CHAPTER 4

THE SPLIT EDGE DOMINATION AND SPLIT EDGE NEIGHBOURHOOD NUMBERS OF A GRAPH
CHAPTER 4

THE SPLIT EDGE DOMINATION AND SPLIT EDGE 
NEIGHBOURHOOD NUMBERS OF A GRAPH

ABSTRACT:

An edge dominating set (edge neighbourhood set) $S$ of a graph $G = (V, E)$ is a split edge dominating set (split edge neighbourhood set) if $<E - S>$ is disconnected. The split edge domination number (split edge neighbourhood number) $\gamma_{sp}(G)$ ($n_{sp}(G)$) of $G$ is the minimum cardinality of a split edge dominating set (split edge neighbourhood set). In this chapter we study split edge dominating number (split edge neighbourhood number) and investigate the relationship of $\gamma_{sp}(G)$ ($n_{sp}(G)$) with other known parameters of $G$.

The graphs considered here are finite, undirected, without loops or multiple edges and have at least one component which is not complete or at least two components neither of which are isolated vertices. Unless otherwise stated, all graphs are assumed to have 'p' vertices and 'q' edges.

A set $S$ of vertices in graph $G$ is a neighbourhood set (n-set) of $G$ if $G = \cup_{u \in S} <N[u]>$ where $<N[u]>$ is the
subgraph of $G$ induced by $u$ and all vertices adjacent to $u$. The *neighbourhood number* $n_o(G)$ of $G$ is the minimum cardinality of an $n$-set of $G$. This parameter is introduced in [5].

For an edge $x = uv$ in $G$, let $N[x] = N(u) \cup N(v)$ where $N(u)$ is the set of vertices adjacent to $u$. A set $T$ of edges in a graph $G$, is an *edge neighbourhood set* (ln-set) if $G = \cup_{u \in T} N[x]$. The *edge (line) neighbourhood number* $n'_o(G)$ of $G$ is the minimum cardinality of an ln-set. This parameter is introduced in [6].

A dominating set $D$ of a graph $G = (V,E)$ is a *split dominating set* if the induced subgraph $\langle V-D \rangle$ is disconnected. Then *split domination number* $\gamma_s(G)$ of $G$ is the minimum cardinality of a split dominating set. This parameter is introduced in [4].

An *edge dominating set* (edge neighbourhood set) $S$ of a graph $G = (V,E)$ is a *split edge dominating set* (split edge neighbourhood set) if $\langle E-S \rangle$ is disconnected. The *split edge domination number* (split edge neighbourhood number) $\gamma'_{sp}(G)$ ($n'_{sp}(G)$) of $G$ is the minimum cardinality of a split edge dominating set (split edge neighbourhood set).
A \gamma'-set is a minimum edge dominating set. A \gamma'_{sp}-set (n'_{sp}-set) can be defined similarly. We note that \gamma'_{sp}-sets (n'_{sp}-sets) exist if the graph is not complete and either contains a non-complete component or contains at least two non-trivial components. Thus, for the rest of this chapter we will assume that G is a non-complete connected graph.

For the graph G in figure 1,
\gamma'(G) = \gamma'_{sp}(G) = 2 = n'_o(G) = n'_sp(G)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Fig. 1}
\end{figure}

However for the graph G in figure 2,
\gamma'(G) = 1 = n'_o(G) = n'_sp(G), \gamma'_{sp}(G) = 2

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Fig. 2}
\end{figure}
For the graph G in figure 3,
$n'_o(G) = 1, \; \gamma'(G) = 2, \; n'_sp(G) = 3, \; \gamma'_sp(G) = 4$

Thus, we observe that, for any graph G,
\[ \gamma'(G) \leq \gamma'_sp(G) \]  
\[ n'_o(G) \leq n'_sp(G) \]  
\[ n'_sp(G) \leq \gamma'_sp(G) \]  
\[ \lambda(G) \leq \gamma'_sp(G) \]  
\[ \lambda(G) \leq n'_sp(G) \]

where \( \lambda(G) \) is the edge connectivity of G.

Next, we determine \( \gamma'_sp \) and \( n'_sp \) for some standard graphs.

**Proposition 4.1**: Let \( t = \gamma'_sp \) or \( n'_sp \)
b) For any path $P_n$ on $n$ vertices,

$$
t(P_{3n-1}) = t(P_{3n}) = t(P_{3n+1}) = n
$$

for $n = 1, 2, 3, \ldots$

c) For a cycle $C_n$ of length $n$,

$$
t(C_{3n-2}) = t(C_{3n-1}) = t(C_{3n}) = n,
$$

for $n = 2, 3, 4, \ldots$

d) For a wheel $W_n$ with $n \geq 4$ vertices,

$$
n_{sp}(W_n) = 3, \gamma_{sp}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 3
$$

e) For complete bipartite graph $K_{m,n}$

$$
t(K_{m,n}) = n \quad \text{for} \quad m \geq n \geq 2
$$

f) For any tree $T$,

$$
\gamma'(T) = n'_O(T) = t(T) \leq q - e
$$

Next, we find some bounds for $\gamma'_{sp}$ and $n'_{sp}$.

**Proposition 4.2 :** For any graph $G$,

$$
n'_{sp}(G) \leq \gamma'_{sp}(G) \leq \alpha_1(G) \quad \ldots \quad (2)
$$

where $\alpha_1(G)$ is the edge covering number of $G$.

**Proof :** Let $T$ be a maximum independent set of edges in $G$. Then $T$ has at least two edges and every edge in $T$ is adjacent to some edge in $E - T$. This implies that $E - T$ is a split edge dominating set of $G$. 
Therefore \( \gamma'_{sp}(G) \leq \alpha_{1}(G) \). The result (2) follows since \( n'_{sp}(G) \leq \gamma'_{sp}(G) \).

Note that for an edge \( x = uv \) in \( G \),
\[
\deg(x) = \deg u + \deg v - 2.
\]

Now we consider the results of \( \gamma' \) as obtained in [3].

**Theorem A[3]**: A line dominating set \( S \) of a graph \( G \) is minimal if and only if for each \( e \in S \), one of the following two conditions holds:

a) \( N(e) \cap S = \emptyset \)

b) there exists a line \( f \in E(G) - S \), such that,
\[
N(f) \cap S = \{e\}.
\]

**Theorem B[3]**: If \( G \) does not have an isolated line then for every minimal line dominating set \( S \) of \( G \).

**Theorem C[3]**: For a graph \( G \),
\[
\gamma'(G) \leq q - \Delta'(G)
\]

Correspondingly we have the following two results.

**Proposition 4.3**: A split edge dominating set \( S \) of a graph \( G \) is minimal if and only if for each \( x \in S \), one of the following two conditions is satisfied.

a) \( N[x] \cap S = \emptyset \)

b) there exists an edge \( y \in E-S \) such that
\[
N[y] \cap S = \{x\}.
\]
Proposition 4.4: If $G$ is a connected graph then for every minimal split edge dominating set $S$ of $G$, $E \cap S$ contains a split edge dominating set.

Correspondingly we have the following results.

Proposition 4.5: Let $S$ be a $\gamma_{sp}'$-set of $G$, then

\[ |E - S| \leq \sum_{x \in S} \deg(x) \quad \ldots \quad (3) \]

equality holds if and only if the following conditions (a) and (b) hold.

(a) $S$ is independent

(b) For each $x \in E - S$ there exists only one edge $y \in S$, such that, $N[x] \cap S = \{y\}$.

Proof: Let $S$ be a $\gamma_{sp}'$-set. Since each edge in $E - S$ is adjacent to at least one edge in $S$, it contributes at least one to sum of the degrees of the edges of $S$.

Hence, \[ |E - S| \leq \sum_{x \in S} \deg(x) \quad \ldots \quad (4) \]

Now let \[ |E - S| = \sum_{x \in S} \deg(x) \quad \ldots \quad (5) \]

Suppose $S$ is not independent.

Let $x_1$ and $x_2$ be any two edges of $S$ having a common vertex. Then $x_1$ is counted twice; once in \[ \deg(x_1) \] and once in \[ \deg(x_2) \]. Then the sum of the degrees of edges in $S$ exceeds $|E - S|$ by at least two, a contradiction to the equality in (5).

Hence, $S$ must be independent.
Now let $|E - S| = \sum_{x \in S} \deg(x)$ and (b) does not hold. Then $N[z] \cap S = \emptyset$ or $|N[z] \cap S| \geq 2$ for some $z \in E - S$.

Since $S$ is a $\gamma'_{sp}$-set the former case does not arise.

Let $x_1$ and $x_2$ belong to $N[z] \cap S$. Then $\sum_{x \in S} \deg(x)$ exceeds $|E - S|$ by at least one, since $z$ is counted twice, once in $\deg(x_1)$ and once in $\deg(x_2)$, a contradiction.

Hence if equality in (i) holds, (a) and (b) hold.

Conversely, if (a) and (b) are true, then

$$|E - S| = \sum_{x \in S} \deg(x).$$

**Proposition 4.6** : For a graph $G$,

$$\left[ \frac{q}{\Delta'(G) + 1} \right] \leq \gamma'_{sp}(G) \leq q - \beta_1(G)$$

where $\beta_1(G)$ is the maximum number of independent edges in $G$.

**Proof** : From Proposition 4.5,

$$|E - S| \leq \sum_{x \in S} \deg(x) \leq \Delta'(G) \gamma'_{sp}(G)$$

hence $q - \gamma'_{sp}(G) \leq \Delta'(G) \gamma'_{sp}(G)$
$$q \leq \gamma'_{sp}(G) + \Delta'(G) \gamma'_{sp}(G)$$
$$= (1 + \Delta'(G)) \gamma'_{sp}(G)$$

$$\therefore \left[ \frac{q}{1+\Delta'(G)} \right] \leq \gamma'_{sp}(G)$$ and hence the lower bound.

Let $S$ be a maximum independent set of edges of $G$. Hence $S$ is also a split edge dominating set of $G$. But we know that every split edge dominating set is an edge dominating set.

Therefore $\gamma'(G) \leq \gamma'_{sp}(G) \leq |E - S| \leq q - (\beta'_{sp}(G))$, hence the upper bound.

**Proposition 4.7**: For a graph $G$, if $t(G) = \frac{q}{\Delta'(G) + 1}$, then $\Delta'(G) + 1$ divides $q$, for $t(G) = \gamma'_{sp}(G)$ or $n'_{sp}(G)$.

**Proof**: Let $S$ be a split edge dominating set (split edge neighbourhood set) of $G$ with $|S| = t(G)$ and let $t(G) = \frac{q}{\Delta'(G) + 1}$. It is clear that $S$ is independent.

Otherwise, from Proposition 4.5,

$$|E - S| < \sum \text{deg}(x) \leq t(G) \Delta'(G)$$

implies $q - t(G) < t(G) \Delta'(G)$

implies $\frac{q}{\Delta'(G) + 1} < t(G) = \frac{q}{\Delta'(G) + 1}$, a contradiction to the assumption.
Now we claim that for each $x \in E - S$ there exists only one edge $y \in S$, such that, $N(x) \cap S = \{y\}$. Suppose not, then again we get $q - \Delta'(G) < t(G) \Delta'(G)$ leading to the same contradiction. Hence $|E - S| = \sum_{x \in S} \deg(x)$ for each $x \in S$. Suppose there exists $w \in S$ such that $\deg(w) < \Delta'(G)$. Since $\deg(z) \leq \Delta'(G)$, for all $z \in E(G)$ from the above inequality we get $q - t(G) < t(G) \Delta'(G)$ leading to the same contradiction. Hence $\deg(x) = \Delta(G)$ for each $x \in S$.

Hence $q - t(G) = t(G) \Delta'(G)$ implies $\Delta'(G) + 1$ divides $q$.

The converse of the above result is not true. For the graph $G$ in figure 2,

$q(G) = 4$, $\Delta'(G) = 3$, $\frac{q}{\Delta'(G) + 1} = 1$ but $\gamma_{sp}'(G) = 2$, $n_{sp}'(G) = 1$.

Corollary 4.7.1: Let $G$ be a graph and let $S$ be a minimum split edge dominating set (split edge neighbourhood set) of $G$, such that, $|S| = 1$, then the following conditions hold.

a) $S$ is independent

b) $|E - S| = \sum_{x \in S} \deg(x)$

c) $\frac{q}{\Delta'(G) + 1} = 1$
We next consider the case when the diameter of $G$ is 2.

**Proposition 4.8**: If $\text{diam}(G) = 2$ then $\gamma'_{sp}(G) \leq \delta(G)$ where $\delta(G)$ is the minimum degree of $G$.

**Proof**: Let $y$ be an edge of minimum degree in $G$. Since $\text{diam}(G) = 2$ there exists an edge $x$ not adjacent to $y$. Hence $x$ must be adjacent to some edge in $N[y]$. Hence it follows that $N[y]$ is a split edge dominating set of $G$.

Thus $\gamma'_{sp}(G) \leq \delta(G)$

**Corollary 4.8.1**: If $\text{diam}(G) = 2$ then $n'_{sp}(G) \leq \delta(G)$

**Proposition 4.9**: For a tree with $p \geq 4$ vertices and $q \geq 3$ edges, $b \leq \gamma' \leq t(T) \leq q - e$ where $'b'$ is the number of support edges in $T$ and $t = \gamma'_{sp}$ or $n'_{sp}$.

**Proof**: In a tree always there exists a minimum edge dominating set which contains one support edge incident with each support. Hence $b \leq \gamma'$.

Also from Proposition 4.1 (f) we have

$$b \leq \gamma' \leq t(T) \leq q - e$$

Hence the result follows.
Proposition 4.10 : Let $G$ be a graph, such that, both $G$ and its complement $\bar{G}$ are connected.

Then $t(G) + t(\bar{G}) \leq q(q - 3)$

where $t(G) = \gamma'_sp(G)$ or $n'_sp(G)$

Further, the bound is attained if and only if $G = P_4$.

Proof : By (2), $t(G) \leq \alpha_1(G)$

Since both $G$ and $\bar{G}$ are connected, $\Delta(G), \Delta(\bar{G}) \leq q - 1$. This implies that $\beta_1(G) \beta_1(\bar{G}) \geq 2$. Hence $t(G) \leq q - 2 \leq 2(q - 1) - q \leq 2p - q$.

Similarly $t(\bar{G}) \leq 2\bar{p} - q$.

Thus $t(G) + t(\bar{G}) \leq 2(p + \bar{p}) - 2q$

$\leq q(q - 1) - 2q$

$\leq q(q - 3)$

Suppose that the bound is attained, then it follows that $t(G) = 2p - q$ and $t(\bar{G}) = 2\bar{p} - q$.

This implies that $p, \bar{p} < q$ and hence both $G$ and $\bar{G}$ are trees. Thus $G = P_4$. The converse is obvious.
REFERENCES

1. F. Harary,

Graph Theory,
Addison-Wesley, Reading Massachusetts, 1969.

2. S.T. Hedetniemi and R. Laskar,

Bibliography on Domination in Graphs and some Basic Definitions of Domination Parameters,

3. S.R. Jayaram,

Line Domination Number in Graphs,

4. V.R. Kulli and Janakiram,

The split domination number of a graph,

5. E. Sampathkumar and P.S. Neeralagi,

The Neighbourhood Number of a Graph,

6. E. Sampathkumar and P.S. Neeralagi,

The Line Neighbourhood Number of a Graph,