CHAPTER - IV

DIFFERENTIAL POLYNOMIALS AND THEIR DEFICIENT VALUES
4.1 Introduction and preliminaries:

Let $f(z)$ be a nonconstant meromorphic function in the complex plane and $m(r,a,f), N(r,a,f) = N(r,\frac{1}{f-a}), \overline{N}(r,a,f), N(r,f), T(r,f)$ etc, have the usual meaning. We assume familiarity with

$$\delta(a,f) = 1 - \limsup_{r \to \infty} \frac{N(r,a)}{T(r,f)}, \lambda(a,f) = 1 - \liminf_{r \to \infty} \frac{N(r,a,f)}{T(r,f)}$$

$$\Theta(a,f) = \liminf_{r \to \infty} \frac{N(r,f) - \overline{N}(r,a)}{T(r,a)}$$

Let $\Psi$ be as in the introduction.

**Definition 4.1:** If $r_0$ is any positive number, $\alpha \geq 0$ then, as in chapter III, let

$$T_\alpha(r,r_0,f) = \int_{t_0}^{t_{r_0}} \frac{T(r,f)}{t^{1+\alpha}} dt, \quad N_\alpha(r,r_0,f) = \int_{t_0}^{t_{r_0}} \frac{N(t,a)}{t^{1+\alpha}} dt, \quad m_\alpha(r,r_0,f) = \int_{t_0}^{t_{r_0}} \frac{m(t,a)}{t^{1+\alpha}} dt,$$

$$\overline{N}_\alpha(r,r_0,f) = \int_{t_0}^{t_{r_0}} \frac{\overline{N}(t,a)}{t^{1+\alpha}} dt$$

and $S_\alpha(r,r_0,f) = \int_{t_0}^{t_{r_0}} \frac{S(t,a)}{t^{1+\alpha}} dt$, for any complex number $a$, finite or not. $\delta_\alpha(a,f), \Delta_\alpha(a,f)$ and $\Theta_\alpha(a,f)$ are defined as follows

$$\delta_\alpha(a,f) = \liminf_{r \to \infty} \frac{m_\alpha(r,a)}{T_\alpha(r,a)}, \quad \Delta_\alpha(a,f) = \limsup_{r \to \infty} \frac{m_\alpha(r,a)}{T_\alpha(r,a)},$$

$$\Theta_\alpha(a,f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}_\alpha(r,a)}{T_\alpha(r,a)}$$
Here we are improving relations between Nevanlinna characteristic of $f$ and $\Psi$, and after we shall use these to find varies relations between deficient values of homogeneous differential polynomial. We require some following lemmas.

Lemma 4.1: [26] If $\Psi$ is a homogeneous differential polynomial in $f$ of degree

$$n \geq 1$$

then $m(r, \frac{\Psi}{f^n}) = S(r, f)$.

Lemma 4.2: [11](i) $m(r, \frac{\Psi}{f^n}) \leq S(r, f) + \left\{ \sum_{j=1}^{s} (\gamma_{\Psi} - \gamma_{M_j}) \right\} m(r, 0, f)$

(ii) $m(r, \frac{\Psi}{f^n}) \leq S(r, f) + \left\{ \sum_{j=1}^{s} (\gamma_{M_j} - \gamma_{\Psi}) \right\} m(r, f)$

(iii) $\gamma_{\Psi} \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq T_{\alpha}(r, \Psi(f)) - N_{\alpha}(r, 0, \Psi(f)) + S_{\alpha}(r, f)$

(iv) $\gamma_{\Psi} \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) + N_{\alpha}(r, \frac{1}{\Psi}) + S_{\alpha}(r, f) \leq T_{\alpha}(r, \Psi)$

\[ \leq \gamma_{\Psi} T_{\alpha}(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) N_{\alpha}(r, f) + S_{\alpha}(r, 1) \]

Lemma 4.3: [11] $N(r, \Psi(f)) \leq \gamma_{\Psi} N(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) \overline{N}(r, f) + S(r, f)$
Lemma 4.4. [11]

(i) \( T(r, \Psi(f)) \leq \gamma_{\Psi} T(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) N(r, f) + \left( \sum_{j=1}^{s} (\gamma_{Mj} - \gamma_{\Psi}) \right) m(r, 0, f) + S(r, f) \)

and

(ii) \( T(r, \Psi(f)) \leq \gamma_{\Psi} + \left( \sum_{j=1}^{s} (\gamma_{Mj} - \gamma_{\Psi}) \right) m(r, f) + \gamma_{\Psi} N(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) N(r, f) + S(r, f) \)

This lemma is a combination of Lemma 4.2 and Lemma 4.3.

Lemma 4.5. [11] Let \( f \) be of positive or infinite lower order and

\[ \lim_{r \to \infty} \frac{T_\alpha(r, \Psi(f))}{T_\alpha(r, f)} > 0. \text{ Then } \lim_{r \to \infty} \frac{T(r, \Psi(f))}{T(r, f)} = \lim_{r \to \infty} \frac{T_\alpha(r, \Psi(f))}{T_\alpha(r, f)}. \]

Lemma 4.6. [11] \( T_\alpha(r, \Psi(f)) = 0(T_\alpha(r, f)) + S_\alpha(r, f) \)

Lemma 4.7. [11] \( \lim_{r \to \infty} \frac{S_\alpha(r, f)}{T_\alpha(r, f)} = 0. \)

Lemma 4.8: Let \( f \) be a either of finite order or of non-zero lower order such that \( \Theta_\alpha(\infty, f) = \sum_{a \neq -} S^\alpha_k(a, f) = 1. \) Then for homogenous \( \Psi(f), \)

\[ \lim_{r \to \infty} \frac{T_\alpha(r, \Psi(f))}{T_\alpha(r, f)} = \gamma_{\Psi}. \]

Proof: As in Theorem 3.1.1

In this section we consider only the non-constant homogeneous differential polynomial and we denote by \( \Psi(f) \) a differential polynomial not containing \( f. \)
Theorem 4.1 Let $f$ be a meromorphic function and $\Psi(f)$ be a homogeneous differential polynomial in $f$ as defined in introduction.

If $\sum_{\alpha} \delta_{\alpha}(\infty, f) \geq 1 - \zeta; \delta_{\alpha}(\infty, f) \geq 1 - \zeta, (0 \leq \zeta \leq 1)$, then and $\Gamma_{\Psi} \geq 2\gamma_{\Psi}$, then

$$\gamma_{\Psi} - (\Gamma_{\Psi} - \gamma_{\Psi}) \zeta \leq \liminf_{r \to \infty} \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \leq \limsup_{r \to \infty} \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi}) \zeta$$

where $f^{(k)}$ is the highest derivative occurring in $\Psi$.

Proof. From Lemma 4.4(i), we get, as in $\Psi$ is homogeneous

$$T_{\alpha}(r, \Psi(f)) \leq \gamma_{\Psi} T_{\alpha}(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) N_{\alpha}(r, f) + S_{\alpha}(r, f)$$

Hence

$$\limsup_{r \to \infty} \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi}) \limsup_{r \to \infty} \frac{N_{\alpha}(r, f)}{T_{\alpha}(r, f)} + \limsup_{r \to \infty} \frac{S_{\alpha}(r, f)}{T_{\alpha}(r, f)}$$

$$\leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi})(1 - \Theta_{\alpha}(\infty, f))$$

$$\leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi})(1 - \delta_{\alpha}(\infty, f))$$

$$= \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi}) \zeta.$$  \hspace{1cm} (\because \delta_{\alpha}(\infty, f) \geq 1 - \zeta)

And considering the first part of inequality of Theorem 3.1.1, we have for any $\alpha$

$$T_{\alpha}(r, \Psi) \geq \gamma_{\Psi} \sum_{i=1}^{q} m_{\alpha}(r, a_{i}, f) + N_{\alpha}(r, \frac{1}{\Psi}) + S_{\alpha}(r, f)$$

which easily yields

$$\liminf_{r \to \infty} \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \geq \gamma_{\Psi} \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m_{\alpha}(r, a_{i}, f)}{T_{\alpha}(r, f)}.$$
\[
\lim \inf_{r \to \infty} \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \geq \gamma_\Psi - (\Gamma_\Psi - \gamma_\Psi)\zeta
\]

this along with (2) completes the proof of the theorem.

\[
\gamma_\Psi - (\Gamma_\Psi - \gamma_\Psi)\zeta \leq \lim \inf_{r \to \infty} \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \leq \lim \sup_{r \to \infty} \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \leq \gamma_\Psi + (\Gamma_\Psi - \gamma_\Psi)\zeta
\]

**Theorem 4.2.** If \( \Psi(f) \) is a homogeneous differential polynomial defined by (1) of finite order and satisfying \( N_\alpha(r, f) + N_\alpha(r, \frac{1}{f}) = S_\alpha(r, f) \), then

\[
\gamma_\Psi T_\alpha(r, f) \sim N_\alpha(r, \frac{1}{\Psi - b}) \text{ for all } b \text{ except possibly } 0 \text{ and } \infty.
\]

**Proof:** With the same hypothesis (4), it is already proved in Theorem 2 of [7] that \( T_\alpha(r, \Psi) \sim N_\alpha(r, \frac{1}{\Psi - b}) \). Our aim here is to prove the remaining part of the asymptotic relation. However for sake of completeness we shall outline some of the steps of Theorem 2 of [7]

Now

\[
\gamma_\Psi m_\alpha(r, \frac{1}{f}) = m_\alpha(r, \frac{1}{f^\Psi}) \leq m_\alpha(r, \frac{1}{\Psi}) + m_\alpha(r, \frac{\Psi}{f^\Psi}) \leq m_\alpha(r, \frac{1}{\Psi}) + S_\alpha(r, f) \text{ by Lemma 4.1}
\]

Adding \( \gamma_\Psi N_\alpha(r, \frac{1}{f}) \) to both sides and using (4), we have

\[
\gamma_\Psi T_\alpha(r, \frac{1}{f}) \leq m_\alpha(r, \frac{1}{\Psi}) + S_\alpha(r, f) \leq T_\alpha(r, \frac{1}{\Psi}) + S_\alpha(r, f).
\]
So,

$$\gamma_{\Psi} T_{\alpha}(r, f) \leq T_{\alpha}(r, \Psi) + S_{\alpha}(r, f)$$

from which it follows that

$$S_{\alpha}(r, f) = S_{\alpha}(r, \Psi). \quad \text{.. (5)}$$

Since the poles of $\Psi$ can occur only at the poles of $f$ or at the poles of the coefficient $A_j[f]$ of $\Psi$ and $T_{\alpha}(r, A_j) = S_{\alpha}(r, f)$ we have

$$N_{\alpha}(r, \Psi) \leq N_{\alpha}(r, f) + S_{\alpha}(r, f)$$

so, by (4) and (5)

$$N_{\alpha}(r, \Psi) = S_{\alpha}(r, \Psi) \quad \text{.. (6)}$$

by Lemma 4.3,

$$N_{\alpha}(r, \Psi) \leq \gamma_{\Psi} N_{\alpha}(r, f) + (\Gamma_{\Psi} - \gamma_{\Psi}) N_{\alpha}(r, f) + S_{\alpha}(r, f)$$

where $\gamma_{\Psi}$ is the degree of $\Psi$, $\Gamma_{\Psi}$ is weight of $\Psi$ and $f^{(k)}$ is the highest derivative of $f$ occurring in $\Psi$.

Hence by (4) and (6), we get

$$N_{\alpha}(r, \Psi) = S_{\alpha}(r, \Psi) \quad \text{.. (7)}$$

Again

$$N_{\alpha}(r, \frac{1}{\Psi}) = N_{\alpha}(r, \frac{f^{\gamma_{\Psi}}}{\Psi})$$

$$N_{\alpha}(r, \frac{1}{\Psi}) \leq N_{\alpha}(r, \frac{f^{\gamma_{\Psi}}}{\Psi})$$

$$\leq N_{\alpha}(r, \frac{f^{\gamma_{\Psi}}}{\Psi}) + T_{\alpha}(r, \frac{f^{\gamma_{\Psi}}}{\Psi})$$

$$= N_{\alpha}(r, \frac{f^{\gamma_{\Psi}}}{\Psi}) + O(1)$$

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Thus

\[ \text{Na}(r, \frac{1}{f}) = \text{Sa}(r, \Psi) \]  

If \( b \in \mathbb{C} \) and \( b \neq 0 \) then by Nevanlinna's second fundamental theorem, we have

\[ T_a(r, \Psi) \leq \text{Na}(r, \Psi) + \text{Na}(r, \frac{1}{\Psi}) + \text{Na}(r, \frac{1}{\Psi - b}) + \text{Sa}(r, f) \]

for \( b \in \mathbb{C} - \{0\} \)

\[ = \text{Na}(r, \frac{1}{\Psi - b}) + \text{Sa}(r, \Psi), \text{ by (7) and (9)} \]

Therefore by (9), we get

\[ \gamma T_a(r, f) \leq \text{Na}(r, \frac{1}{\Psi}) + \text{Sa}(r, \Psi) \]

(Since \( \text{Sa}(r, f) = \text{Sa}(r, \Psi) \) always).

Now by the first fundamental theorem of Nevanlinna, we have

\[ T_a(r, \Psi) \geq \text{Na}(r, \frac{1}{\Psi - b}) + o(1) \]

So, from (10) and (11), we have
Again from Lemma 4.4(i), we have

\[
T_a(r, \Psi) \leq \gamma \Psi T_a(r, f) + (\Gamma \Psi - \gamma \Psi)N_a(r, f) + S_a(r, f),
\]

because \( \Psi \) is homogeneous

\[
\leq \gamma \Psi T_a(r, f) + S_a(r, f)
\]

by hypothesis

hence from (11) and (13) one has

\[
\liminf \frac{N_a(r, \frac{1}{\Psi - b})}{T_a(r, f)} \leq \liminf \frac{T_a(r, \Psi)}{T_a(r, f)} \leq \gamma \Psi
\]

and hence the theorem follows from (12) and (14).

We now prove that if \( \Psi(f) \) is a monomial of degree \( \gamma \Psi \) containing \( f \) with the exponent of highest derivative as \( n_u \) then under certain conditions

\[
T_a(r, \Psi) \sim (\gamma \Psi + kn_k)T_a(r, f).
\]

**Theorem 4.3:** For \( \Psi(f) \) as defined in the introduction. Then

\[
\begin{align*}
(i) \quad & \limsup_{r \to \infty} \frac{N_a(r, \Psi)}{T_a(r, \Psi)} \leq \frac{\Gamma \Psi \zeta}{\gamma \Psi - (\Gamma \Psi - \gamma \Psi) \zeta} \\
(ii) \quad & \limsup_{r \to \infty} \frac{N_a(r, \frac{1}{\Psi})}{T_a(r, \Psi)} \leq \frac{\Gamma \Psi \zeta}{\gamma \Psi + (\Gamma \Psi - \gamma \Psi) \zeta}
\end{align*}
\]

where \( \zeta \) is as in the above theorem.
Proof: From Lemma 4.3, we obtain

\[ N(r, \Psi(f)) \leq \gamma_\Psi N(r, f) + (\Gamma_\Psi - \gamma_\Psi) \overline{N}(r, f) + S(r, f), \] this implies

\[ N_\alpha(r, \Psi(f)) \leq \gamma_\Psi N_\alpha(r, f) + (\Gamma_\Psi - \gamma_\Psi) \overline{N}_\alpha(r, f) + S_\alpha(r, f). \]

We obtain

\[ \frac{N_\alpha(r, \Psi)}{T_\alpha(r, \Psi)} \leq \frac{T_\alpha(r, f)}{T_\alpha(r, \Psi)} \left( \gamma_\Psi \frac{N_\alpha(r, f)}{T_\alpha(r, f)} + (\Gamma_\Psi - \gamma_\Psi) \frac{\overline{N}_\alpha(r, f)}{T_\alpha(r, f)} + \frac{S_\alpha(r, f)}{T_\alpha(r, f)} \right) \]

Using the first part of inequality of Theorem 4.1, we obtain

\[ \limsup_{r \to \infty} \frac{N_\alpha(r, \Psi)}{T_\alpha(r, \Psi)} \leq \frac{1}{\gamma_\Psi - (\Gamma_\Psi - \gamma_\Psi) \zeta} \left( \gamma_\Psi (1 - \delta_\alpha(\infty, f)) + (\Gamma_\Psi - \gamma_\Psi)(1 - \Theta_\alpha(\infty, 1)) \right) \]

by (1)

\[ \leq \frac{1}{\gamma_\Psi - (\Gamma_\Psi - \gamma_\Psi) \zeta} (\gamma_\Psi \zeta + (\Gamma_\Psi - \gamma_\Psi) \zeta), \text{ by hypothesis} \]

\[ \leq \frac{\Gamma_\Psi \zeta}{\gamma_\Psi - (\Gamma_\Psi - \gamma_\Psi) \zeta} \] hence the result.

To prove (ii) we consider the first part of inequality in Theorem 3.1.1 Dividing by \( T_\alpha(r, \Psi) \) and taking superior limit as \( r \to \infty \) we obtain on simplifying

\[ \limsup_{r \to \infty} \frac{N_\alpha(r, \frac{1}{\Psi})}{T_\alpha(r, \Psi)} \leq 1 - \liminf_{r \to \infty} \frac{T_\alpha(r, f)}{T_\alpha(r, \Psi)} \times \gamma_\Psi \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m_\alpha(r, a_i, f)}{T_\alpha(r, f)} \]

\[ \leq 1 - \frac{\gamma_\Psi}{\gamma_\Psi + (\Gamma_\Psi - \gamma_\Psi) \zeta} \sum_{i=1}^{q} \delta_\alpha(a_i, f) \]

\[ \leq 1 - \frac{\gamma_\Psi (1 - \zeta)}{\gamma_\Psi + (\Gamma_\Psi - \gamma_\Psi) \zeta} \]

\[ = \frac{\Gamma_\Psi \zeta}{\gamma_\Psi + (\Gamma_\Psi - \gamma_\Psi) \zeta}. \]
Theorem 4.4: Let $|a_i| < \infty \ (i=1,2...q)$. If $\Psi$ is homogeneous differential polynomial defined by (1), then

\[
(1 - \delta_\alpha(0,\Psi)) \sum_{i=1}^{q} \delta_\alpha(a_i, f) \leq \left\{ 1 + \left( \frac{\Gamma_\psi - \gamma_\psi}{\gamma_\psi} \right) \left( 1 - \Theta_\alpha(\infty, f) - \sum_{i=1}^{q} \delta_\alpha(a_i, f) \right) \right\} \lambda_\alpha(0, \phi)
\]

where $f^{(k)}$ is the highest derivative of $f$ occurring in $\Psi$.

Proof: Let $\lim_{r \to \infty} \sup \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} = \frac{a}{b}$.

Adding $\gamma_\psi \sum_{i=1}^{q} N_\alpha(r,a_i,f)$ on both sides of inequality Theorem 3.1.1, we obtain:

\[
\gamma_\psi q T_\alpha(r, f) + N_\alpha(r, f) + S_\alpha(r, f) \leq T_\alpha(r, \Psi) + \gamma_\psi \sum_{i=1}^{q} N_\alpha(r,a_i,f)
\]

So

\[
\gamma_\psi q \frac{N_\alpha(r, f)}{T_\alpha(r, f)} + \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} + \frac{S_\alpha(r, f)}{T_\alpha(r, f)} \leq \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} + \gamma_\psi \sum_{i=1}^{q} \frac{N_\alpha(r,a_i,f)}{T_\alpha(r, f)}
\]

Hence

\[
\gamma_\psi q + \lim_{r \to \infty} \inf \frac{N_\alpha(r, f)}{T_\alpha(r, f)} \times \lim_{r \to \infty} \inf \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \leq \lim_{r \to \infty} \inf \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} + \gamma_\psi \sum_{i=1}^{q} \lim_{r \to \infty} \sup \frac{N_\alpha(r,a_i,f)}{T_\alpha(r, f)}
\]
we have

\[ \gamma \psi q + (1 - \delta \alpha (0, \Psi)) B \leq B + \sum_{i=1}^{q} (1 - \delta \alpha (a_i, f)) \]

where \( \lim \inf \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} = B \),

which reduces on simplifying to

\[ \gamma \psi \sum_{i=1}^{q} \delta \alpha (a_i, f) \leq B \lambda _\alpha (0, \Psi) \]

Again from (15) we have

\[ \gamma \psi q + \lim \sup_{r \to \infty} \left( \frac{N_\alpha (r, \Psi)}{T_\alpha(r, \Psi)} \times \lim \inf_{r \to \infty} \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \right) \leq \lim \sup_{r \to \infty} \left( \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} \right) + \gamma \psi \sum_{i=1}^{q} \lim \sup_{r \to \infty} \left( \frac{N_\alpha (r, \Psi)}{T_\alpha(r, f)} \right) \]

and so

\[ \gamma \psi q + (1 - \delta \alpha (0, \Psi)) B \leq A + \gamma \psi \sum_{i=1}^{q} (1 - \delta \alpha (a_i, f)) \]

where \( \lim \sup_{r \to \infty} \frac{T_\alpha(r, \Psi)}{T_\alpha(r, f)} = A \).

Therefore on rearranging, \( (1 - \delta \alpha (0, \Psi)) B \leq A + \gamma \psi \sum_{i=1}^{q} (1 - \delta \alpha (a_i, f)) \) and

\( 1 - \delta \alpha (0, \Psi) \geq 0 \), we find on multiplying this with the corresponding inequalities of (16)

\[ \gamma \psi (1 - \delta \alpha (0, \Psi)) \sum_{i=1}^{q} \delta \alpha (a_i, f) \leq (A + \gamma \psi \sum_{i=1}^{q} \delta \alpha (a_i, f)) \lambda _\alpha (0, \Psi) \].
But from Lemma 4.4(i), we have

\[ T(r, \Psi(f)) \leq \gamma_{\Psi} T(r, f) + \sum_{a = 1}^{q} \delta_{\alpha}(a_i, f) \leq \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi})(1 - \Theta_{\alpha}(\infty, f)) \]

\[ \therefore A = \lim_{r \to \infty} \sup \frac{T_{\alpha}(r, \Psi)}{T_{\alpha}(r, f)} \leq \gamma_{\Psi} + (\Gamma_{\Psi} - \gamma_{\Psi})(1 - \Theta_{\alpha}(\infty, f)). \tag{18} \]

Therefore from (17) and (18), we get

\[ (1 - \delta_{\alpha}(0, \Psi)) \sum_{i=1}^{q} \delta_{\alpha}(a_i, f) \leq \left( 1 + \left( \frac{\Gamma_{\Psi} - \gamma_{\Psi}}{\gamma_{\Psi}} \right)(1 - \Theta_{\alpha}(\infty, f)) \right) \delta_{\alpha}(0, \Psi) \]

which on rearranging its terms gives the above inequality (17), and thus completes the proof.

**Theorem 4.5:** Let \( f \) be a meromorphic function of finite order and let \( \Psi \) be a homogeneous differential polynomial defined by (1). Further, let \( S_{\alpha}(r, f) = S_{\alpha}(r, \Psi) \), then

\[ \delta_{\alpha}(0, \Psi) \geq \frac{\gamma_{\Psi}}{\Gamma_{\Psi}} \sum_{a \neq \infty} \delta_{\alpha}(a, f) \tag{19} \]

where \( f^{(k)} \) is the highest derivative occurring in \( \Psi \). Further if \( f \) is entire then

\[ \sum_{a \neq \infty} \lim_{r \to \infty} \inf \frac{m_{\alpha}(r, a_i, f)}{T_{\alpha}(r, f)} \leq \delta_{\alpha}(0, \Psi) \]

\[ \sum_{a \neq \infty} \delta_{\alpha}(a, f) \leq \delta_{\alpha}(0, \Psi). \tag{16} \]
Proof: Let \( a_1, a_2, a_3 \ldots a_q \) be distinct finite complex number and

Let \( F = \sum_{i=1}^{q} \frac{1}{(f - a_i)^{\psi}} \).

Then it follows that theorem 3.1.1 that

\[
\gamma_{\psi} \sum_{i=1}^{q} m_{\alpha}(r, a_i, f) \leq m_{\alpha}(r, \frac{1}{\psi}) + S_{\alpha}(r, f).
\]

So, dividing by \( T_{\alpha}(r, \Psi) \) on both sides and using \( S_{\alpha}(r, f) = S_{\alpha}(r, \Psi) \)

we deduce that

\[
\gamma_{\psi} \sum_{i=1}^{q} \liminf_{r \to 0} \frac{m_{\alpha}(r, a_i, f)}{T_{\alpha}(r, \Psi)} \leq \delta_{\alpha}(0, \Psi).
\]

But from Lemma 1.4(i), we obtain

\[
T_{\alpha}(r, \Psi) \leq \Gamma_{\psi} T_{\alpha}(r, f) + S_{\alpha}(r, f)
\]

\[
\quad = \Gamma_{\psi} T_{\alpha}(r, f) + S_{\alpha}(r, f)
\]

and using this in (21), one has

\[
\delta_{\alpha}(0, \Psi) \geq \gamma_{\alpha} \sum_{i=1}^{q} \liminf_{r \to 0} \frac{m_{\alpha}(r, a_i, f)}{T_{\alpha}(r, f)}, \text{ which yields } \delta_{\alpha}(0, \Psi) \geq \frac{\gamma_{\psi}}{\Gamma_{\psi}} \sum_{i=1}^{q} \delta_{\alpha}(a_i)
\]

on making \( q \to \infty \), we obtain (19).
Next, if $f$ is entire functions of finite order, then

\[
T_\alpha(r, \Psi) = m_\alpha(r, \Psi)
\]

\[
\leq m_\alpha(r, \frac{\Psi}{f^\gamma}) + m_\alpha(r, f^\gamma)
\]

\[
\leq \gamma_\alpha m_\alpha(r, f) + S_\alpha(r, f) \quad \text{by Lemma 4.1.}
\]

\[
\leq \gamma_\alpha T_\alpha(r, f) + S_\alpha(r, f) \quad \text{and hence from (21), we have}
\]

\[
\sum_{q=1}^{\infty} \liminf_{r \to \infty} \frac{m_\alpha(r, \frac{\Psi}{f})}{T_\alpha(r, f)} \leq \delta_\alpha(0, \Psi).
\]

On letting $q \to \infty$, we obtain (20) let us note that if $\Psi$ is a monomial then the condition $S_\alpha(r, f) = S_\alpha(r, \Psi)$ is automatically satisfied, since if

\[
\Psi(f) = (f')^{n_0_1} (f')^{n_0_2} \cdots (f^{(k)})^{n_0_k} \quad \text{where} \quad n_0_0 + n_0_1 + \ldots + n_0_k = m_{j0}
\]

Then clearly

\[
T_\alpha(r, \Psi) \leq C T_\alpha(r, f) + S_\alpha(r, f) \quad \text{for some constant } C.
\]

Also

\[
\Psi(f) = f^{\gamma_\Psi} \left( \frac{f'}{f} \right)^{n_0_1} \left( \frac{f''}{f} \right)^{n_0_2} \cdots \left( \frac{f^{(k)}}{f} \right)^{n_0_k} \quad \text{and so}
\]

\[
\gamma_\Psi T_\alpha(r, f) \leq T_\alpha(r, \Psi') + n_1 T_\alpha(r, \frac{f}{f'}) + \ldots + n_k T_\alpha(r, \frac{f^{(k)}}{f^{(k)}})
\]

\[
\leq T_\alpha(r, \Psi') + n_1 T_\alpha(r, \frac{f'}{f}) + \ldots + n_k T_\alpha(r, \frac{f^{(k)}}{f}) + S_\alpha(r, f).
\]
Thus using Milloux's theorem

\[ \gamma \psi T_\alpha(r, f) \leq T_\alpha(r, \Psi) + n_1 N_\alpha(r, \frac{f'}{f}) + \cdots + n_k N_\alpha(r, \frac{f^{(k)}}{f}) \]

but

\[ N_\alpha(r, \frac{f^{(k)}}{f}) = 1 \left[ N_\alpha(r, \frac{1}{f}) + N_\alpha(r, f) \right] \text{ for } l=1,2,\ldots,k \]

\[ \leq \left[ N_\alpha(r, \frac{1}{f}) + N_\alpha(r, \Psi) \right] \]

\[ \leq \left[ T_\alpha(r, \Psi) + T_\alpha(r, \Psi) \right] + S_\alpha(r, \Psi) \]

\[ \leq 21T_\alpha(r, \Psi) + S_\alpha(r, \Psi). \]

Thus \( \gamma \psi T_\alpha(r, f) \leq BT_\alpha(r, \Psi) + S_\alpha(r, f) \), for some constant B.

It follows that \( S_\alpha(r, f) = S_\alpha(r, \Psi) \), also since \( \lambda_\alpha(0, f^{(k)}) \geq \delta_\alpha(0, f^{(k)}) \).

Theorem 3 of Kamthan [13] becomes a particular case of our theorem. We now use the above Theorem to find an upper bound for \( \delta_\alpha(\infty, \Psi) \).