CHAPTER - 8

SET NEIGHBOURHOOD AND GLOBAL SET NEIGHBOURHOOD NUMBERS OF A GRAPH
ABSTRACT:

A neighbourhood set $S$ of a connected graph $G$ is a set neighbourhood set (sn-set) if for every set $T \subseteq V-S$ there exists a set $W \subseteq S$, such that, $\langle W \cup T \rangle$ is connected. The set neighbourhood number $n_S(G)$ of $G$ is the minimum cardinality of an sn-set. Further, $S$ is a global sn-set if $S$ is an sn-set of both $G$ and $\bar{G}$. The global set neighbourhood number $n_{sg}(G)$ of $G$ is the minimum cardinality of a global sn-set. Graphs considered here are co-connected (i.e. both $G$ and $\bar{G}$ are connected). In this chapter we study these two parameters.

Set Neighbourhood Number:

A neighbourhood set $S$ of a connected graph $G$ is a set neighbourhood set (sn-set) if for every set $T \subseteq V-S$ there exists a set $W \subseteq S$, such that, $\langle W \cup T \rangle$ is connected. The set neighbourhood number $n_S(G)$ of $G$ is the minimum cardinality of an sn-set.

For example,

- $n_S(P_5) = 3$, $n_S(C_6) = 3$ and $n_{\bar{G}} = 2$
In this section some properties of this parameter are established. Besides investigating some relationships of $n_s(G)$ with other known parameters of $G$, many bounds for $n_s(G)$ are obtained.

Throughout this section, by a graph we mean a connected graph.

We now consider the following observations.

\[
\gamma(G) \leq \gamma_s(G) \leq \gamma_c(G) \quad (\text{see [8]}) \quad \ldots \quad (1)
\]

\[
n_o(G) \leq n_c(G) \quad (\text{see [7]}) \quad \ldots \quad (2)
\]

\[
\gamma_c(G) \leq n_c(G) \quad (\text{see [7]}) \quad \ldots \quad (3)
\]

**Proposition 8.1:** For any connected graph $G$ we have

\[
n_o(G) \leq n_s(G) \leq n_c(G) \quad \ldots \quad (4)
\]

and \[\gamma(G) \leq \gamma_s(G) \leq n_s(G) \quad \ldots \quad (5)\]

**Proof:** (4) follows from the fact that every $sn$-set is a $n$-set and every connected $n$-set is an $sn$-set. (5) follows from (1) and since every $sn$-set is a set-dominating set.

For $P_4$, $n_o = n_s = n_c = 2$

Set neighbourhood numbers of some standard graphs can be easily found and are given as follows:

[a] For any graph with a vertex of full degree (in particular $K_{p', K_{1,n}}$ and $W_{p'}$, a wheel of order $p$)

\[
n_o = n_c = n_s = 1
\]
For a cycle $C_n$ of length $n \geq 6$, and $n_s(C_4) = 2$, $n_s(C_5) = 3$, $n_s(C_n) = n - 3$

For a path $P_n$ of order $n \geq 4$, $n_s(P_n) = n - 2$

For complete bipartite graph $K_{m,n}$ with $m \leq n$, $n_s(K_{m,n}) = m$

For a tree $T$, $n_s(T) = p - e$.

Let $n_o$-set be a minimum $n$-set. $n_c$-set and $n_s$-set are defined similarly. It is known that every $n_c$-set contains all its cut vertices (see [7]). But it is not so in the case of $n_s$-sets. For example, for the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$ of order 5 the set $\{v_2, v_4, v_5\}$ is a $n_s$-set which does not contain the cut vertex $v_3$.

We now obtain two sufficient conditions for a cut vertex to be in every $n_s$-set.

**Proposition 8.2:** Let $v$ be a cut vertex of a graph $G$. Then $v$ is in every $n_s$-set $S$ of $G$ if either

i) $G - v$ has at least three components or

ii) $G - v$ has exactly two components $G_1$ and $G_2$ and neither $<G_1 \cup \{v\}>$ or $<G_2 \cup \{v\}>$ is a path.
Proof: Let S be an $n_s$-set. Suppose $v \notin S$. Consider vertices $u$ and $w$ in different components of $G - v$ which are not in $S$. Since $v$ is on every $u - w$ path in $G$ and $v \notin S$, there is no set $W \subset S$, such that, the subgraph $<W \cup \{u,w\}>$ is connected which is a contradiction. Hence we have the following observation.

(A): All vertices in all except possibly one component of $G-v$ belong to $S$.

We now consider different cases.

Case 1: $G - v$ has at least three components say $G_1$, $G_2$ and $G_3$. By the observation (A) all vertices in all but one component say $G_1$ belongs to $S$. Let $x$ and $y$ be two vertices adjacent to $v$, such that, $x \in G_2$ and $y \in G_3$. Then both $x$ and $y$ are in $S$. Clearly the set $S' = S - \{x,y\} \cup \{v\}$ is also an $n_s$-set of $G$ with $|S'| = |S| - 1$, which is a contradiction to the fact that $S$ is an $n_s$-set. This proves $v \in S$.

Case 2: $G-v$ has exactly two components say $G_1$ and $G_2$, such that, neither $H_1 = G_1 \cup \{v\}$ nor $H_2 = G_2 \cup \{v\}$ is a path.

If $v \notin S$ then by observation (A), all vertices in either $G_1$ or $G_2$ say $G_1$ are in $S$. Let $T$ be a spanning tree of $H_1$. 


i) If there exist two end vertices \( x \) and \( y \) other than \( v \) in \( T \) which are not adjacent in \( H_1 \) then,

\[
S' = \{ S - \{ x, y \} \cup \{ v \} \}
\]

is also an \( n_s \)-set of \( G \) which, \(|S'| = |S| - 1\), which is a contradiction to the fact that \( S \) is an \( n_s \)-set. This proves \( v \in S \).

ii) If there does not exist two end vertices \( x \) and \( y \) other than \( v \) in \( T \) which are non-adjacent in \( H_1 \) then, \( H_1 \) must contain a cycle. Let \( x \) and \( y \) be two vertices in spanning tree \( T \) of \( H_1 \). If they lie on \( C_3 \) in \( H_2 \) then

\[
|S'| = S - \{ x, y \} \cup \{ v \}
\]

is an \( n_s \)-set. \(|S'| = |S| - 1\) which is a contradiction to the fact that \( S \) is an \( n_s \)-set. Hence \( v \in S \).

If the end vertices lie on \( C_n, n \geq 4 \), then there exist two non-adjacent vertices \( x \) and \( y \) other than cut vertex \( v \) in \( H_1 \), clearly \( S' = S - \{ x, y \} \cup \{ v \} \) is an \( n_s \)-set which is a contradiction to the fact that \( S \) is an \( n_s \)-set and \(|S'| = |S| - 1\). Hence \( v \in S \).

Note that if one of \( H_1 \) or \( H_2 \) is a path then cut vertex \( v \) may not belong to every \( n_s \)-set as the example \( C_5 \) cited above shows. However, one can always find an \( n_s \)-set containing all cut vertices.
Proposition 8.3: Let $G$ be a graph having cut-vertices then there exists an $n_s$-set of $G$ containing all cut vertices.

Proof: Let $v$ be a cut vertex of $G$ and $S$ be an $n_s$-set of $G$. If $v \not\in S$, then by proposition 2, $G - v$ has exactly two components $G_1$ and $G_2$, such that, at least one of the graphs $H_1 = G_1 \cup \{v\}$ or $H_2 = G_2 \cup \{v\}$ is a path. Let $H_1$ be a path. By observation (A) and by the argument as in Case 2 of Proposition 8.2, we find that all vertices in $G_1$ belong to $S$. Let $u$ be an end vertex of $H_1$ different from $v$. Then the set $S_1 = (S - \{u\}) \cup \{v\}$ is an $n_s$-set with $|S'| = |S| = n_s$. Repeating this process for all cut vertices $v$, we ultimately obtain an $n_s$-set containing all cut vertices.

Corollary 8.3.1: Let cut denote the number of cut vertices of a graph $G$. Then

$$\text{cut } G \leq n_s$$

Corollary 8.3.2: For any tree $T$ with 'p' vertices and 'e' end vertices,

$$n_s = n_c = p - e$$

Proof: It is known that the set of all non-pendant vertices of a tree $T$ (i.e. cut vertices) form an $n_c$-set.
and hence \( n_c = p - e \) (see [7]). Thus by (6) \( n_c \leq n_s \).

Now (7) follows from (4).

For trees though \( n_s = n_c \), every \( n_s \)-set need not be a \( n_c \)-set. For example, the end vertices of the path \( P_4 \) of order 4, form an \( n_s \)-set which is not \( n_c \)-set. Using Proposition 8.2, one can say when this is true.

**Proposition 8.4**: In a tree \( G \), every \( n_s \)-set is an \( n_c \)-set if for every vertex \( v \) with \( \deg v = 2 \), neither \( G_1 \cup \{v\} \) nor \( G_2 \cup \{v\} \) is a path, where \( G_1 \) and \( G_2 \) are the two components of \( G - v \).

**Proof**: The result follows from Proposition 8.2, since the given condition implies that every \( n_s \)-set contains all cut vertices of \( G \).

Now we consider the two lower bounds of \( \gamma_s \) as obtained in [8].

**Theorem A[8]**: (i) For any graph \( G \)

\[ \text{diam } G - 1 \leq \gamma_s \]

(ii) If \( G \) is a graph of order \( p \) and maximum degree \( \Delta \), then \( \frac{p}{1+\Delta} \leq \gamma_s \) and equality holds if and only if \( \Delta = p - 1 \).
From the result (5) and Theorem A we have the following lower bounds for $n_s$.

**Proposition 8.5**: (i) For any graph $G$, 
$$\text{diam } G - 1 \leq n_s$$

(ii) If $G$ is a graph of order $p$ and maximum degree $\Delta$, then 
$$\frac{p}{1+\Delta} \leq n_s$$
and the equality holds if and only if $\Delta = p - 1$.

Now we consider some upper bounds of $n_c$ as obtained in [7].

**Theorem B[7]**: For any connected graph $G$, 

(i) $n_c \leq 2 \beta_1$

where $\beta_1$ is the matching number of $G$.

(ii) $n_c \leq p - e + \left\lfloor \frac{e}{2} \right\rfloor$, where $e$ is the number of pendant vertices in any spanning tree of $G$.

(iii) Let $G(\neq K_2)$ be a non-trivial connected $(p, q)$ graph. If $G$ is not a cycle then $n_c \leq p - 2$ and $n_c \leq 2q - p$.

By virtue of Theorem B and the result (4) we have upper bounds for $n_s$ as follows.

**Proposition 8.6**: For any connected graph $G$, 

(i) $n_s \leq 2 \beta_1$

where $\beta_1$ is the matching number of $G$ and
ii) \( n_s \leq p - e + \left\lceil \frac{e}{2} \right\rceil \)

where \( e \) is the number of pendant vertices in any spanning tree of \( G \).

iii) Let \( G(\neq K_2) \) be a non-trivial connected \((p,q)\) graph. If \( G \) is not a cycle then

\[
\begin{align*}
    n_s & \leq p - 2 \quad \ldots \ (8) \\
    \text{and} \quad n_s & \leq 2q - p \quad \ldots \ (9)
\end{align*}
\]

The bounds in (8) and (9) are attained for \( G = P_4 \), a path of order 4 and \( G \) is a path respectively.

**Theorem C[7]**: Let both \( G \) and \( \tilde{G} \) be connected and \( \tilde{n}_c = n_c(\tilde{G}) \). Then \( n_c + \tilde{n}_c \leq 2p - 2 \) and if \( G \) is a cycle, \( n_c + \tilde{n}_c \leq 2p - 4 \).

By Theorem C and the result (4) we have

**Proposition 8.7** : Let both \( G \) and \( \tilde{G} \) be connected and \( \tilde{n}_s = n_s(\tilde{G}) \). Then

\[
\begin{align*}
    n_s + \tilde{n}_s & \leq 2p - 2 \quad \ldots \ (10) \\
    \text{and if} \ G \ \text{is a cycle,} \ n_s + \tilde{n}_s & \leq 2p - 4 \quad \ldots \ (11)
\end{align*}
\]

(11) is attained for \( C_5 \).

A graph may not have an \( n_s \)-set which is independent. For example \( P_6 \) and cycle of length 5.

A necessary condition for \( G \) to have an independent \( n_s \)-set is as follows.
Proposition 8.8: A graph $G$ has an independent $n_S$-set then $\text{diam } G \leq 4$.

Proof: Suppose $S$ is an $n_S$-set which is independent. We consider the following different cases.

Case 1: Let $u, v \in V-S$. Since $S$ is an independent $n_S$-set, both $u$ and $v$ are adjacent to a common vertex in $S$ or $u$ and $v$ are adjacent. In either case $d(u, v) \leq 2$.

Case 2: Let $u, v \in S$. Since $G$ is connected and $S$ is independent, there exist vertices $u_1$ and $v_1$ in $V-S$ such that, $u_1u$ and $v_1v$ are edges. Hence, by Case 1, we have

$$d(u, v) \leq d(u, u_1) + d(u_1, v_1) + d(v_1, v) \leq 2 + d(u_1, v_1) \leq 4.$$ 

Case 3: Let $u \in S$ and $v \in V-S$. Then there exists $u_1 \in V-S$, such that, $u_1$ is adjacent to $u$ and $d(u, v) \leq d(u, u_1) + d(u_1, v) \leq 1 + 2$ by Case 1.

Thus for all $u, v \in V(G)$, $d(u, v) \leq 4$.

Hence $\text{diam } G \leq 4$.

Note that for the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$ whose diameter is 4, the set $\{v_1, v_3, v_5\}$ is an independent $n_S$-set.
But, however not all graphs with diameter \( \leq 4 \) have an independent \( n_s \)-set. For example \( C_5 \).

Note that, let \( G \) be a connected graph having some cut vertices. Then \( n_s \) need not be equal to \( n_c \).

For example,

![Diagram of a graph with vertices \( v_1, v_2, v_3, v_4, v_5 \).]

Here \( n_c = 4, n_s = 3 \)

Proposition 8.9 : For a connected graph \( G \) of order \( p \),

\[ n_s(G) \leq p - \Delta(G) \quad \ldots \quad (12) \]

Proof : Let \( v_o \) be a vertex of \( G \), such that \( \deg v_o = \Delta = r \). Let \( S = \{v_1, v_2, \ldots, v_r\} \) be the set of vertices adjacent to \( v_o \). Then clearly \( V(G) - S \) is an sn-set. Hence \( n_s(G) \leq p - \Delta(G) \).

The bound in (12) is attained for \( C_5 \), the cycle of length five.
Proposition 8.10: For a connected graph $G$ of order $p$,
\[ n_s(G) + n_s(\overline{G}) \leq p + 1 \quad \ldots \quad (13) \]
\[ n_s(G) n_s(\overline{G}) \leq p + \delta(G) \Delta(\overline{G}) \quad \ldots \quad (14) \]

Proof: From result (12),
\[ n_s(G) \leq p - \Delta(G) \]
Hence $n_s(G) \leq p - \delta(G)$
Also we have $n_s(G) \leq p - \delta(G)$
Thus $n_s(G) + n_s(\overline{G}) \leq 2p - [\Delta(G) + \delta(\overline{G})]$
\[ = 2p - (p - 1) \]
\[ = p + 1 \]
Equality holds for $G = K_p$, $\overline{K_p}$, or $C_5$.
Similarly $n_s(G) n_s(\overline{G}) \leq (p - \delta(G)) (p - \Delta(\overline{G}))$
\[ = p + \delta(G) \Delta(\overline{G}) \]
The equality holds for $G = C_5$.

Proposition 8.11: For a tree $T$ of order $p$ with $e$ end vertices,
\[ n_s(T) + n_s(\overline{T}) \leq p - e + 2 \]

Proof: By the result (7),
\[ n_s'(T) = p - e \]
Note that $\overline{T}$ contains a vertex of degree $p - 2$ so that $\Delta(\overline{T}) = p - 2$.
By the result (12),
\[ n_s(T) \leq p - \Delta(\overline{T}) = p - (p - 2) = 2 \]
Hence \( n_s(T) + n_s(\overline{T}) \leq p - e + 2 \)
The equality follows for \( G = P_4 \).

**Corollary 8.11.1**: For a path \( P_n \) of order \( n > 3 \),
\[ n_s(P_n) + n_s(\overline{P}_n) \leq n \]
The equality follows for \( P_4 \).

**Global Set Neighbourhood Number of a Graph**: 

A set neighbourhood set \( S \) is said to be a global set neighbourhood set if \( S \) is an \( s_n \)-set of both \( G \) and \( \overline{G} \). Suppose \( G \) is a co-connected graph (i.e. both \( G \) and its complement \( \overline{G} \) are connected). The global set neighbourhood number \( n_{sg}(G) \) of \( G \) is the minimum cardinality of a global \( s_n \)-set.

In this section some properties of this parameter are studied. Besides investigating some relationship of \( n_{sg}(G) \) with other known parameters of \( G \), many bounds for \( n_{sg}(G) \) are obtained.

Throughout this section, we consider only co-connected graphs.

We observe that,

i) For a cycle \( C_n \) of length \( n \geq 6 \),
\[ n_{sg}(C_n) = n - 2 \]
where as \( n_{sg}(C_5) = 4 \)
ii) For a path $P_n$ of order $n \geq 4$,

$$n_{sg}(P_n) = n - 1$$

Since $n_g \geq 2$ and a global set neighbourhood set is a global neighbourhood set, also since global neighbourhood set is a set neighbourhood set we have the following results.

$$2 \leq n_g \leq n_{sg} \quad \ldots \quad (15)$$

and

$$n_s \leq n_g \leq n_{sg} \quad \ldots \quad (16)$$

For the cycle $C_5$, $n_g = n_{sg} = 4$ and for the path $P_5$, $n_g = 3$ and $n_{sg} = 4$.

**Proposition 8.12**: For any graph $G$,

i) $n_{sg} = \tilde{n}_{sg}$ \quad \ldots \quad (17)

ii) $\frac{n_g + \tilde{n}_g}{2} \leq n_{sg} \leq n_g + \tilde{n}_g \quad \ldots \quad (18)$

**Proof**: (i) follows as a direct consequence of definition of $n_{sg}$.

We prove only (ii)

We have $n_g \leq n_{sg}$ by (15)

and $\tilde{n}_g \leq \tilde{n}_{sg} = n_{sg}$ by (17)

\[ \therefore n_g + \tilde{n}_g \leq 2n_{sg} \]

Thus the lower bound of (18) follows.

Next, if $W$ and $\tilde{W}$ are the minimum sn-sets of $G$ and $\tilde{G}$ respectively. Then $W \cup \tilde{W}$ is a global sn-set of $G$. 

\[ n_{sg} \leq |W \cup \tilde{W}| = n_g + \tilde{n}_g \]

Hence (ii) follows.

**Proposition 8.13**: Let G be a co-connected graph of order \( p \geq 4 \). Then \( 2 \leq n_{sg} \leq p - 1 \) ... \((19)\)

**Proof**: Let \( u \) be a vertex of degree at least two (such vertex clearly exists). Then \( V \setminus \{u\} \) is a global sn-set of \( G \), so \( n_{sg} \leq p - 1 \).

**Proposition 8.14**: In a tree \( T \) with \( p \) vertices and \( e \) end vertices, that is not a star, then \( n_{sg} = p - e + 1 \).
REFERENCES

1. E.J. Cockayne and S.T. Hedetniemi,
Towards a Theory of Domination in Graphs,

2. S.T. Hedetniemi and R. Laskar,
Connected Domination in Graphs,
Graph Theory and Combinatorics, Academic Press,

3. R. Laskar and H.B. Walikar,
On Domination Related Concepts in Graph Theory,
Combinatorics and Graph Theory,
Lecture Notes in Mathematics, 885, Springer,

4. E. Sampathkumar,
The Global Domination Number of a Graph,

5. E. Sampathkumar and H.B. Walikar,
The Connected domination number of a graph,
6. E. Sampathkumar and P.S. Neeralagi, 
The Neighbourhood Number of a Graph, 

7. E. Sampathkumar and P.S. Neeralagi, 
Independent, Perfect and Connected Neighbourhood 
Numbers of a Graph, 
Jour. of Combinatorics, Information and System 

8. E. Sampathkumar and L. Pushpa Latha, 
Set Domination in Graphs, 

9. E. Sampathkumar and L. Pushpa Latha, 
The Global Set-Domination Number of a Graph, 