CHAPTER - III
ON MODIFIED DEFICIENCIES OF MEROMOPHIC FUNCTIONS AND EXCEPTIONAL VALUES OF DIFFERENTIAL POLYNOMIALS
3.1 On Modified Deficiencies of Meromorphic Functions

**Section I**

Introduction:

Let \( f(z) \) be a non-constant meromorphic function. N.Toda [27] (1970) introduced modified characteristic function and deficiencies to the Nevanlinna theory. Sarangi and Patil [21] introduced the bounds for \( \frac{T_{\alpha}(r,f^{(n)})}{T_{\alpha}(r,f)} \). Also the bounds for

\[
K_{\alpha}(f) = \limsup_{r \to \infty} \frac{N_{\alpha}(r,f^{-1}) + N_{\alpha}(r,f)}{T_{\alpha}(r,f)}
\]

were obtained in [21].

Toda defines \( T_{\alpha}(r,f), N_{\alpha}(r,f), \delta_{\alpha}(a,f) \) etc, as follows.

If the order of \( f(z) \) is infinite then the so-called “Exceptional Set” appears in the second fundamental theorem of Nevanlinna.

Let \( f(z) \) be a meromorphic function in the finite plane of order \( \rho \) \((0 \leq \rho \leq \infty)\) and lower order \( \mu \). Denote by \( \alpha \) any non-negative number smaller than \( \rho \) if \( \rho \) is not zero and zero if \( \rho = 0 \). The symbols \( T(r,f), M(r,a) = m(r, a, f), N(r,a) = N(r,a,f), N(r,f) \) etc have the usual meanings of the Nevanlinna theory of meromorphic functions (Hayman 1964). In contrast to the
case of finite order we have to face many difficulties in the investigation of value distribution of meromorphic functions of infinite order.

If \( r_0 \) is any positive number, then for any complex \( a \), finite or infinite, we define

\[
T_a(r,a,f) = \int_{r_0}^{r} \frac{T(t,a,f)}{t^{1+a}} \, dt, \quad N_a(r,a,f) = \int_{r_0}^{r} \frac{N(t,a,f)}{t^{1+a}} \, dt.
\]

\[
m_a(r,a,f) = \int_{r_0}^{r} \frac{m(t,a,f)}{t^{1+a}} \, dt, \quad \bar{N}_a(r,a,f) = \int_{r_0}^{r} \frac{\bar{N}(t,a,f)}{t^{1+a}} \, dt.
\]

and \( S_a(r,a,f) = \int_{r_0}^{r} \frac{S(t,a,f)}{t^{1+a}} \, dt. \)

\( \delta_a(a,f), \Delta_a(a,f) \) and \( \Theta_a(a,f) \) are defined as follows:

\[
\delta_a(a,f) = \liminf_{r \to \infty} \frac{m_a(r,a,f)}{T_a(r,f)}, \quad \Delta_a(a,f) = \limsup_{r \to \infty} \frac{m_a(r,a,f)}{T_a(r,f)},
\]

\[
\Theta_a(a,f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_a(r,a,f)}{T_a(r,f)}.
\]

A monomial in \( f \) is an expression of the form

\[
M_j(f) = f^{n_0} f^{\nu_0} \ldots f^{(k)} f^{\nu_k} \text{ where } n_0, n_1, n_2, \ldots n_k \text{ are non-negative integers.} \quad \gamma_{M_j} = n_0 + n_1 + n_2 + \ldots + n_k \text{ is called the degree of the monomial and} \quad \Gamma_{M_j} = n_0 + 2n_1 + 3n_2 + \ldots + (k+1)n_k = \sum_{i=1}^{k} (i+1)n_i, \text{ the weight of } M_j(f).
\]

\[
P(f) = \sum_{j=1}^{q} a_j M_j(f(z)), \text{ where } a_i (i = 1,2,\ldots,n) \text{ are constants. Then } P(f) \text{ is called a differential polynomial in } f \text{ of degree } \gamma_P \text{ and the weight } \Gamma_P. \]P(f) are defined
as follows: \( \gamma_p = \max_{\theta \in \mathbb{Q}} \gamma_{M_j} \) and \( \Gamma_p = \max_{\theta \in \mathbb{Q}} \Gamma_{M_j} \), also we call the number \( \gamma_p \) the lower degree of \( P(f) \). If \( \gamma_p = \gamma_p \), \( P(f) \) is called a homogeneous differential polynomial in \( f \).

Toda [27] (1970) proved the following result.

**Theorem A.** If \( f(z) \) is a transcendental meromorphic function in \( |z| < \infty \), then

\[
\sum_{a \neq 0} \delta_a(a) \leq \liminf_{r \to \infty} \frac{T_a(r,f)}{T_a(r,f)} \leq \limsup_{r \to \infty} \frac{T_a(r,f)}{T_a(r,f)} \leq 2 - \Theta_{\alpha}(\infty,f).
\]

Later, Sarangi and Patil [21] proved the following result.

**Theorem B.** If \( f(z) \) is a transcendental meromorphic function in \( |z| < \infty \), then for any positive integer \( l \),

\[
\sum_{a \neq 0} \delta_a(a) \leq \liminf_{r \to \infty} \frac{T_a(r,f^{(l)})}{T_a(r,f)} \leq \limsup_{r \to \infty} \frac{T_a(r,f^{(l)})}{T_a(r,f)} \leq (1 + l) - l\Theta_{\alpha}(\infty,f).
\]

We extend the above results to the above mentioned homogeneous differential polynomials.

**Theorem 3.1.1:** If \( f(z) \) is a transcendental meromorphic function in \( |z| < \infty \), then

\[
\gamma_p \sum_{a \in \mathbb{C}} \delta_a(a) \leq \liminf_{r \to \infty} \frac{T_a(r,P(f))}{T_a(r,f)} \leq \limsup_{r \to \infty} \frac{T_a(r,P(f))}{T_a(r,f)} \leq (\Gamma_p - (\Gamma_p - \gamma_p)\Theta_{\alpha}(\infty,f)).
\]

where \( P(f) \) is a homogeneous differential polynomial, not involving the \( f \) term.

To prove this we require the following lemmas.
Lemma 3.1.1: If $f(z)$ is a transcendental meromorphic function and $a_1, a_2, \ldots, a_q$ are distinct elements, then

$$
\gamma_P \sum_{j=1}^{q} m_\alpha (r, a_j, f) \leq T_\alpha (r, P(f)) - N(r, 0, P(f)) + S_\alpha (r, f),
$$

or

$$
\gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq m_\alpha (r, \frac{1}{P(f)}) + S_\alpha (r, f),
$$

where $P(f)$ is a homogeneous differential polynomial of degree $\gamma_P$.

Proof: Without loss of generality we may assume that $q \geq 2$.

Let $F(z) = \sum_{j=1}^{q} \frac{1}{(f(z) - a_j)^{\gamma_P}}$ and proceeding as in (Hayman 1964, 32-33)

we get

$$
\gamma_P \sum_{i=1}^{q} m(r, a_i, f) \leq m(r, F) + O(1)
$$

$$
\leq \sum_{i=1}^{q} m(r, \frac{P(f-a_j)}{(f-a_j)^{\gamma_P}}) + m(r, 0, P(f)) + O(1),
$$

which gives by the first fundamental theorem and Milloux (Hayman [9], 55) that

$$
\gamma_P \sum_{i=1}^{q} m(r, a_i, f) \leq T(r, P(f)) - N(r, 0, P(f)) + S(r, f).
$$

This implies,

$$
\gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq T_\alpha (r, P(f)) - N_\alpha (r, 0, P(f)) + S_\alpha (r, f).
$$
Or, using the first fundamental theorem, we have

$$\gamma_r \sum_{j=1}^{n} m_{a_j}(r, a_j, f) \leq m_{a}(r, \frac{1}{P(f)} + S_{a}(r, f)).$$

**Lemma 3.1.2:** [2] If $Q[f]$ is a differential polynomial in $f$ with arbitrary meromorphic coefficients $q_j \quad 1 \leq j \leq n$, then

$$m(r, Q[f]) \leq \gamma_q m(r, f) + \sum_{j=1}^{n} m(r, a_j) + S(r, f)$$

**Lemma 3.1.3:** [11] $N(r, P(f)) \leq \gamma_p N(r, f) + (\Gamma_p + \gamma_p) \overline{N}(r, f) + S(r, f)$

**Proof of Theorem 3.1.1:** By Lemma 3.1.2 and Lemma 3.1.3, we have,

$$m(r, P[f]) \leq \gamma_p m(r, f) + S(r, f).$$

This implies,

$$m_{a}(r, P[f]) \leq \gamma_p m_{a}(r, f) + S_{a}(r, f).$$

and

$$N(r, P(f)) \leq \gamma_p N(r, f) + (\Gamma_p - \gamma_p) \overline{N}(r, f) + S(r, f)$$

This implies,

$$N_{\alpha}(r, P(f)) \leq \gamma_p N_{\alpha}(r, f) + (\Gamma_p - \gamma_p) \overline{N}_{\alpha}(r, f) + S_{\alpha}(r, f).$$

Thus, we get

$$T_{a}(r, P(f)) = \gamma_p T_{a}(r, f) + (\Gamma_p - \gamma_p) \overline{T}_{a}(r, f) + S_{a}(r, f).$$

Dividing by $T_{a}(r, f)$ and taking limit sup on both sides, we get
\[
\limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} = \gamma_p + (\Gamma_p - \gamma_p) \limsup_{r \to \infty} \frac{N_a(r, f)}{T_a(r, f)}
\]
\[
\limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \gamma_p + (\Gamma_p - \gamma_p)(1 - \Theta_a(\infty, f))
\]
\[
\leq \Gamma_p - (\Gamma_p - \gamma_p) \Theta_a(\infty, f) \quad \ldots (2)
\]

on the other hand by Lemma 3.1.1, we have
\[
\gamma_p \sum_{j=1}^{q} m_a(r, a_j, f) \leq T_a(r, P(f)) + S_a(r, f)
\]
Hence
\[
\gamma_p \sum_{j=1}^{q} \delta_a(a_j) \leq \liminf_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)}.
\]
\[
\ldots (3)
\]
From (2) and (3), the result follows
\[
\gamma_p \sum_{a \in C} \delta_a(a) \leq \liminf_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq (\Gamma_p - (\Gamma_p - \gamma_p) \Theta_a(\infty, f))
\]

Corollary 3.1.1: If \( f(z) \) is meromorphic function with \( \sum \delta_a(a) = 1 \) and \( \delta_a(\infty) = 1 \), then
\[
T_a(r, P(f)) \sim \gamma_p T_a(r, f).
\]

Proof: By theorem 3.1.1,
\[
\gamma_p \sum_{a \in C} \delta_a(a) \leq \liminf_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq (\Gamma_p - (\Gamma_p - \gamma_p) \Theta_a(\infty, f))
\]
\[
\gamma_p \leq \liminf_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \gamma_p
\]
\[
\therefore \lim_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} = \gamma_p \quad \Leftrightarrow T_a(r, P(f)) \sim \gamma_p T_a(r, f).
\]

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Corollary 3.1.2: If $\Gamma_p = 2$, $\gamma_p = 1$ and $\sum_{a \in \mathbb{C}} \delta_\alpha(a,f) = 2$, then

$$\lim_{r \to \infty} \frac{T_\alpha(r, P(f))}{T_\alpha(r,f)} = 2 - \Theta_\alpha(\infty,f).$$

Proof: Using the hypothesis the above theorem takes the form

$$2 - \Theta_\alpha(\infty,f)) \leq 2 - \delta_\alpha(\infty,f) \leq \lim_{r \to \infty} \inf \frac{T_\alpha(r, P(f))}{T_\alpha(r,f)}$$

$$\leq \lim_{r \to \infty} \sup \frac{T_\alpha(r, P(f))}{T_\alpha(r,f)} \leq 2 - \Theta_\alpha(\infty,f)) .$$

Sarangi and Patil [21] (1979) have proved the following result.

Theorem C. If $f(z)$ is a transcendental meromorphic function in $|z| < \infty$, then

$$\frac{1}{(1+1) - 1\Theta_\alpha(\infty)} \sum_{a \neq \infty} \delta_\alpha(a) \leq \delta_\alpha(0, f^{(1)}).$$

We wish to extend the above result to homogeneous differential polynomials of

Theorem 3.1.1.

Theorem 3.1.2: If $f(z)$ is a transcendental meromorphic function in the finite plane, $P(f)$ is a homogeneous differential polynomial, then

$$\frac{\gamma_p}{(\Gamma_p - (\Gamma_p - \gamma_p)\Theta_\alpha(\infty,f))} \sum_{a \in \mathbb{C}} \delta_\alpha(a) \leq \delta_\alpha(0, P(f)).$$

Proof: By Lemma 3.1.1.

$$\gamma_p \sum_{i=1}^{q} m_\alpha(r, a_i, f) \leq T_\alpha(r, P(f)) - N_\alpha(r, O, P(f)) + S_\alpha(r, f).$$
Dividing by $T_a(r,f)$ and taking limit sup on both sides, we get

\[ \gamma_p \sum_{a \in C} \delta_a(a, f) \leq \lim_{r \to \infty} \sup \frac{T_a(r, P(f))}{T_a(r, f)} \frac{N_a(r, \frac{1}{P(f)})}{T_a(r, f)} + \lim_{r \to \infty} \sup \frac{S_a(r, f)}{T_a(r, f)} \]

\[ \leq \lim_{r \to \infty} \sup \frac{T_a(r, P(f))}{T_a(r, f)} \frac{N_a(r, \frac{1}{P(f)})}{T_a(r, P(f))} \frac{T_a(r, P(f))}{T_a(r, f)} + \lim_{r \to \infty} \sup \frac{S_a(r, f)}{T_a(r, f)} \]

\[ \leq \left( 1 - \lim_{r \to \infty} \sup \frac{N_a(r, \frac{1}{P(f)})}{T_a(r, P(f))} \right) \lim_{r \to \infty} \sup \frac{T_a(r, P(f))}{T_a(r, f)} \]

Or,

\[ \gamma_p \sum_{a \in C} \delta_a(a) \leq \delta_a(0, P(f)) \times \lim_{r \to \infty} \sup \frac{T_a(r, P(f))}{T_a(r, f)} \]

By Theorem 3.1.1, we have

\[ \gamma_p \sum_{a \in C} \delta_a(a) \leq \delta_a(0, P(f)) \times \lim_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \]

\[ \leq \delta_a(0, P(f)) (\Gamma_p - (\Gamma_p - \gamma_p) \Theta_\alpha(\infty, f)) \]

Hence the proof of the theorem.

**Corollary 3.1.3:** If $f(z)$ is a meromorphic function with

\[ \delta_\alpha(\infty) = 1 = \Theta_\alpha(\infty, f), \]

then

\[ \sum_{a \in C} \delta_a(a) \leq \delta_a(0, P(f)) \]

Another immediate consequences of theorem 3.1.2, is the following.
Corollary 3.1.4: If \( f(z) \) is a meromorphic function with at least one modified deficient value then '0' is the modified deficient value of \( P(f) \).

Singh and Sarangi [24] (1973) have proved the following result:

Theorem D. If \( f(z) \) is a meromorphic function of finite order \( \rho \) with
\[
\sum_{a \in C} \delta_a(a, f^{(k)}) = 2,
\]
then \( \rho \) is a positive integer. This theorem was proved even for functions of \( f(z) \) infinite order.

Sarangi and Patil [21] (1979) have also proved the following result.

Theorem E. If \( f(z) \) is a meromorphic function in \( |z| < \infty \) with
\[
\sum_a \delta_a(a, f') = 2,
\]
the order \( \rho \) of \( f \) is a positive integer if \( \rho < \infty \) or \( \rho \) is infinite.

We shall prove similar results for homogeneous differential polynomials.

Theorem 3.1.3: If \( f(z) \) is a meromorphic function in the finite plane, with
\[
\sum_a \delta_a(a, P(f)) = 2.
\]
If \( P(f) \) is a homogeneous differential polynomial (with \( \Gamma \rho = 1 \)), then the order \( \rho \) of \( f \) is a positive integer if \( \rho < \infty \) or \( \rho \) is infinite.

For the proof of this result, we require the following lemma.

Lemma 3.1.4: If \( f(z) \) is a meromorphic function in the finite plane with
\[
\sum_{a \in C} \delta_a(a) = 1 \quad \text{and} \quad \delta_a(\infty) = 1,
\]
then \( \delta_a(O, P(f)) = 1 \) and \( \delta_a(\infty, P(f)) = 1 \).
Proof: By Theorem 3.1.2, we have

\[ \frac{\gamma_p}{(\Gamma_p - (\Gamma_p - \gamma_p)\Theta_\alpha(\infty,f)) \sum_{a \in C} \delta_\alpha(a) \leq \delta_\alpha(0,P(f)).} \]

Since, \(1 = \delta_\alpha(\infty) \leq \Theta_\alpha(\infty) \leq 1, \Theta_\alpha(\infty,f) = 1,\) we have

\[ \sum_{a \in C} \delta_\alpha(a) \leq \delta_\alpha(0,P(f)). \]

Since, \(\sum_{a \in C} \delta_\alpha(a) = 1\) we have \(\delta_\alpha(0,P(f)) = 1\)

By Corollary 3.1.1, we have

\[ T_\alpha(r,P(f)) \sim \gamma_p T_\alpha(r,f), \] (4)

also we know that, by Lemma 3.1.3

\[ N(r,P(f)) \leq \gamma_p N(r,f) + (\Gamma_p - \gamma_p)N(r,f) + S(r,f) \]

This implies,

\[ N_\alpha(r,P(f)) \leq \gamma_p N_\alpha(r,f) + (\Gamma_p - \gamma_p)N_\alpha(r,f) + S_\alpha(r,f) \]

\[ \leq \gamma_p N_\alpha(r,f) + (\Gamma_p - \gamma_p)N_\alpha(r,f) + S_\alpha(r,f) \]

\[ \leq \Gamma_p N_\alpha(r,f) + S_\alpha(r,f) \] (5)

Now by (4), (5) and \(\delta_\alpha(\infty) = 1,\) it follows that

\[ \frac{N_\alpha(r,P(f))}{T_\alpha(r,P(f))} \times \frac{T_\alpha(r,P(f))}{T_\alpha(r,f)} \leq \Gamma_p \frac{N_\alpha(r,f)}{T_\alpha(r,f)} + \frac{S_\alpha(r,f)}{T_\alpha(r,f)}. \]
Hence
\[
\limsup_{r \to \infty} \frac{N_\alpha(r, P(f))}{T_\alpha(r, P(f))} \times \gamma_p \leq \Gamma_p \left(1 - \delta_\alpha(\infty, P(f))\right)
\]
\[
\limsup_{r \to \infty} \frac{N_\alpha(r, P(f))}{T_\alpha(r, P(f))} \times \gamma_p \leq 0
\]
\[
\gamma_p (1 - \delta_\alpha(\infty, P(f))) \leq 0
\]
\[
1 \leq \delta_\alpha(\infty, P(f)), \quad \therefore \delta_\alpha(\infty, P(f)) = 1.
\]

**Proof of Theorem 3.1.3:**

Since, \(2 = \sum \delta_\alpha(a, P(f)) \leq \sum \Theta_\alpha(a, P(f)) \leq 2\),

we have \(\delta_\alpha(\infty, P(f)) = \Theta_\alpha(\infty, P(f))\)

By Lemma 3.1.3

\[
N_\alpha(r, \infty, P(f)) \geq \gamma_p \overline{N}_\alpha(r, f) + (\Gamma_p - \gamma_p) \overline{N}_\alpha(r, f) + S_\alpha(r, f)
\]
\[
\geq \Gamma_p \overline{N}_\alpha(r, \infty, f) = \Gamma_p \overline{N}_\alpha(r, \infty, P(f))
\]

Hence \(1 - \delta_\alpha(\infty, P(f)) \geq \Gamma_p (1 - \Theta_\alpha(\infty, P(f)))\),

Or, \(1 - \delta_\alpha(\infty, P(f)) \geq \Gamma_p - \Gamma_p \delta_\alpha(\infty, P(f))\), by (6)

Thus, \((\Gamma_p - 1) \delta_\alpha(\infty, P(f)) \geq \Gamma_p - 1\)

\(\Rightarrow \delta_\alpha(\infty, P(f)) \geq 1\), since \(\Gamma_p \neq 1\).

Therefore, \(\delta_\alpha(\infty, P(f)) = 1\). Hence \(\sum_{a \in \mathbb{C}} \delta_\alpha(a, P(f)) = 1\)
Now by Lemma 3.1.4., we have

\[ \delta_\alpha(\infty, P(f)) = 1 = \delta_\alpha(0, P(f)) \].

Hence

\[ K_\alpha(P(f)) \leq 2 - \delta_\alpha(\infty, P(f)) - \delta_\alpha(0, P(f)) = 0 \]

where

\[ K_\alpha(P(f)) = \limsup_{r \to \infty} \frac{N_\alpha(r, P(f)) + N_\alpha(r, \frac{1}{P(f)})}{T_\alpha(r, P(f))} \]

Hence by, Toda [27] (1970) \( \rho \) is a positive integer if \( \rho < \infty \) or \( \rho \) is infinite.
Section II

In this section we prove theorems on sum of the modified deficiencies. P(f) is a general differential polynomial of degree one and weight two. This P(f) is not containing f.

Sarangi and Patil [21] proved the following results.

Theorem F. Let f(z) be a meromorphic function in \(|z|<\infty\) with \(\{a_i\}\) as modified deficient values such that \(\sum_{a \in \mathbb{C}} \delta_\alpha(a_i) = \beta, \ a_i \neq \infty\), \(\sum_{a \in \mathbb{C}} \delta_\alpha(a_i) = 2\) where a_i's are distinct \((0 \leq |a| \leq \infty)\), then \(T_\alpha(r,f') \sim T_\alpha(r,f)\).

Theorem G. If f(z) is a transcendental meromorphic function in \(|z|<\infty\) with \(\{a_i\}\) as modified deficient values, such that \(\sum_{a \in \mathbb{C}} \delta_\alpha(a_i) = \beta, \ a_i \neq \infty\) and \(\delta_\alpha(\infty) = 2 - \beta\), then \(\frac{\beta-1}{\beta} \leq K_\alpha(f') \leq \frac{2(\beta-1)}{\beta}\),

where \(K_\alpha(f') = \lim_{r \to \infty} \sup_{T_\alpha(r,f')} \frac{N_\alpha(r,f') + N_\alpha(r,f')^{-1}}{T_\alpha(r,f')}\).

Here we prove the above results for differential polynomials.

Theorem 3.1.4: Let f(z) be a meromorphic function in the finite plane with \(\{a_i\}\) as modified deficient values such that \(\sum_{a \in \mathbb{C}} \delta_\alpha(a_i) = \beta, \ \sum_{a \in \mathbb{C}} \delta_\alpha(a_i) = 2\) where a_i’s are distinct \((0 \leq |a_i| \leq \infty)\), and in addition \(\Gamma_p = 2, \ \gamma_p = 1\), then
\[ T_\alpha (r, P(f)) \sim \beta T_\alpha (r, f). \]

**Proof:** From Lemma 3.1.1, we have
\[ \gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq m_\alpha (r, \frac{1}{P(f)}) + S_\alpha (r, f). \]

Dividing by \( T_\alpha (r, f) \) and simplifying, we get
\[ \gamma_P \sum_{i=1}^{q} \delta_\alpha (a_i, f) \leq \Delta_\alpha (0, P(f)) \lim \inf_{r \to \infty} \frac{T_\alpha (r, P(f))}{T_\alpha (r, f)} + S_\alpha (r, f). \]

Now given \( \varepsilon > 0 \), we can choose \( a_1, a_2, \ldots, a_q \) (\( q \geq 3 \)) so that \( \sum_{i=1}^{q} \delta_\alpha (a_i) < \varepsilon \) (\( a_i \neq \infty \))

Hence \( \sum_{i=1}^{q} \delta_\alpha (a_i) > \beta - \varepsilon \) (\( a_i \neq \infty \))

Since \( 1 \leq \beta \leq 2 \) and \( \delta_\alpha (\infty) = 2 - \beta \), it follows that
\[ \sum_{i=1}^{q} \delta_\alpha (a_i) > \beta - \varepsilon > 0 \]

So by (7),
\[ \frac{\gamma_P (\beta - \varepsilon)}{\Delta_\alpha (0, P(f))} \leq \lim \inf_{r \to \infty} \frac{T_\alpha (r, P(f))}{T_\alpha (r, f)} \]

Also we know that
\[ \limsup_{r \to \infty} \frac{T_\alpha (r, P(f))}{T_\alpha (r, f)} \leq \Gamma_P - (\Gamma_P - \gamma_P) \Theta_\alpha (\infty, f) \]

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\[ \leq 2 - \Theta_a(\infty, f), \text{ by hypothesis} \]
\[ \leq 2 - \delta_a(\infty, f) \]
\[ = 2 - (2 - \beta) \quad (\because \delta_a(\infty) = 2 - \beta) \]
\[ \leq \beta \quad (1 \leq \beta \leq 2) \]

Hence from (9), we have

\[ \frac{\gamma_p (\beta - \varepsilon)}{\Delta_a(0, P(f))} \leq \liminf_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \limsup_{r \to \infty} \frac{T_a(r, P(f))}{T_a(r, f)} \leq \beta. \]

Thus, \( T_a(r, P(f)) \sim T_a(r, f) \) and \( \Delta_a(0, P(f)) = 1, \ (\gamma_p = 1, \text{ by hypothesis}) \)

**Corollary 3.1.5:** If \( \sum_{a \in \mathbb{C}} \delta_a(a) = 1 \) and \( \delta_a(\infty) = 1 \), in the above corollary, then

\[ T_a(r, P(f)) \sim T_a(r, f). \]

**Theorem 3.1.5:** If \( f(z) \) is a transcendental meromorphic function in the finite plane with \( \{a_i\} \) as modified deficient values, such that \( \sum \delta_a(a_i) = \beta > 0 \)

\( (a_i \neq \infty) \) and \( \delta_a(\infty) = 2 - \beta \), \( \gamma_p = 1 \) and \( \Gamma_p = 2 \), then

\[ \frac{\beta - 1}{\beta} \leq K_\alpha(P(f)) \leq \Gamma_p \frac{\beta - 1}{\beta} \]

where \( K_\alpha(P(f)) = \lim_{r \to \infty} \sup \frac{N_\alpha(r, P(f)) + N_\alpha(r, \frac{1}{P(f)})}{T_a(r, P(f))} \)

**Proof:** By Lemma 3.1.3

\[ N(r, P(f)) \leq \gamma_p N(r, f) + (\Gamma_p - \gamma_p)N(r, f) + S(r, f) \]

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This implies,
\[ N_\alpha(r, P(f)) \leq \gamma_P N_\alpha(r, f) + (\Gamma_P - \gamma_P) \bar{N}_\alpha(r, f) + S_\alpha(r, f) \]
\[ \leq \gamma_P N_\alpha(r, f) + (\Gamma_P - \gamma_P) N_\alpha(r, f) + S_\alpha(r, f) \]
\[ \leq \Gamma_P N_\alpha(r, f) + S_\alpha(r, f) \quad \ldots \quad (1') \]

So
\[ \frac{N_\alpha(r, P(f))}{T_\alpha(r, P(f))} \leq \frac{N_\alpha(r, f)}{T_\alpha(r, f)} + \frac{S_\alpha(r, f)}{T_\alpha(r, f)}. \]

The equalities \( \sum \delta_\alpha(a_i) = \beta \) and \( \sum \delta_\alpha(a_i) = 2, \)

This imply that
\[ T_\alpha(r, P(f)) \sim \beta T_\alpha(r, f), \text{ by Theorem 4.1.1} \]

Hence we have
\[
\beta \lim_{r \to \infty} \sup \frac{N_\alpha(r, P(f))}{T_\alpha(r, P(f))} \leq \Gamma_P (1 - \delta_\alpha(\infty, f))
\]
\[ = \Gamma_P (1 - (2 - \beta)) \]
\[ = \Gamma_P (\beta - 1) \]

Therefore,
\[
\lim_{r \to \infty} \sup \frac{N_\alpha(r, P(f))}{T_\alpha(r, P(f))} \leq \frac{\Gamma_P (\beta - 1)}{\beta}.
\]

Also we know that
\[
\lim_{r \to \infty} \frac{N_\alpha(r, \frac{1}{P(f)})}{T_\alpha(r, P(f))} = 0
\]
Hence

\[ K_\alpha (P(f)) \leq \frac{\Gamma_p (\beta - 1)}{\beta} \]  \quad \ldots \quad (11)

Further we have

\[ N(r, f) \leq N(r, P(f)) . \]

This implies,

\[ N_\alpha (r, f) \leq N_\alpha (r, P(f)) \]

Therefore

\[ \frac{N_\alpha (r, f)}{T_\alpha (r, f)} \leq \frac{N_\alpha (r, P(f))}{T_\alpha (r, P(f))} \times \frac{T_\alpha (r, P(f))}{T_\alpha (r, f)} \]

\[ = \beta \frac{N_\alpha (r, P(f))}{T_\alpha (r, P(f))} . \]

Hence

\[ \lim_{r \to \infty} \sup \frac{N_\alpha (r, P(f))}{T_\alpha (r, P(f))} \geq \frac{1}{\beta} \lim_{r \to \infty} \sup \frac{N_\alpha (r, f)}{T_\alpha (r, f)} , \]

\[ = \frac{1}{\beta} (1 - \delta_\alpha (\infty, f) \]

\[ = \frac{1}{\beta} (1 - (2 - \beta)) \]

\[ = \frac{\beta - 1}{\beta} \]

Therefore,

\[ K_\alpha (P(f)) \geq \frac{\beta - 1}{\beta} \]  \quad \ldots \quad (12)

Combining (11) and (12), we get

\[ \frac{\beta - 1}{\beta} \leq K_\alpha (P(f)) \leq \Gamma_p \frac{\beta - 1}{\beta} . \]

Hence the proof of the theorem.
3.2 Exceptional and Deficient Values of Differential Polynomials

Bhoosnurmath [1] has proves the following results. For the sake of completeness, we record the results again.

**Theorem A.** Let \( f \) be a meromorphic function of finite order.

(a) If \( P \) is a homogeneous differential polynomial in \( f \) of degree \( n \), then

\[
\limsup_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq n[(m + 1) - m\Theta(\infty, f)],
\]

where \( f^{(m)} \) is the highest derivative of \( f \) occurring in \( P \) \((m \geq 0)\). As usual, \( f^{(0)} \) stands for \( f \).

(b) If \( P \) is a homogeneous differential polynomial in \( f \) of degree \( n \) and if \( P \) does not involve \( f \), then

\[
\sum_{b \in \mathbb{C}} \delta(b, f) \leq \delta(O, P) \limsup_{r \to \infty} \frac{T(r, P)}{T(r, f)}
\]

and

\[
\sum_{b \in \mathbb{C}} \delta(b, f) \leq \Delta(O, P) \liminf_{r \to \infty} \frac{T(r, P)}{T(r, f)}.
\]

Provided that \( P \) does not reduce to a constant.

**Theorem B:** Let \( f \) be a meromorphic function of finite order and \( P \) is a homogeneous differential polynomial in \( f \) of degree \( n \).

Let \( \alpha = \limsup_{r \to \infty} \frac{\overline{N}(r, f) + N(r, \frac{1}{f})}{T(r, f)}. \)
Then
\[ n(1 - m\alpha) \leq \liminf_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq \limsup_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq n(1 + m\alpha). \]

where \( f^{(m)} \) is the highest derivative of \( f \) occurring in \( P \), provided that \( P \) does not reduce to a constant.

We wish to prove the above results for general differential polynomials, that is, not necessarily for homogeneous differential polynomials.

**Theorem 3.2.1:** Let \( f \) be a meromorphic function of finite order.

(a) If \( P \) is a differential polynomial in \( f \) of degree \( \gamma_p \), then
\[
\limsup_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} \leq \Gamma_p - (\Gamma_p - \gamma_p)\Theta_\alpha(\infty, f) + (\gamma_p - \gamma_p') + \Delta_\alpha(0, f).
\]

where \( \gamma_p \) is the degree and \( \Gamma_p \) the weight of \( P(f) \) and \( \gamma_p' \) is lower degree of \( P \).

(b) If \( P \) is a differential polynomial in \( f \) of degree \( \gamma_p \) and if \( P \) does not involve \( f \), then
\[
\gamma_p \sum \delta(a_i, f) \leq \delta_\alpha(0, P) \limsup_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} \quad \ldots(2)
\]

and
\[
\gamma_p \sum \delta(a_i, f) \leq \Delta_\alpha(0, P) \liminf_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} \quad \ldots(3)
\]

provided that \( P \) does not reduce to a constant.

To establish the above results we require the following lemmas.
Lemma 3.2.1: [2] Suppose \( f(z) \) is a transcendental meromorphic function and \( P(f) \) is a differential polynomial, then

\[
m_\alpha(r, f) \leq (\gamma_P - \gamma_P) m_\alpha(r, \frac{1}{f}) + S_\alpha(r, f)
\]

Lemma 3.2.2: [11] \( N_\alpha(r, P(f)) \leq \gamma_P(N_\alpha(r, f) + (\Gamma_P - \gamma_P) N_\alpha(r, f) + S_\alpha(r, f)) \)

Lemma 3.2.3: If \( f(z) \) is a transcendental meromorphic function and \( a_1, a_2, \ldots, a_q \) are distinct elements, then

\[
\gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq \sum_{j=1}^{q} (\gamma_P - \gamma_P) m_\alpha (r, \frac{1}{f - a_i}) + m_\alpha (r, 0, P(f)) + S_\alpha (r, f).
\]

where \( P(f) \) is a differential polynomial of degree \( \gamma_P \) and \( P(f) \) is not involving in \( f \).

**Proof:** Without loss of generality we may assume that \( q \geq 2 \).

Let \( F(z) = \sum_{j=1}^{q} \frac{1}{(f(z) - a_j)^{\gamma_P}} \) and proceeding as in Hayman [9] (Pp. 32-33), we get

\[
\gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq m_\alpha (r, F) + O(1)
\]

\[
\leq \sum_{i=1}^{q} m_\alpha (r, \frac{P(f - a_i)}{(f - a_i)^{\gamma_P}}) + m_\alpha (r, 0, P(f)) + O(1).
\]

\[
\therefore \gamma_P \sum_{i=1}^{q} m_\alpha (r, a_i, f) \leq \sum_{j=1}^{q} (\gamma_P - \gamma_P) m_\alpha (r, \frac{1}{f - a_i}) + m_\alpha (r, 0, P(f)) + S_\alpha (r, f).
\]
Proof of Theorem 3.2.1: By using Lemma 3.2.1 and Lemma 3.2.2, we have
\[ m_{\alpha}(r, P(f)) \leq \gamma_{p} m_{\alpha}(r, f) + (\gamma_{p} - \gamma_{p}) m_{\alpha}(r, f) + S_{\alpha}(r, f). \]

Also
\[ N_{\alpha}(r, P(f)) \leq \gamma_{p} N_{\alpha}(r, f) + (\Gamma_{p} - \gamma_{p}) N_{\alpha}(r, f) + S_{\alpha}(r, f), \]
we get
\[ T_{\alpha}(r, P(f)) \leq \gamma_{p} T_{\alpha}(r, f) + (\Gamma_{p} - \gamma_{p}) N_{\alpha}(r, f) + (\gamma_{p} - \gamma_{p}) m_{\alpha}(r, f) + S_{\alpha}(r, f). \]

Dividing by \( T_{\alpha}(r, f) \) and taking \limsup \ on both sides, we get
\[ \limsup_{r \to \infty} \frac{T_{\alpha}(r, P(f))}{T_{\alpha}(r, f)} \leq \gamma_{p} + (\Gamma_{p} - \gamma_{p}) \limsup_{r \to \infty} \frac{N_{\alpha}(r, f)}{T_{\alpha}(r, f)} + (\gamma_{p} - \gamma_{p}) \limsup_{r \to \infty} \frac{m_{\alpha}(r, f)}{T_{\alpha}(r, f)}. \]

\[ \limsup_{r \to \infty} \frac{T_{\alpha}(r, P(f))}{T_{\alpha}(r, f)} \leq \gamma_{p} + (\Gamma_{p} - \gamma_{p})(1 - \Theta_{\alpha}(\infty, f)) + (\gamma_{p} - \gamma_{p}) \Delta_{\alpha}(0, f) \]
\[ = \Gamma_{p} - (\Gamma_{p} - \gamma_{p}) \Theta_{\alpha}(\infty, f) + (\gamma_{p} - \gamma_{p}) \Delta_{\alpha}(0, f) \]
which proves (a).

(b) Suppose \( P \) does not involve \( f \) and is not a constant. Let \( \{a_i\} \) be an infinite sequence of distinct element of \( C \) which includes every \( a \in C \) for which \( \delta(a, f) > 0 \).
Let $q$ be a positive integer, then by Lemma 3.2.3

$$\gamma_p \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq \sum_{j=1}^{q} \left( \gamma_p - \frac{1}{P(f)} \right) m_{\alpha}(r, \frac{1}{f - a_i}) + m_{\alpha}(r, 0, P(f)) + S_{\alpha}(r, f).$$

Or,

$$\gamma_p \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq (\gamma_p - \frac{1}{P(f)}) \sum_{j=1}^{q} m_{\alpha}(r, a_i, f) + m_{\alpha}(r, \frac{1}{P(f)}) + S_{\alpha}(r, f).$$

Or,

$$\gamma_p \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq m_{\alpha}(r, \frac{1}{P(f)}) + S_{\alpha}(r, f)$$

Or,

$$\gamma_p \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) + N_{\alpha}(r, \frac{1}{P(f)}) \leq T_{\alpha}(r, P) + S_{\alpha}(r, f)$$

Adding $m_{\alpha}(r, \frac{1}{P(f)})$ on both sides, and using

$$T_{\alpha}(r, \frac{1}{P(f)}) = T_{\alpha}(r, P) + O(1),$$

we get

$$\gamma_p \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq m_{\alpha}(r, \frac{1}{P(f)}) + S_{\alpha}(r, f).$$

Hence

$$\gamma_p \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m_{\alpha}(r, a_i, f)}{T_{\alpha}(r, f)} \leq \liminf_{r \to \infty} \frac{m_{\alpha}(r, \frac{1}{P(f)})}{T_{\alpha}(r, f)} \leq \liminf_{r \to \infty} \frac{m_{\alpha}(r, \frac{1}{P(f)})}{T_{\alpha}(r, P)} \limsup_{r \to \infty} \frac{T_{\alpha}(r, P)}{T_{\alpha}(r, f)}$$

and

$$\gamma_p \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m_{\alpha}(r, a_i, f)}{T_{\alpha}(r, f)} \leq \liminf_{r \to \infty} \frac{m_{\alpha}(r, \frac{1}{P(f)})}{T_{\alpha}(r, f)}$$
that is
\[
\gamma_p \sum_{i=1}^{q} \delta_{\alpha}(a_i, f) \leq \delta_{\alpha}(0, P) \limsup_{r \to \infty} \frac{T_a(r, P)}{T_a(r, f)}
\]

and
\[
\gamma_p \sum_{i=1}^{q} \delta_{\alpha}(a_i, f) \leq \Delta_{\alpha}(0, P) \liminf_{r \to \infty} \frac{T_a(r, P)}{T_a(r, f)},
\]

letting \( q \to \infty \), we obtain, (2) and (3) respectively.

The following corollaries are immediate consequences of this Theorem.

**Corollary 3.2.1:** If \( f \) is meromorphic function of finite order

with \( \sum_{a \in C} \delta_{\alpha}(a, f) = 1 \) and \( \Theta_{\alpha}(\infty, f) = 1 \) and if \( P \) is differential polynomial in \( f \) of degree \( \gamma_P \) containing \( f \), then

\[ \delta_{\alpha}(0, P) = \Delta_{\alpha}(0, P) = 1 \]...

(4)

and

\[ T_a(r, P) \sim \gamma_P T_a(r, f) , \]...

(5)

provided that \( P[f] \) does not reduce to a constant. In particular (2) and (3) hold,

if \( f \) is an entire function and \( \sum_{a \in C} \delta_{\alpha}(a, f) = 1 \).

**Corollary 3.2.2:** If \( f \) is a meromorphic function of finite order with

\( \Theta_{\alpha}(\infty, f) = 1 \) and if \( P \) is a differential polynomial in \( f \) and not involving \( f \), then
\[ \sum_{\alpha \in \mathbb{C}} \delta_{\alpha}(a,f) \leq \delta_{\alpha}(0,P) \] (6)

provided that \( P[f] \) is not a constant.

If \( f \) is an entire function, then (6) holds since \( \Theta(\infty,f) = 1 \).

We have proved theorem 3.2.1(b), when \( P \) is a differential polynomial in \( f \) and \( P \) does not involve \( f \), if \( P \) involves \( f \), then we have the following

**Theorem 3.2.2:** Let \( f \) be a meromorphic function of finite order and \( P \) is a differential polynomial in \( f \) of degree \( \gamma_P \).

Let

\[ \beta = \limsup_{r \to \infty} \frac{\overline{N}_\alpha(r,f) + \overline{N}_\alpha(r,\frac{1}{f})}{T_\alpha(r,f)}. \]

Then

\[ \gamma_P - k\beta \gamma_P \leq \liminf_{r \to \infty} \frac{T_\alpha(r,P)}{T_\alpha(r,f)} \leq \limsup_{r \to \infty} \frac{T_\alpha(r,P)}{T_\alpha(r,f)} \leq \gamma_P[k\beta + 2] - \gamma_P \]

where \( f^{(k)} \) is the highest derivative of \( f \) occuring in \( P \), provided that \( \beta \) does not reduce to a constant.

**Proof:** Since a zero or pole of \( f \), which is not a pole of any coefficient \( a(\alpha) \), is not a pole of \( \frac{P}{f^{\gamma_P}} \), \( P \), is a pole of \( \frac{P}{f^{\gamma_P}} \) of degree \( k\gamma_P \) at most, we have

\[ N_\alpha(r,\frac{P}{f^{\gamma_P}}) \leq k\gamma_P \left[ \overline{N}_\alpha(r,f) + \overline{N}_\alpha(r,\frac{1}{f}) \right] + S_\alpha(r,f) \] (7)
Now,

\[ T_a\left(r, \frac{P}{f^{\gamma r}}\right) = m_a\left(r, \frac{P}{f^{\gamma r}}\right) + N_a\left(r, \frac{P}{f^{\gamma r}}\right). \]

\[ T_a\left(r, \frac{P}{f^{\gamma r}}\right) \leq k\gamma_p \left( \overline{N}_a(r,f) + \overline{N}_a(r,\frac{1}{f}) \right) + (\gamma_p - \gamma_p)m_a(r,\frac{1}{f}) + S_a(r,f) \quad \text{(9)} \]

(by (8) and Lemma 3.2.1).

Therefore

\[ T_a(r,P) \leq T_a\left(r, \frac{P}{f^{\gamma r}}\right) + T_a(r,f^\gamma) \]

\[ \leq k\gamma_p \left( \overline{N}_a(r,f) + \overline{N}_a(r,\frac{1}{f}) \right) + (\gamma_p - \gamma_p)m_a(r,\frac{1}{f}) + \gamma_p T_a(r,f) + S_a(r,f) \]

\[ \limsup_{r \to \infty} \frac{T_a(r,P)}{T_a(r,f)} \leq k\gamma_p \left( \limsup_{r \to \infty} \frac{\overline{N}_a(r,f) + \overline{N}_a(r,\frac{1}{f})}{T_a(r,f)} \right) + (\gamma_p - \gamma_p) \limsup_{r \to \infty} \frac{T_a(r,f)}{T_a(r,f)} \]

\[ + \gamma_p \limsup_{r \to \infty} \frac{T_a(r,f)}{T_a(r,f)} \]

\[ \leq k(\beta)\gamma_p + (\gamma_p - \gamma_p) + \gamma_p \]

\[ \leq \gamma_p(k\beta + 2) - \gamma_p \]

\[ \limsup_{r \to \infty} \frac{T_a(r,P)}{T_a(r,f)} \leq \gamma_p(k\beta + 2) - \gamma_p \quad \text{(11)} \]

The right inequality in (7) follows from (11).
On the other hand,

\[ \gamma_p T_\alpha(r, f) = T_\alpha(r, f^\gamma) \]

\[ \leq T_\alpha(r, f^\gamma) + T_\alpha(r, P) \]

\[ \leq T_\alpha(r, P) + T_\alpha(r, \frac{P}{f^\gamma}) + O(1) \]

\[ \leq T_\alpha(r, P) + k\gamma_p \left( \overline{N}_\alpha(r, f) + \overline{N}_\alpha(r, \frac{1}{f}) \right) + (\gamma_p - \gamma_p) \alpha(r, \frac{1}{f}) + S_\alpha(r, f) \]

Dividing by \( T_\alpha(r, f) \) on both sides, taking limit inf, we get

\[ \gamma_p \leq \liminf_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} + k\gamma_p \left( \limsup_{r \to \infty} \frac{\overline{N}_\alpha(r, f) + \overline{N}_\alpha(r, \frac{1}{f})}{T_\alpha(r, f)} \right) + (\gamma_p - \gamma_p) \]

\[ \gamma_p \leq \liminf_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} + k\gamma_p \beta + (\gamma_p - \gamma_p) \text{, by hypothesis} \]

\[ \gamma_p - k\gamma_p \beta \leq \liminf_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} \]

which gives the left hand inequality in (7).

As an immediate consequence of Theorem 3.2.2, we have the following:

**Corollary 3.2.3:** Let \( f \) be a meromorphic function of finite order and \( P \) a differential polynomial in \( f \) of degree \( \gamma_p \). If \( \Theta_\alpha(\infty, f) = \Theta_\alpha(0, f) = 1 \), then for \( a \in \overline{C} \), we define with Milloux [14], the **relative modified defect** as follows:

\[ \delta^p_{\alpha, r}(a, f) = 1 - \limsup_{r \to \infty} \frac{N_\alpha(r, a, P)}{T_\alpha(r, f)} \]

\[ \delta^p_{\alpha, r}(a, f) = 1 - \limsup_{r \to \infty} \frac{N_\alpha(r, a, P)}{T_\alpha(r, f)} \]

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where $P$ is a differential polynomial in $f$ (Milloux considered only the special case $P = f'$). $\delta_{\alpha, f}(a, f)$ is called the relative defect of $a$ for $P$ with respect to $f$. We investigate the conditions under which the defects and the relative defects are equal. In this connection, we prove the following theorem.

Bhoosnurmath [1] has proved following result.

**Theorem C:** Let $f$ be a meromorphic function of finite order

a) If $P$ is a homogeneous differential polynomial in $f$ of degree $n$, then for $a \in \mathbb{C}$,

$$1 - \delta_{\alpha, f}(a, f) \leq (1 - \delta(a, P)n[(m + 1) - m\Theta(\infty, f)],$$

and

$$1 - \delta_{\alpha, f}(a, f) \geq (1 - \delta(a, P)n(1 - m\alpha),$$

where $\alpha$ is as in Theorem C

b) If $P$ is a homogeneous differential polynomial in $f$ of degree $n$, not involving in $f$, then

$$\Delta(0, P)(1 - \delta_{\alpha, f}(a, f)) \geq (1 - \delta(a, P)n \sum_{b \in \mathbb{C}} \delta(b, f))$$

for $a \in \mathbb{C}$, provided $P$ does not reduce to a constant.

We shall prove the above result.
Theorem 3.2.3: Let $f$ be a meromorphic function of finite order

(a) If $P$ is a differential polynomial in $f$ of degree $\gamma_P$, then $a \in \mathbb{C}$,

$$1 - \delta_{a,r}^{p}(a,f) \leq (1 - \delta_{a}(a,P))(\Gamma_{P} - (\Gamma_{P} - \gamma_{P})\Theta_{a}(\infty,f) + (\gamma_{P} - \gamma_{P})\Delta_{a}(0,f))... \ (14)$$

$$1 - \delta_{a,r}^{p}(a,f) \leq (1 - \delta_{a}(a,P))\gamma_{P}(k\beta + 2) - \gamma_{P} \quad ... \ (15)$$

and

$$1 - \delta_{a,r}^{p}(a,f) \geq (1 - \delta_{a}(a,P))(\gamma_{P} - k\beta\gamma_{P}) \quad ... \ (16)$$

where $\beta$ is as in Theorem 3.2.2

(b) If $P$ is a differential polynomial in $f$ of degree $\gamma_P$, not involving $f$, then

$$\Delta_{a}(0,P)(1 - \delta_{a,r}^{p}(a,f)) \geq (1 - \delta_{a}(a,P))\left(\gamma_{P} \sum_{a \in \mathbb{C}} \delta_{a}(a,f)\right) \quad ... \ (17)$$

for $a \in \mathbb{C}$, provided $P$ does not reduce to a constant.

Proof: (a) Now,

$$1 - \delta_{a,r}^{p}(a,f) = \limsup_{r \to \infty} \frac{N_{a}(r,a,P)}{T_{a}(r,f)}$$

$$\leq \limsup_{r \to \infty} \frac{N_{a}(r,a,P)}{T_{a}(r,P)} \limsup_{r \to \infty} \frac{T_{a}(r,P)}{T_{a}(r,f)}$$

$$\leq (1 - \delta_{a}(a,P))(\Gamma_{P} - (\Gamma_{P} - \gamma_{P})\Theta_{a}(\infty,f) + (\gamma_{P} - \gamma_{P})\Delta_{a}(0,f))$$

by using definition of $\delta_{a}(a,P)$ and Theorem 3.1.1 (a). Thus (14) is proved.

Similarly, using (7), we can prove (15) and (16).
(b) Now, $\Delta_\alpha(0, P)(1-\delta_{\alpha, r}(a, f))$

\[= \limsup_{r \to \infty} \frac{N_a(r, a, P)}{T_\alpha(r, f)} \Delta_\alpha(0, P)\]

\[\geq \limsup_{r \to \infty} \frac{N_a(r, a, P)}{T_\alpha(r, P)} \liminf_{r \to \infty} \frac{T_\alpha(r, P)}{T_\alpha(r, f)} \Delta_\alpha(0, P)\]

\[\geq (1-\delta_a(a, P)) \gamma_p \sum \delta_\alpha(a, f),\]

by using definition of $\delta_\alpha(a, P)$ and Theorem 3.2.1(b). Thus (b) is proved.

We then have the following corollaries.

**Corollary 3.2.4:** If $f$ is a meromorphic function of the finite order with

\[\sum_{a \in \mathbb{C}} \delta_\alpha(a, f) = 1 \text{ and } \Theta_\alpha(\infty, f) = 1, \text{ and if } P \text{ is a homogeneous differential polynomial in } f \text{ of degree } \gamma_p \text{ and if } P \text{ does not involve } f, \text{ then}

\[1-\delta_{\alpha, r}(a, f) = \gamma_p(1-\delta_\alpha(a, P))\]

for $a \in \overline{\mathbb{C}}$, provided that $P$ does not reduce to a constant.

**Proof:** Since

\[\sum_{a \in \mathbb{C}} \delta_\alpha(a, f) = 1 \text{ and } \Theta_\alpha(\infty, f) = 1, \text{ by Corollary 1.1.1., } \Delta_\alpha(0, P) = 1 \text{ and } \Delta_\alpha(0, f) = 1 \text{ from (14) and (17) the result follows.}

\[1-\delta_{\alpha, r}(a, f) \leq (1-\delta_a(a, P))(\Gamma_p - (\Gamma_p - \gamma_p)\Theta_\alpha(\infty, f) + (\gamma_p - \gamma_p))\Delta_\alpha(0, f)\]

\[\leq (1-\delta_a(a, P))(\Gamma_p - (\Gamma_p - \gamma_p)),\]

since $P$ is homogeneous $\gamma_p = \gamma_p$. 

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\[ \leq (1 - \delta_{a}(a,P))((\Gamma_p - \Gamma_P + \gamma_P)) \]

\[ \leq (1 - \delta_{a}(a,P))\gamma_P = (1 - \delta_{a}(a,P))\gamma_P \]

and by (17) of Corollary 3.2.1, we have

\[
\Delta_a(0,P)(1 - \delta_{a,r}(a,f)) \geq (1 - \delta_{a}(a,P))\left(\gamma_p \sum_{a \in \mathbb{C}} \delta_{a}(a,f)\right)
\]

Hence, \( (1 - \delta^P_{a,r}(a,f)) \geq (1 - \delta_{a}(a,P))\gamma_P \),

Thus, the result is proved.

If, in the above corollary, the degree of \( P = 1 \), then

\[ \delta^P_{a,r}(a,f) \geq \delta_{a}(a,P) \]

again from (15) and (16), we obtain the following.

**Corollary 3.2.5:** If \( f \) is a meromorphic function of finite order with \( \Theta_a(\infty, f) = \Theta_a(0, f) = 1 \) and if \( P \) is a homogeneous differential polynomial in \( f \) of degree \( \gamma_P \), then \( (1 - \delta^P_{a,r}(a,f)) = \gamma_p ((1 - \delta_{a}(a,P)) \).

Bhoosnurmath [1] has proved the following result.

**Theorem D:** Let \( f \) be a meromorphic function of finite order \( p \) and lower order \( \lambda \). Let \( P \) be a Differential polynomial in \( f \) of degree \( \gamma_P \geq 1 \), with each term in \( P \) containing the factor \( f \) and suppose that \( P \) is not identically zero. Let the non-zero terms of \( P \) have degree \( \gamma_1, \gamma_2, \ldots, \gamma_k \) with \( 1 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k = \gamma_P \).
and suppose that \( k = 1 \) or \( k > 1 \) and \( \sum_{i=1}^{k-1} \gamma_{E_i} > (k-2)\gamma_p \). Also let \( \rho_p, \lambda_p \) be the order and lower order \( P \) respectively. Then \( \rho_p = \rho \) and \( \lambda_p = \lambda \).

To prove the above Theorem, we use the following result of Bhoosnurmath [1]

\[
\left( \sum_{i=1}^{k-1} \gamma_{E_i} - (k-2)\gamma_p \right) T_\alpha(r,f) \leq (\gamma_p + 2)T_\alpha(r,P) + S_\alpha(r,f)
\]

...(18)

and

\[
T_\alpha(r,P) \leq (\gamma_p(k+1) + 2\gamma_p)T_\alpha(r,f) + S_\alpha(r,f)
\]

...(19)

**Theorem 3.2.4:** Let \( f \) be a meromorphic function of finite order with \( \Theta_\alpha(\infty,f) = 1 \) and \( P \) is a homogeneous differential polynomial in \( f \) of degree \( \gamma_p \). Let each term of \( P \) contain the factor \( f \). Then

(1) \( \Theta_\alpha(0,f) = 1 \) if and only if \( \Theta_\alpha(0,P) = 1 \)

and (2) \( \overline{\Delta}_\alpha(0,f) = 1 \) if and only if \( \overline{\Delta}_\alpha(0,P) = 1 \),

where

\[
\overline{\Delta}_\alpha(0,f) = 1 - \liminf_{r \to \infty} \frac{N_\alpha(r,0,f)}{T_\alpha(r,f)} \quad \text{and} \quad \overline{\Delta}_\alpha(0,P) \quad \text{is defined similarly}.
\]

**Proof:** Since \( f \) is finite order, we have

\[
S_\alpha(r,f) = O(T_\alpha(r,f)) \quad \text{and so, by (18) and (19)}
\]

\[
T_\alpha(r,f) \leq \alpha T_\alpha(r,P)
\]

...(21)
and
\[ T_{\alpha}(r, P) \leq \beta T_{\alpha}(r, f) \] 
for all large \( r \), where \( \alpha, \beta \) are positive constants.

Now,
\[ N_{\alpha}(r, -\infty) \leq N_{\alpha}(r, \infty) + N_{\alpha}(r, -\infty), \]
\[ \leq N_{\alpha}(r, \infty) + T_{\alpha}(r, \frac{P}{f^{\gamma_P}}) + 0(1) \]
\[ \leq N_{\alpha}(r, \infty) + \sum_{i=1}^{q} (\gamma_P - \gamma_P) m_{\alpha}(r, \frac{1}{f}) + N_{\alpha}(r, \frac{P}{f^{\gamma_P}}) + S_{\alpha}(r, f) \]
by Lemma 3.2.

Hence by (8)
\[ N_{\alpha}(r, \frac{1}{P}) \leq N_{\alpha}(r, \frac{1}{f}) + \sum_{i=1}^{q} (\gamma_P - \gamma_P) T_{\alpha}(r, \frac{1}{f}) + k\gamma_P (N_{\alpha}(r, f) + N_{\alpha}(r, \frac{1}{f})) + S_{\alpha}(f) \]

Since a zero or a pole of \( f \) is a pole of \( \frac{P}{f^{\gamma_P}} \) of order at most \( \gamma_P k \) and since \( P \) is homogeneous, \( \gamma_P = \gamma_P \) and hence
\[ N_{\alpha}(r, \frac{1}{P}) \leq (1 + k\gamma_P) N_{\alpha}(r, \frac{1}{f}) + 0(T_{\alpha}(r, f)) \]
since \( \Theta_{\alpha}(\infty, f) = 1 \) so that \( N_{\alpha}(r, f) = 0(T_{\alpha}(r, f)) \).

Thus \( \Theta_{\alpha}(0, f) = 1 \) implies \( \Theta_{\alpha}(0, P) = 1 \)

Since \( 0(T_{\alpha}(r, f)) = 0(T_{\alpha}(r, P)) \) an account of (20)
On the other hand, we have

\[ \overline{N}_\alpha(r, \frac{1}{f}) \leq \overline{N}_\alpha(r, \frac{1}{P}) \quad \text{and so} \quad \Theta_\alpha(0, P) = 1 \implies \Theta_\alpha(0, f) = 1 \text{ since} \]

\[ 0(T_a(r, P)) = 0(T_a(r, f)) \text{ on account of (21).} \]

Part (2) of theorem is obtained in a similar manner.

**Theorem 3.2.5:** Let \( f \) be a meromorphic function satisfying

\[ \overline{N}_\alpha(r, f) + \overline{N}_\alpha(r, \frac{1}{f}) = S_\alpha(r, f) \]

and let \( P \) be a differential polynomial in \( f \).

Suppose that \( P \) is not a constant. Then the order of \( P \) is equal to the order of \( f \)

and \( \overline{N}_\alpha(r, P) + \overline{N}_\alpha(r, \frac{1}{P}) = S_\alpha(r, P) \) so that \( \overline{N}_\alpha(r, \frac{1}{P - a}) \neq S_\alpha(r, P) \) and \( \Theta_\alpha(a, P) = 0 \) for all \( a \neq 0 \neq \infty \) and there exists no evB for \( P \) for distinct zeros in \( \mathbb{C} - \{0, \infty\} \).

**Proof:** If the degree of \( P \) is \( \gamma_p \), then by (12), we have

\[ \gamma_p T_a(r, f) \leq T_a(r, P) + k \gamma_p \left( \overline{N}_\alpha(r, f) + \overline{N}_\alpha(r, \frac{1}{f}) \right) + (\gamma_p - \gamma_p) m_\alpha(r, \frac{1}{f}) + S_\alpha(r, f) \]

\[ = T_a(r, P) + (\gamma_p - \gamma_p) m_\alpha(r, \frac{1}{f}) + S_\alpha(r, f) \]

\[ = T_a(r, P) + (\gamma_p - \gamma_p) T_a(r, f) + S_\alpha(r, f), \]

by hypothesis

from which it follows that

\[ S_\alpha(r, f) = S_\alpha(r, P) \quad \ldots (22) \]

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on the other hand, by (10)

\[ T_\alpha (r, P) \leq k\gamma_p \left( \overline{N}_\alpha (r, f) + \overline{N}_\alpha (r, f) \right) + (\gamma_p - \gamma_p) m_\alpha (r, f) + \gamma_p T_\alpha (r, f) + S_\alpha (r, f) \]

\[ \leq (\gamma_p - \gamma_p) T_\alpha (r, f) + \gamma_p T_\alpha (r, f) + S_\alpha (r, f) \]

by hypothesis

Hence the order of \( P \) and \( f \) are equal.

Now,

\[ \overline{N}_\alpha (r, P) \leq \overline{N}_\alpha (r, f) + S_\alpha (r, f) \]

\[ = S_\alpha (r, f) \quad \text{by hypothesis} \]

So by (22),

\[ \overline{N}_\alpha (r, P) = S_\alpha (r, P) . \]

As in the proof of above theorem, we have

\[ \overline{N}_\alpha (r, \frac{1}{P}) \leq \overline{N}_\alpha (r, \frac{1}{f}) \]

\[ + k\gamma_p \left( \overline{N}_\alpha (r, f) + \overline{N}_\alpha (r, f) \right) \]

\[ = (1 + k\gamma_p) \overline{N}_\alpha (r, \frac{1}{f}) + S_\alpha (r, f) \]

\[ + \sum_{i=1}^{q} (\gamma_p - \gamma_p) T_\alpha (r, \frac{1}{f}) + S_\alpha (r, f) \]

by hypothesis and \( \gamma_p = \gamma_p \)

\[ = S_\alpha (r, f) \]

by (22) and (24), we get

\[ \overline{N}_\alpha (r, \frac{1}{P}) = S_\alpha (r, P) \quad \text{this proves the theorem.} \]