CHAPTER - 6

THE GLOBAL NEIGHBOURHOOD NUMBER OF A GRAPH
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ABSTRACT:

A neighbourhood set (n-set) $S$ of $G$ is a global neighbourhood set (g.n. set) if it is an n-set of the complement $\bar{G}$ of $G$. The global neighbourhood number $n_g(G)$ of $G$ is the minimum cardinality of a g.n. set. This parameter is determined for various known graphs and some bounds are obtained for it.

INTRODUCTION:

A dominating set $S$ of $G$ is a global dominating set (g.d. set) of $G$ if it is also a dominating set of the complement $\bar{G}$ of $G$. The global domination number $\gamma_g=\gamma_g(G)$ of $G$ is the minimum cardinality of a g.d. set. This parameter is introduced in [5].

Now in this chapter we define global neighbourhood number and study this parameter.

An n-set $S$ of $G$ is a global n-set (g.n. set) of $G$ if $S$ is an n-set of $\bar{G}$ also. The global neighbourhood number $n_g=n_g(G)$ of $G$ is the minimum cardinality of a g.n. set of $G$. 
For example,

For the graph in figure 1, \( n_0 = 3 = n_g = \gamma_g \)
and in figure 2, \( n_0 = 2 = \gamma_g, n_g = 3 \)

\[ \therefore n_0 \neq n_g \]

It is clear that \( \gamma_g \leq n_g \). . . . (1)

For a graph \( G \), let \( \tilde{\gamma} = \gamma(\bar{G}), \tilde{\gamma}_g = \gamma_g(\bar{G}) \),
\( \tilde{n}_0 = n_0(\bar{G}), \tilde{n}_g = n_g(\bar{G}) \),

We start with some basic properties of \( n_g \).
Proposition 6.1: For any graph $G$,

(i) $n = n_g$ ...(2)
(ii) $n_o \leq n_g$ ...(3)
(iii) $n_o + \tilde{n}_o \leq n_g \leq n_o + \tilde{n}_o$ ...(4)

Proof: (i) and (ii) follow as a direct consequences of the definition of $n_g$.

We prove only (iii).

We have $n_o \leq n_g$ by (3)
and $\tilde{n}_o \leq \tilde{n}_g = n_g$ by (2)

$\therefore n_o + \tilde{n}_o \leq 2 n_g$

Thus the lower bound of (4) follows.

Next if $S$ and $\tilde{S}$ are the minimum $n$-sets of $G$ and $\tilde{G}$ respectively. Then $S \cup \tilde{S}$ is a g.n. set of $G$.

$\therefore n_g \leq |S \cup \tilde{S}| = n_o + \tilde{n}_o$

Hence (4) follows.

In Proposition 6.2, we determine $n_g$ for some standard graphs.

Proposition 6.2: i) For a graph $G$ with $p$ vertices,

$n_g(G) = p$ if and only if $G = K_p$ or $\tilde{K}_p$.

ii) For $1 \leq m \leq n$, $n_g(K_m, n) = m + 1$

In particular, $n_g(K_1, n) = 2$
iii) \( n_g(C_{2n}) = n + 1 \)
    \( n_g(C_{2n+1}) = n + 2 \)
    for \( n = 2, 3, 4, \ldots \)

iv) For path \( P_n \) on \('n'\) vertices,
    \( n_g(P_3) = 2 \)
    \( n_g(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \)

v) For a wheel \( W_n \) on \('n'\) vertices,
    \( n_g(W_n) = 3 \).

Remark: \( n_g \geq 2 \).

Now we obtain some bounds for \( n_g \).

Proposition 6.3: Let \( S \) be a minimum \( n \)-set of \( G \). If
there exists a vertex \( v \) in \( V-S \) adjacent to only
vertices in \( S \) then,
    \[ n_g \leq n_o + 1 \]  \( \ldots \)  (5)

Proof: This follows since \( S \cup \{v\} \) is a g.n. set.

Corollary 6.3.1: Let \( G = (V_1 \cup V_2, E) \) be a bipartite
graph without isolates, where \( |V_1| = m, |V_2| = n \) and
\( m \leq n \).

Then \( n_g \leq m + 1 \).

Proof: The result follows from (5) since \( n_o \leq m \).
Corollary 6.3.2: For any graph with pendant vertex \( v \) holds. In particular for a tree.

Let \( \alpha_o \) and \( \beta_o \) respectively denote the covering and independence numbers of a graph.

The following results appear in [4].

Theorem A[4]: i) For any graph without isolated vertices,
\[
\eta_o \leq \alpha_o = p - \beta_o
\]

ii) For any graph \( G \),
\[
n_o \leq p - \Delta
\]

Now as a consequence of Theorem A and result (4) we have

Proposition 6.4: For any graph \( G \) such that \( G \) and \( \bar{G} \) have no isolates,

i) \( n_g \leq \alpha_o(G) + \alpha_o(\bar{G}) \)

ii) For any graph \( G \),
\[
n_g(G) \leq p - \Delta + p - \bar{\Delta} = 2p - (\Delta + \bar{\Delta})
\]

where \( \bar{\Delta} = \Delta(\bar{G}) \)

The following result appears in [5].

Theorem B[5]: For a \((p, q)\) graph \( G \) without isolates,
\[
\frac{2q - p(P-3)}{2} \leq \gamma_g \leq p - \beta_o + 1
\]
Proposition 6.5: For a \((p, q)\) graph \(G\) without isolates,
\[
\frac{2q - p(P-3)}{2} \leq n_g \leq p - \beta_0 + 1 = \alpha_0 + 1 \quad \ldots (6)
\]

Proof: The lower bound follows from Theorem B. To establish the upper bound, let \(S\) be a maximum independent set of \(G\), since \(G\) has no isolates, \(V - S\) is an \(n\)-set and hence \((V-S) \cup \{v\}\), for any \(v \in S\) is a g.n. set of \(G\).

\[\therefore n_g \leq p - \beta_0 + 1.\]

The independent neighbourhood number \(n_i(G)\) of \(G\) is the minimum cardinality of an \(n\)-set which is independent. The graph is said to be IN-graph if there exists an independent neighbourhood set. For every graph, \(n_i\) may not exist, for example \(C_5\). This parameter is studied in [6].

Clearly we have, for any IN-graph \(G\),
\[
n_o(G) \leq n_i(G) \leq \beta_0 \quad \ldots (7)
\]

Corollary 6.5.1: For any graph \(G\), if there exists a vertex \(v\) in \(V - S\), adjacent to any vertices of \(S\), where \(S\) is a minimum \(n\)-set of \(G\), then
\[
n_o(G) + n_g(G) \leq 2\alpha_0 + 1
\]

Proof: The proof follows from Proposition 6.5 and Theorem A(i).
As a consequence of (6) and (7) we have Corollary 6.5.2.

Corollary 6.5.2: For any IN-graph of order $p$, without isolates,

i) $n_o(G) + n_g(G) \leq p + 1$

ii) $n_i(G) + n_g(G) \leq p + 1$

The following results appear in [3].

Theorem C[3]: Let $T$ be a tree $\gamma_g(T) = \gamma(T) + 1$ if and only if $T$ is a star or $T$ is a tree of diameter four which is constructed from two or more stars, each having at least two leaves, by connecting the centers of these stars to a common vertex.

We observe that for the tree as stated in Theorem C, $\gamma(T) = n_o(T)$ and $\gamma_g(T) = n_g(T)$

Hence, we have the following result.

Proposition 6.6: Let $T$ be a tree. $n_g(T) = n_o(T) + 1$ if and only if $T$ is a star or $T$ is a tree of diameter four which is constructed from two or more stars, each having at least two leaves, by connecting the centers of these stars to a common vertex.
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