CHAPTER 5

CONNECTED EDGE NEIGHBOURHOOD NUMBER OF A GRAPH
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ABSTRACT:

A set $T \subseteq E$ is a connected edge neighbourhood set if $\langle T \rangle$ is connected. The connected edge neighbourhood number $n'_c(G)$ of $G$ is the minimum cardinality of a connected edge neighbourhood set (dn-set). Besides investigating the relationship of $n'_c(G)$ with other known parameters of $G$, some bounds for $n'_c(G)$ are obtained.

In this chapter, we define connected edge neighbourhood set of a connected graph.

A connected neighbourhood set of a connected graph $G$ is an $n$-set $S$ of $G$, such that, the subgraph $\langle S \rangle$ induced by $S$ is connected. A minimum cardinality of a connected n-set is the connected neighbourhood number $n_c(G)$ of $G$. This parameter is studied in [8].

A connected edge neighbourhood set of a connected graph $G$ is an edge neighbourhood set $T$ of $G$, such that, the subgraph $\langle T \rangle$ induced by $T$ is connected. The connected edge neighbourhood number $n'_c(G)$ of a
connected graph $G$ is the minimum cardinality of a connected edge neighbourhood set of $G$.

Clearly, for a connected graph $G$,

$$n'_o(G) \leq n'_c(G)$$

where $n'_o(G)$ is the edge neighbourhood number of $G$.

For example, in $G_1$, $n'_o = n'_c = 2$ and in $G_2$, $n'_o = 2$, $n'_c = 3$.

Fig. 1
We observe that for $C_4$, a cycle of length 4, $n_Q(C_4) = 2$ and $n'(C_4) = 1$. But for $C_7$, $n_Q(C_7) = 4$ and $n'(C_7) = 5$. Thus there is no relation between $n_Q$ and $n'$. 

Also we observe that $n'_C(G) \neq n'_C(L(G))$. For $G = C_4$, $n'_C(L(G)) = n'_C(C_4) = 3$. But $n'_C(C_4) = 1$.

A connected domination number $\gamma_C(G)$ (the connected edge domination number $\gamma'_C(G)$) of a connected graph $G$ is the minimum cardinality of a dominating set (edge dominating set) $D$, such that, the subgraph $<D>$ is connected. $\gamma_C$ is introduced in [6] and studied in [3] and $\gamma'_C(G)$ is studied in [9].

It is observed that $\gamma' \leq \gamma'_C$ \hspace{1cm} (1) \hspace{1cm} where $\gamma'$ is the edge domination number.

Since every connected edge dominating set is a connected edge neighbourhood set, we have 

$n'_C \leq \gamma'_C$ \hspace{1cm} (2)

We first determine $n'_C$ for some known graphs.

**Proposition 5.1** : (a) $n'_C = 1$ for the following graphs,

(i) $K_p$ (ii) $K_p-x$ (iii) $K_{1,p-1}$ (iv) The wheel on $p$ points.
(b) Suppose $x = uv$ is an edge in $G$ with $N(u) \cap N(v) \neq \emptyset$ and $\deg u + \deg v \geq p$, then $n'_c = 1$; for example, $n'_c(K_{m,n}) = 1$

(c) If $P_n$ is a path on $n$ vertices then
- $n'_c(P_n) = 1$ for $n = 2, 3, 4$
- $n'_c(P_n) = n - 3$ for $n \geq 5$

(d) For a cycle $C_n$ on $n$ vertices,
- $n'_c(C_n) = 1$ if $n = 3, 4$
- $n'_c(C_n) = n - 2$ if $n \geq 5$

(e) For a tree $T$ with $q$ edges $n'_c = q - e$ where $e$ is the number of pendant vertices in $T$.

**Proposition 5.2**: If $S$ is a minimum connected edge neighbourhood set of a connected graph $G$, then $\langle S \rangle$ is a tree.

**Proof**: It is sufficient to prove that $\langle S \rangle$ is acyclic. Suppose $\langle S \rangle$ has a cycle $z$. Let $x$ be an edge on the cycle $z$. Then every edge adjacent to $x$ in $S$ is adjacent to some edge on $z$. Hence $\langle S \rangle - x$ is a connected edge neighbourhood set of $G$, a contradiction. Hence the result.

**Corollary 5.2.1**: If $S$ is a minimum edge neighbourhood set of a graph $G$ then $\langle S \rangle$ is a forest.
**Proposition 5.3**: If $S$ is minimum connected edge dominating set of a connected graph $G$, then $<S>$ is a tree.

**Proof**: Similar to Proposition 5.2.

**Corollary 5.3.1**: If $S$ is a minimum edge dominating set of a graph $G$, then $<S>$ is a forest.

**Proposition 5.4**: For a connected graph $G$,\[\gamma_c - 1 \leq n'_c(G)\]

**Proof**: Let $S = \{x_1, x_2, \ldots, x_k\}$ be a minimum connected edge neighbourhood set of a connected graph $G$ so that, $|S| = n'_c(G) = k$. By Proposition 5.2, $<S> = G'$ is a tree. Thus $G'$ has $k + 1$ vertices each of which is incident to some $x_i$, $i = 1, 2, \ldots, k$.

Hence, these $k + 1$ vertices constitute a connected dominating set of $G$, and thus\[\gamma_c = k + 1 = n'_c + 1\]

or \[\gamma_c - 1 \leq n'_c\]

**Corollary 5.4.1**: $\gamma - 1 \leq n'_c$

**Proposition 5.5**: For any connected graph $G$,\[b \leq n'_c(G)\]

where 'b' is the number of non-pendant bridges of $G$. 
Proof: Let $S$ be a connected edge neighbourhood set of $G$. Let $x = uv$ be a non-pendant bridge of $G$.

Suppose $x \notin S$.

Then $u \in B_1$ and $v \in B_2$ for two different blocks $B_1$ and $B_2$, such that, every path containing $u$ and $v$ contains $x$ also.

Since $x \notin S$, this implies that $\langle S \rangle$ is not connected, a contradiction.

Hence $x \in S$. Thus it follows that $b \leq n'_c(G)$.

Note: The above bound is attained for a tree other than $K_{1,n}$.

The following lower bounds of $\gamma_c$ are established in [3] and [6].

Theorem A(i) [6]: For any connected graph $G$ with $p$ points and maximum degree $\Delta$,

$$\frac{p}{\Delta+1} \leq \gamma_c$$

(ii) [3] $\text{diam}(G) - 1 \leq \gamma_c$

where $\text{diam}(G)$ denote the diameter of $G$.

Proposition 5.6: For any graph $G$ with $p$ points and maximum degree $\Delta$, 

...
\[ \frac{\delta}{\Delta+1} \leq n'_c \]

where \( \delta \) is the minimum degree of \( G \).

**Proof :** From Theorem A(i) and Proposition 5.4 we have

\[ \frac{p}{\Delta+1} - 1 \leq \gamma_c - 1 \leq n'_c \]

i.e.

\[ \frac{p - 1 - \Delta}{\Delta+1} \leq n'_c \]

i.e.

\[ \frac{\delta}{\Delta+1} \leq n'_c \]

\[ \therefore p - 1 - \Delta = \tilde{\delta} \]

Hence

\[ \frac{\delta}{\Delta+1} \leq n'_c \]

**Proposition 5.7 :** For any connected graph \( G \),

\[ \text{diam}(G) - 2 \leq n'_c \]

where \( \text{diam}(G) \) denote the diameter of \( G \).

**Proof :** The result follows from Theorem A (ii).

**Proposition 5.8 :** For any connected graph \( G \) with \( p \) vertices,

\[ n'_c + \epsilon_T \geq p - 1 \]

where \( \epsilon_T \) is maximum number of end vertices in spanning tree \( T \) of \( G \).
**Proof**: Let $S$ be a minimum edge connected neighbourhood set with $k$ edges
\[ \text{i.e. } n'_c = |S| = k. \]

Then, by Proposition 5.2, $< S >$ is a tree with $k + 1$ vertices. We can now form a spanning tree $T$ of $G$ by adding remaining $p - (k + 1)$ vertices of $G$ to $< S >$ and joining each of these vertices to one vertex of $S$ to which it is adjacent.

In this way $T$ will have at least $p - (k + 1)$ end vertices.

So \[ e_T \geq p - (k + 1) \]
\[ \text{i.e. } e_T + k \geq p - 1 \]
\[ \text{i.e. } e_T + n'_c \geq p - 1 \]

The above result attains for all tree. i.e., for any tree $T$ with 'p' vertices and 'e' end edges,

\[ n'_c(T) + e = p - 1 \]
\[ \text{i.e. } n'_c(T) = (p - 1) - e \]
\[ = q - e. \]

**Proposition 5.9**: For a graph $G$, $n'_c = \gamma'_c$ if and only if there exists a minimum connected edge neighbourhood set $S$ which is a connected edge dominating set.

**Proof**: Let $n'_c = \gamma'_c$ and $S$ be a minimum connected edge dominating set. Then $S$ is a connected edge neighbour-
hood set and since $n'_c = \gamma'_c$, $S$ is a minimum connected edge neighbourhood set.

Conversely, let $S$ be a minimum connected edge neighbourhood set, which is a connected edge dominating set. Hence $\gamma'_c \leq n'_c = |S|$ and we know $n'_c \leq \gamma'_c$ (from (2)). So we have $n'_c = \gamma'_c$.

**Corollary 5.9.1**: For any tree $T$, $n'_c = \gamma'_c$

**Proof**: The result is obvious from above Proposition 5.9.

**Proposition 5.10**: For any graph $G$, such that, $G$ and $\bar{G}$ are connected,

(i) $\gamma'_c(G) + \gamma'_c(\bar{G}) \leq 2(p - 2)$ (see [9])

(ii) $n'_c(G) + n'_c(\bar{G}) \leq 2(p - 2)$

**Proof of (ii)**: From (i) and (2) result follows.

Notice that for the cycle $C_5$ of length 5, if $G = C_5$ then $\bar{G} = C_5$ and $n'_c(G) + n'_c(\bar{G}) = 3 + 3 = 6 = 2(p - 2)$. Thus the upper bound in Proposition 5.10(ii) is best possible.

**Corollary 5.10.1**: For any tree $T$ with $q > 2$ edges, $T \neq K_1, n'_c(T) + n'_c(\bar{T}) \leq q$. 
Proposition 5.11: If G has no cycles of length ≤ 4 then \( n'_c = \gamma'_c \).

Proof: Let \( S = \{x_1, x_2, \ldots, x_n\} \) be the collection of minimum number of edges, such that, \( <S> \) is connected and \( G = \bigcup_{x_i \in S} <N(x_i)> \) where \( i = 1, 2, \ldots, n \).

Clearly, \( |S| = n'_c \), since G has no cycles of length ≤ 4.

So, \( S \) covers all edges of \( G \), hence \( S \) is minimum connected dominating set, hence \( n'_c = \gamma'_c \).

Edge Independent and Edge Perfect Neighbourhood Numbers of a Graph

Let \( T \) be an edge neighbourhood set of \( G \). Then \( T \) is an edge independent neighbourhood set (IN'-set) if \( T \) is edge independent. Also, \( T \) is an edge perfect neighbourhood set (PN'-set) if for all \( x, y \in T, x \neq y \), the subgraphs \( <N(x)> \) and \( <N(y)> \) are edge disjoint.

For example in figure 2, \( \{x, y\} \) is both IN'-set and PN'-set.

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Fig. 2
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*Fig. 2: Example of edge independent and edge perfect neighbourhood set.*
We observe that a graph may not have an IN'-set or a PN'-set, for example any cycle of odd length \( n > 5 \). Also, any PN'-set is an IN'-set but not conversely. For example, for the graph in figure 3, the IN'-set \( \{a, b\} \) is not a PN'-set.

DEFINITION 5.1: A graph \( G \) is an IN'-graph (PN'-graph) if \( G \) has an IN'-set (PN'-set).

In view of the above remark, it follows that every PN'-graph is an IN'-graph but not conversely. The graph in figure 2 is an IN'-graph but not a PN'-graph.

DEFINITION 5.2: Let \( G \) be an IN'-graph. The edge independent neighbourhood number \( n_i(G) \) of \( G \) is the minimum cardinality of an IN'-set of \( G \).

DEFINITION 5.3: Let \( G \) be a PN'-graph. The edge perfect neighbourhood number \( n_p(G) \) of \( G \) is the minimum cardinality of a PN'-set of \( G \).
To start with, we deduce some elementary properties of $n'_1$ and $n'_p$.

Since every IN'-set is an edge neighbourhood set, every PN'-set is an IN'-set and every PN'-set is edge independent, we have

**Proposition 5.12**: If $G$ is a PN'-graph, then

$$n'_o \leq n'_i \leq n'_p \leq \beta_1$$

where $\beta_1$ is the edge independence number of $G$.

For the graph $G$, in figure 4,

$$n'_o = n'_i = n'_p = \beta_1 = 1$$

\[\text{Fig. 4}\]

However, for the graph in figure 5,

- $n'_o = |\{a, b\}| = 2$
- $n'_i = |\{a, c, d\}| = 3$
- $n'_p = |\{e, f, c, g\}| = 4$
- $\beta_1 = |\{b, c, e, g, h\}| = 5$

\[\therefore n'_o < n'_i < n'_p < \beta_1\]
**Proposition 5.13:** Every cycle $C_n$, $n \geq 3$ where $n \equiv 0 \pmod{3}$ is PN*-graph.

**Proof:** Let $x_1, x_2, x_3, \ldots$ denote the edges of $C_n$. Then $x_1, x_4, x_7, \ldots x_{3r-2}$ form a PN*-set.

We observe that, if $n'_o = 1$ for a graph $G$ then $n'_o = n'_1 = n'_p = 1$.

**Proposition 5.14:**

i) If $G$ is of order $p$ with maximum degree $\Delta(G) = p - 1$ then $n'_o = n'_1 = n'_p = 1$.

ii) For the complete bipartite graph $K_{m,n}$

$$n'_o = n'_1 = n'_p = 1.$$

iii) For a path $P_n$ of $n$ vertices, $n \geq 2$

$$n'_o = n'_1 = n'_p = \left\lceil \frac{n-1}{3} \right\rceil$$
iv) For a cycle \( C_n \) of \( n \) vertices, \( n \geq 3 \)
\[
\begin{align*}
    n'_1(C_{3r}) &= n'_1(C_{3r+1}) = r, \quad r = 1, 2, 3, \ldots \nonumber \\
    n'_2(C_{3r}) &= r, \quad r = 2, 3, \ldots \nonumber 
\end{align*}
\]
It is not hard to prove.

An edge dominating set \( T \) is an \textit{independent edge dominating set}, if set \( T \) is edge independent. The minimum cardinality of an independent edge dominating set is the \textit{independent edge domination number} and it is denoted by \( i'(G) \).

We observe that \( n'_1 \leq i' \quad \ldots \quad (13) \)
For example, \( n'_1(C_5) = i'(C_5) = 2 \)
But for the graph in figure 6,
\[
    n'_1(G) = 1, \quad i'(G) = 2
\]

![Fig. 6](image)

**Proposition 5.15** : For a graph \( G \), \( n'_1 = i' \) if and only if there exists a minimum IN'-set \( T \), such that, every edge in \( E(G) - T \) is adjacent to some edge \( x \) in \( T \).

**Proof** : Let \( n'_1 = i' \) and \( T \) be a minimum independent edge dominating set. Then \( T \) is a IN'-set also and since
n'_i = i'. T is a minimum IN'-set. Clearly every edge in E(G) - T is adjacent to some edge x in T.

For the converse, suppose there is a minimum IN'-set T of G, such that, every edge in E(G) - T is adjacent to some edge x in T. Then T is an edge dominating set. Hence i' ≤ |T| = n'_i. Since n'_i ≤ i' it follows that n'_i = i'.

**Corollary 5.15.1**: If G has no cycles of length ≤ 4 then n'_o = n'_i = i'.

**Proof**: For an edge x in G, let G_x be the subgraph of G containing precisely x and all edges adjacent to x. Since G has no cycles of length ≤ 4. We have G_x = <N(x)>.

If T is any minimum IN'-set of G, then since G = U_{x∈T} <N(x)> = U_{x∈T} G_x. So it follows that, every edge of E - T is adjacent to some edge in T. Hence by above Proposition 5.16, n'_i = i'.

The following results are immediate.

**Corollary 5.15.2**: If G is a bipartite graph without the quadrilaterals, then n'_i = i'.

It is observed that if a given graph is PN'-graph then t ≤ γ' ≤ i' where t = n'_o or n'_i or n'_p.

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Proposition 5.16: i) For a graph,
\[ n' \leq \gamma' \leq p/2 \] (see [7])

ii) For a \( P_{n'} \)-graph,
\[ t \leq \gamma' < t' \leq p/2 \]
i.e. \( t \leq p/2 \) where \( t = n'_1, n'_o, n'_p \).

Proposition 5.17: If both \( G \) and \( \tilde{G} \) are \( P_{n'} \)-graphs

then \( t(G) + t(\tilde{G}) \leq p \)
and \( t(G) t(\tilde{G}) \leq \left( \frac{p}{2} \right)^2 \)

Proof: From the Proposition 5.16 it follows.
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