{2\pi}{k}) \end{bmatrix}, \quad (5.14)
$$

Denoting the matrix in the left side by $Q$ and vector on the right side by $d$,

$$Qb = d. \quad (5.15)$$

Assume that $N_p = (M+1)$, then equation (5.14) is an overdetermined set of linear equations whose least-square solution is given by

$$b = (Q'Q)^{-1} Q'd. \quad (5.16)$$

Based on above description, we propose an iterative modified MEA to determine coefficient vector $b$ for higher-order DD's as follows:

**Step 1** read higher-order DD length $N$, order $k$ and $\omega_p$ is the highest frequency for which differentiating action is required,
Step 2 use least-square solution in equation (5.9) as the initial guess solution and search the extremal frequencies in the range \([0, \omega_p]\) as \(\Omega_i (i=1,2,\ldots M_p)\), using equation (5.8).

Repeat

Step 3 set \(\omega_i = \Omega_i\).

Step 4 compute average peak error

\[
\delta = \frac{1}{N_p} \left[ \sum_{i=1}^{N_p} |A(\omega_i) - D(\omega_i)| \right],
\]

(5.17)

Step 5 compute the new coefficient vector \(b\) using equation (5.16),

Step 6 search for new set of extremal frequencies \(\Omega_i\) of \(E_a(\omega)\) within passband,

Until

satisfy the following inequality for a prescribed small constant \(\varepsilon\):

\[
|\Omega_i - \omega_i| \leq \varepsilon.
\]

(5.18)

5.5 Frequency-selective higher-order DD’s

In many applications differentiation may be needed over a specified range. Thus higher-order differentiators performing differentiation over such a range may be classified as lowpass, highpass, bandpass and bandstop differentiators. In this case the ideal frequency response is given by equation (5.6) where \(D(\omega) = \left( \frac{\omega}{2\pi} \right)^k\) in the passband (P) and zero in the stopband (S). The error function \(E_{ise}\) consists of two terms one reflecting the passband error and other reflecting the stopband error. The error function is given by

\[
E_{ise} = \int_{P} E_{P}^2(\omega) \, d\omega + \int_{S} E_{S}^2(\omega) \, d\omega.
\]

(5.19)
where, \( E_p(\omega) = D(\omega) - A(\omega) \) \( (5.20) \)

\( E_s(\omega) = A(\omega) \) \( (5.21) \)

Minimizing \( E_{se} \) with respect to the filter coefficients results in a system of linear equations given by

\[
(Y_1 + Y_2) \mathbf{b} = Y_3,
\]

where,

\[
Y_1 = \int_{P} C(\omega) C^T(\omega) \, d\omega \tag{5.23}
\]

\[
Y_2 = \int_{S} C(\omega) C^T(\omega) \, d\omega \tag{5.24}
\]

\[
Y_3 = \int_{P} C(\omega) D(\omega) \, d\omega \tag{5.25}
\]

For the sake of exposition, let us consider a higher-order bandpass differentiator for which the passband region is \( P = [\omega_{p1}, \omega_{p2}] \) and that of the stopband is \( S = [0, \omega_{s1}] \cup [\omega_{s2}, \pi] \). Consider the design of even-order bandpass differentiators with length \( N \) odd [case 1]. Consequently the elements of \( Y_1, Y_2 \) and \( Y_3 \) are given as

\[
Y_1 = \int_{\omega_{p1}}^{\omega_{p2}} C(\omega) C^T(\omega) \, d\omega, \tag{5.26}
\]

\[
Y_2 = \int_{0}^{\omega_{s1}} C(\omega) C^T(\omega) \, d\omega + \int_{\omega_{s2}}^{\pi} C(\omega) C^T(\omega) \, d\omega, \tag{5.27}
\]

and

\[
Y_3 = \int_{\omega_{p1}}^{\omega_{p2}} C(\omega) D(\omega) \, d\omega. \tag{5.28}
\]

In the design of higher-order bandpass differentiators, equations (5.20), (5.21) are used to search extremal frequencies and least-square solution equation (5.22) as the initial guess solution in the iterative modified MEA. Let us assume there are \( N_i \)
extrema in the left stopband \([0, \omega_{si}]\), \(N_p\) extrema in the passband \([\omega_{p1}, \omega_{p2}]\) and \(N_r\) extrema in the right stopband \([\omega_{s2}, \pi]\) and select extremal frequencies in the three bands of higher-order bandpass differentiators as follows:

\[
\omega_{1sl} > \omega_{2sl} > \omega_{3sl} > \ldots > \omega_{Nsl} \quad (5.29a)
\]
\[
\omega_{1p} < \omega_{2p} < \omega_{3p} < \ldots < \omega_{Np} \quad (5.29b)
\]
\[
\omega_{1sr} > \omega_{2sr} > \omega_{3sr} > \ldots > \omega_{Nsr} \quad (5.29c)
\]

Find a set of filter coefficients that should satisfy the condition expressed as follows:

\[
A(\omega_{isl}) = D(\omega_i) - (-1)^i \delta_s, \text{ where } i = 1, 2, 3, 4, \ldots, N_{sl} \quad (5.30)
\]
\[
A(\omega_{ip}) = D(\omega_i) - (-1)^i \delta_p, \quad \text{ where } i = 1, 2, 3, 4, \ldots, N_p \quad (5.31)
\]
\[
A(\omega_{isr}) = D(\omega_i) - (-1)^i \delta_s, \quad \text{ where } i = 1, 2, 3, 4, \ldots, N_{sr} \quad (5.32)
\]

To find a solution vector \(\mathbf{b}\), substitute equation (5.5) in equations (5.30), (5.31) and (5.32), we have the following matrix form

\[
\begin{bmatrix}
\text{trig}(\omega_{N_{sl}}, 1) & \ldots & \text{trig}(\omega_{N_{sl}}, M) \\
\vdots & \ddots & \vdots \\
\text{trig}(\omega_{1sl}, l) & \ldots & \text{trig}(\omega_{1sl}, M) \\
\text{trig}(\omega_{1p}, l) & \ldots & \text{trig}(\omega_{1p}, M) \\
\vdots & \ddots & \vdots \\
\text{trig}(\omega_{Np}, l) & \ldots & \text{trig}(\omega_{Np}, M) \\
\text{trig}(\omega_{1sr}, l) & \ldots & \text{trig}(\omega_{1sr}, M) \\
\vdots & \ddots & \vdots \\
\text{trig}(\omega_{Nsr}, l) & \ldots & \text{trig}(\omega_{Nsr}, M)
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_M
\end{bmatrix}
= \begin{bmatrix}
(-1)^{N_{sl}} \delta_s \\
\vdots \\
(-1)^{\delta_s} \\
(-1)^{N_{sr}} \delta_s
\end{bmatrix}
\]

\[
(\frac{\omega_{1sl}}{2\pi})^k - (-1)^{\delta_s} \\
(\frac{\omega_{1p}}{2\pi})^k - (-1)^{\delta_p} \\
(\omega_{1sr})^k - (-1)^{N_{sr}} \delta_s
\]

(5.33)
Letting the matrix in the left side by \( Q \) and the vector on right side by \( d \), then \( Qb = d \).

Assuming that \( N_{sl} + N_p + N_{sr} = (M+1) \), then equation (5.33) is an overdetermined set of linear equations whose least square solution is given by

\[
b = (Q'Q)^{-1} Q'd \tag{5.34}
\]

Based on the above description, we propose an iterative modified MEA to determine coefficient vector \( b \) of higher-order bandpass differentiators as follows:

**step 1** read the higher-order bandpass differentiator specifications as length \( N \), order \( k \), passband region \( P = [\omega_{p1}, \omega_{p2}] \) and stopband \( S = [0, \omega_{s1}] \cup [\omega_{s2}, \pi] \),

**step 2** use least-square solution in equation (5.22) as initial guess and search passband and stopbands extremal frequencies as \( \Omega_{ip}(i = 1, 2, 3, \ldots, N_p) \), \( \Omega_{isl}(i = 1, 2, 3, \ldots, N_{sl}) \) and \( \Omega_{isr}(i = 1, 2, 3, \ldots, N_{sr}) \) of \( A(\omega) \),

**Repeat**

**step 3** set \( \omega_{isl} = \Omega_{isl} \), \( \omega_{isr} = \Omega_{isr} \) and \( \omega_{ip} = \Omega_{ip} \),

**step 4** compute average peak error

\[
\delta = \frac{1}{(N_{sl} + N_p + N_{sr})} \left[ \sum_{i=1}^{N_{sl}} |A(\omega_{isl}) - 0| + \sum_{i=1}^{N_p} A(\omega_{ip}) - \left( \frac{\omega_{ip}}{2\pi} \right)^k + \sum_{i=1}^{N_{sr}} |A(\omega_{isr}) - 0| \right], \tag{5.35}
\]

**step 5** compute the new filter coefficient vector \( b \), by solving equation (5.34),

**step 6** search the extremal frequencies \( \Omega_{isl} \), \( \Omega_{isr} \) and \( \Omega_{ip} \) of \( E_p(\omega) \) and \( E_s(\omega) \),

**Until**

satisfy the following conditions for prescribed small constants \( \varepsilon_1 \), and \( \varepsilon_2 \)

\[
|\Omega_{isl} - \omega_{isl}| \leq \varepsilon_1, \tag{5.36}
\]

\[
|\Omega_{ip} - \omega_{ip}| \leq \varepsilon_2. \tag{5.37}
\]
In this section, design of higher-order DD's using WLS technique is described. Problem formulation for this design technique is same as given in the section 5.2.

### 5.6.1 Error function minimization

The weighted mean-squared error with respect to the higher-order DD passband can be expressed as

\[ E_{\text{mse}} = \sum_{i=1}^{F} W(\omega_i) E_a^2(\omega_i), \tag{5.39} \]

\[ E_a(\omega) = D(\omega) - A(\omega) \tag{5.40} \]

is the error function, where \( W(\omega) \) is the frequency-dependent weighting function and \( F \) is the number of points at which the error function is sampled. In minimizing \( E_{\text{mse}} \), we set \( \frac{\partial E_{\text{mse}}}{\partial b} = 0 \) to obtain a system of linear equations

\[ Qb = d, \tag{5.41} \]

where

\[ Q = \sum_{i=1}^{F} W(\omega_i) C(\omega_i) C^T(\omega_i), \tag{5.42} \]

\[ d = \sum_{i=1}^{F} W(\omega_i) D(\omega_i) C(\omega_i). \tag{5.43} \]
It can be noted that $Q$ is a positive-definite real symmetric matrix for a positive weighting function $W(\omega)$, and thus, a unique solution is guaranteed. As a result, the system of linear equations can be solved by a computationally efficient method like the Cholesky decomposition, which avoids matrix inversion.

5.6.2 Weighted least-squares technique

The variation of a typical error function in the passband of higher-order DD and envelope of the error function with frequency is shown in Fig.5.2.

![Fig.5.2. Variation of a typical error function and envelope of the error function with angular frequency](image)

The shape of the error function is not known a priori and therefore, $W(\omega)$ cannot be found analytically. Consequently, an iterative procedure to identify the appropriate weighting function must be followed.
Let $W_k(\omega)$ be the weighting function at the $k^{th}$ iteration. Then $W_{k+1}(\omega)$ is written as $W_{k+1}(\omega) = W_k(\omega) \chi_k(\omega)$, where the updating function, $\chi_k(\omega)$, is a function of the envelope of the error function $E_a(\omega)$ [19]. Very first step towards obtaining the envelope function is to identify the valley frequencies. The valley frequencies are the frequencies at which $|E_a(\omega)|$ has local minimum. These frequencies are shown as $v_i$ (for $i = 2, 3, \ldots, 14$) in Fig. 5.2. $v_1$ and $v_{15}$ are the band edges.

The maximum values of $|E_a(\omega)|$ between pairs of consecutive $v_i$'s are obtained as

$$r_i = \max \{ |E_a(\omega_i)| \} \text{ for } v_i \leq \omega_i \leq v_{i+1}$$

(5.44)

An updating function can now be defined as

$$\chi_k(\omega_i) = \begin{cases} r_i & v_i \leq \omega_i \leq v_{i+1} \\ r_i & \text{for } i = 2, 3, \ldots, 14. \end{cases}$$

(5.45)

Let us now define

$$q = [\chi_k(\omega_1) \chi_k(\omega_2) \ldots \chi_k(\omega_F)]^t$$

(5.46)

and let $q_{\max} = \max(q)$ and $q_{\min} = \min(q)$.

If

$$R = \frac{q_{\max} - q_{\min}}{q_{\max}} \leq \varepsilon$$

(5.47)

where $\varepsilon$ is a small positive number (say, 0.01), then $|E_a(\omega)|$ is said to be equiripple.

If this is not the case, let us update the weighting function as $W_{k+1}(\omega_i) = W_k(\omega_i) \chi_k(\omega_i)$, where $W_\delta(\omega_i) = 1$ for $0 \leq \omega_i \leq \omega_p$. The weighting function $W_{k+1}(\omega_i)$ is then normalized with respect to its maximum value. It has been observed that in order to obtain faster convergence of the algorithm, the weighting function can be updated as...
$W_{k+1}(\alpha_k) = \chi_k^D(\alpha_k)W_k(\alpha_k)$, where $\sigma$ is a fast converging factor whose value lies between 1.0 and 1.7. The best value of $\sigma$ is 1.5 \[22\].

Based on the above description, we summarize the WLS algorithm for the designing of higher-order DD's as follows:

1) initialize $W_0(\omega_i)$,

2) compute $Q$ and $d$ and solve a system of equations $Qb = d$,

3) evaluate the error function $E_a(\omega_i)$,

4) a) determine $\nu_i$ and $r_i$,

   b) determine $\chi_k(\omega_i)$ and compute $R$,

5) if $R \leq \epsilon$ (0.01), then exit,

6) update the weighting function as $W_{k+1}(\alpha_k) = \chi_k^D(\alpha_k)W_k(\alpha_k)$, and go to step 2),

5.7 Design examples

In this section, two even- and two odd-order DD's have been designed for full- and/ nonfull- band cases using both modified MEA and WLS method, whereas for cases 2 and 4 examples are repeated for different lengths in case of WLS method. Performance of these higher-order DD’s is summarized in Table 5.2. Design examples of frequency-selective higher-order DD’s using modified MEA are also given in this section.
Example 1. *Case1 for even-order and odd length*. A FIR full-band second-order DD is designed with length $N = 25$ and $\omega_p = \pi$. Figs. 5.3(a) and (b) show the amplitude and error response respectively.

**Fig. 5.3(a).** Amplitude response.

**Fig. 5.3 (b).** Error response.
Example 2. Case 2 for even-order and even length. We have designed an even length nonfull-band fourth-order DD with length $N = 34$ and $\omega_p = 0.92\pi$. Figs. 5.4(a) and (b) show the amplitude and error response for both modified MEA and WLS methods respectively.

Fig. 5.4(a). The amplitude responses.

Fig. 5.4(b). The error responses.
Example 3. Case 3 for odd-order and odd length. A nonfull-band FIR third-order DD is designed with length $N = 27$ and $\omega_p = 0.88\pi$. Figs. 5.5(a) and (b) show the amplitude and error response respectively.

Fig. 5.5 (a). The amplitude response

Fig. 5.5 (b). The error response
Example 4. Case 4 for odd-order and even length. A full-band FIR fifth-order DD is designed with length $N = 26$, and $\omega_p = \pi$. Figs. 5.6(a) and (b) show the amplitude and error response respectively.

Fig. 5.6 (a). The amplitude response.

Fig. 5.6 (b). The error response.
Example 5. Case 2 for even-order and even length. An even length non-full band fourth-order DD with length $N = 32$ and $\omega_p = 0.92\pi$. Figs. 5.7(a) and (b) show the amplitude and error response respectively.

Fig. 5.7 (a). The amplitude response.

Fig. 5.7 (b). The error response
Example 6. Case 4 for odd-order and even length. A full-band FIR fifth-order DD is designed with length $N = 32$, and $\omega_p = \pi$. Figs. 5.8(a) and (b) show the amplitude and error response respectively.

Fig. 5.8(a). The amplitude response.

Fig. 5.8(b). The error response.
Example 7. A lowpass second-order DD is designed with $N=31$, $\omega_p = 0.5\pi$, $\omega_s = 0.6\pi$ and ripple ratio $\Delta = 2$. Figs. 5.9(a) and (b) shows the amplitude and error response respectively.

Fig 5.9(a). The amplitude response.

Fig 5.9(b). The error response.
Example 8. A highpass second-order DD is designed with $N=31$, $\omega_s = 0.2\pi$, $\omega_p = 0.3\pi$ and $\Delta = 1$. Figs. 5.10(a) and (b) shows the amplitude and error response respectively.

Fig. 5.10 (a). The amplitude response.

Fig. 5.10 (b). The error response.
Example 9. A bandpass second-order DD is designed with \( N=31, \omega_{s1}=0.2\pi, \omega_{p1}=0.3\pi, \omega_{p2}=0.7\pi, \omega_{s2}=0.8\pi, \) and \( \Delta=1. \) Figs. 5.11(a) and (b) shows the amplitude and error response respectively.

![Amplitude Response](image1)

**Fig. 5.11(a)** The amplitude response.

![Error Response](image2)

**Fig. 5.11(a)** The error response.
Example 10. A bandstop second-order DD is designed with \( N = 31, \omega_{p1} = 0.3\pi, \omega_{s1} = 0.4\pi, \omega_{s2} = 0.6\pi, \omega_{p2} = 0.7\pi, \text{ and } \Delta = 1. \) Figs. 5.12(a) and (b) shows the amplitude and error respectively.

**Fig. 5.12(a)** The amplitude response.

**Fig. 5.12(b)** The error response.
Table 5.2

Comparison of modified MEA and WLS method with Parks-McClellan, eigenfilter, least-squares and GA methods with respect to peak error.

<table>
<thead>
<tr>
<th>DD's order and case</th>
<th>$\omega_p$</th>
<th>N</th>
<th>Modified MEA Method ($x10^{-3}$)</th>
<th>WLS Method ($x10^{03}$)</th>
<th>PM Method ($x10^{03}$)</th>
<th>Eigen Method ($x10^{03}$)</th>
<th>LS Method ($x10^{63}$)</th>
<th>GA Method ($x10^{03}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd case 1</td>
<td>$\pi$</td>
<td>25</td>
<td>3.7</td>
<td>3.7</td>
<td>3.7</td>
<td>8.1</td>
<td>8.101</td>
<td>6.1</td>
</tr>
<tr>
<td>4th case 2</td>
<td>0.92$\pi$</td>
<td>32</td>
<td>-</td>
<td>0.459</td>
<td>0.47</td>
<td>1.46</td>
<td>1.504</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>34</td>
<td>0.341</td>
<td>0.347</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3rd case 3</td>
<td>0.88$\pi$</td>
<td>27</td>
<td>0.296</td>
<td>0.295</td>
<td>0.30</td>
<td>0.91</td>
<td>1.02</td>
<td>0.57</td>
</tr>
<tr>
<td>5th case 4</td>
<td>$\pi$</td>
<td>32</td>
<td>-</td>
<td>0.911</td>
<td>0.90</td>
<td>1.96</td>
<td>1.97</td>
<td>1.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>26</td>
<td>1.1</td>
<td>1.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

5.8 Conclusions

In this chapter, higher-order DD’s have been designed using modified MEA and WLS methods. A new method is described for designing equiripple FIR higher-order DD’s by modified MEA in which extremal frequencies are searched from error response instead of amplitude response. Further, it has been shown that various types of frequency-selective (lowpass, highpass, bandpass and bandstop) DD’s satisfying wide range of specifications are designed. An attempt is made to apply WLS technique for designing equiripple higher-order DD’s. This method involves an iterative algorithm in which appropriate frequency-dependent weighting function that yields an equiripple design is determined. Referring to Table 5.2 it is observed that
peak error obtained by both of these methods are in close agreement with those due to Parks-McClellan (PM) method whereas peak errors are very much lower than eigenfilter, least-squares (LS) and genetic algorithm methods. The latter three methods are optimal in the sense of different minimum norms of error function but much better performance are obtained with the methods under study in the entire frequency band.

Though the methods under study for designing of higher-order DD’s are optimal in minimax sense and having almost same peak error, modified MEA is in general faster in design speed than WLS method. Several numerical design examples are given to illustrate the effectiveness and usefulness of the methods.

The main conclusions drawn on the basis of results reported in chapter III, IV and V are summarized in chapter six. Also, further scope of the work in this area is indicated in that chapter.
References


[10] C.A. Rahenkamp and B.V.K. Vijaya Kumar, "Modifications to the McClellan, Parks and Rabiner computer program for designing higher-order


