CHAPTER 2
\textsc{\textalpha g^*s-Continuous Functions in Topological Spaces}

\section*{§2.1 Introduction}

The class of continuous functions plays an important role in general topological spaces. The stronger and weaker forms of continuity have been introduced and studied by several topologists. In 1960, Levine \cite{54} introduced strongly continuous functions. Some stronger forms of continuous functions were introduced and studied by Noiri \cite{88} in 1980. After that, several mathematicians like Levine \cite{54}, Arya and Gupta \cite{6}, Reilly and Vamanmurthy \cite{102} and Munshi and Bassan \cite{77} have respectively introduced and studied some stronger forms of continuous functions namely, perfectly continuous functions, completely continuous functions and super continuous functions.

In 1963, Levine \cite{56} introduced and the studied the weaker forms of continuity namely semi-continuity. Balachandran et al \cite{9}, Mashhour et al \cite{60, 59}, Abd El-Monsef et al \cite{1}, Biswas \cite{14}, Gnanambal \cite{47}, Devi et al \cite{23, 25}, Dontchev \cite{28}, Maki et al \cite{71}, Sundaram and Sheik John \cite{115}, Veera Kumar \cite{120,121,122,124}, Nono et al \cite{91} and Rajamani and Viswanathan \cite{99}, have introduced g-continuity, \(\alpha\)-continuity and pre-continuity, \(\beta\)-continuity, sg-continuity, gpr-continuity, gs-continuity and \(\alpha g\)-continuity, gsp-continuity, gp-continuity, \(\omega\)-continuity, g*p-continuity, \(\alpha g\)r-continuity, g"s-continuity and g*s-continuity, g"a-continuity and ags-continuity respectively, which are weaker forms continuous functions.
Crossley and Hildebrand [20] introduced the concepts of irresolute functions, which are independent of continuous functions and stronger than semi-continuous functions. Sundaram [112] introduced and investigated gc-irresolute functions. Recently Devi et al [26], Sheik John [106] and Rajamani and Viswanathan [99] introduced and studied $\alpha g$-irresolute, $\omega$-irresolute and $\alpha g s$- irresolute functions respectively.

In 1982, Malghan [73] introduced and studied the concept of generalized closed functions. After that several topologists like Sundaram [112], Noiri [84], Biswas [14], Mashhour et al [59,60], Maki et al [71], Devi et al [26, 22], Gnanambal [47], Sheik John [106], Veera Kumar [121], Nono et al [91] and Rajamani and Viswanathan [99] introduced and studied generalized open functions, semi closed functions, semi open functions, $\alpha$-open functions, gp-closed functions, $\alpha g$-closed and gs-closed functions, gpr-closed functions, $\omega$-closed and $\omega$-open functions, $\alpha g r$-closed functions, $g^#\alpha$-closed and $g^#\alpha$-open functions and $\alpha g s$-closed and $\alpha g s$-open functions.

The concepts of homeomorphism has been generalized by many topologists. Biswas [14] and Crossley and Hilderbrand [20] have introduced and studied semi-homeomorphisms which are strictly weaker than homeomorphisms in topological spaces. Maki et al [69] have introduced and studied g-homeomorphisms and gc-homeomorphisms in topological spaces. Recently many researchers like Devi [24], Gnanambal [47], Sheik John [106] and Nono et al [91] have introduced and investigated several types of homeomorphisms in topological spaces.

This chapter contains four sections. In the second section, we introduce the concept of a new class of continuous functions called $\alpha g^s$-
continuous functions and investigate some of their properties. We also introduce the class of $\alpha g^s$-irresolute functions analogous to irresolute functions. Moreover, we introduce the concepts of $\alpha g^s$-closed functions and $\alpha g^s$-open functions and investigate some of their properties.

In the third section we introduce some stronger forms of continuous functions namely, strongly $\alpha g^s$-continuous, perfectly $\alpha g^s$-continuous and completely $\alpha g^s$-continuous functions in topological spaces and discuss some of their properties.

In the last section, we introduce the new class of homeomorphism known as $\alpha g^s$-homeomorphisms in topological spaces and obtain some of their properties and we also prove that $g$-homeomorphism and $\alpha g^s$-homeomorphisms are independent.

**Definition 2.1.1:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a

i) strongly $\alpha$-continuous [54] if $f^{-1}(G)$ is both open and closed in $(X, \tau)$ for each subset $G$ of $(Y, \sigma)$.

ii) perfectly $\alpha$-continuous [88] if $f^{-1}(G)$ is both open and closed set in $(X, \tau)$ for each open set $G$ of $(Y, \sigma)$.

iii) perfectly $g$-continuous [113] if $f^{-1}(G)$ is both open and closed set in $(X, \tau)$ for each $g$-open set $G$ of $(Y, \sigma)$.

iv) strongly $\omega$-continuous [106] if $f^{-1}(G)$ is $\omega$-open set in $(X, \tau)$ for each subset $G$ of $(Y, \sigma)$.

v) perfectly $\omega$-continuous [106] if $f^{-1}(G)$ is both open and closed set in $(X, \tau)$ for each $\omega$-open set $G$ of $(Y, \sigma)$. 35
vi) completely continuous [6] if $f^1(G)$ is regular open set in $(X, \tau)$ for each open set $G$ of $(Y, \sigma)$.

**Definition 2.1.2**: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a

(i) semi-continuous [56] if $f^1(A)$ is semi-closed in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(ii) pre-continuous [59] if $f^1(A)$ is pre-closed in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(iii) $\alpha$-continuous [60] if $f^1(A)$ is $\alpha$-closed in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(iv) semi-pre-continuous (= $\beta$-continuous [1]) [2] if $f^1(G)$ is semi-pre-open set in $(X, \tau)$ for every open set $G$ of $(Y, \sigma)$.

(v) g-continuous [9] if $f^1(A)$ is g-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(vi) sg-continuous [14] if $f^1(A)$ is sg-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(vii) gs-continuous [112] if $f^1(A)$ is gs-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(viii) $\alpha g$-continuous [25] if $f^1(A)$ is $\alpha g$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(ix) gsp-continuous [28] if $f^1(A)$ is gsp-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(x) gp-continuous [71] if $f^1(A)$ is gp-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$. 36
(xi) gpr-continuous [47] if $f^l(A)$ is gpr-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xii) $g^*$-continuous (= Strongly g-continuous [96]) [119] if $f^l(A)$ is $g^*$-closed (= strongly g-closed) set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xiii) $\omega$-continuous [106] if $f^l(A)$ is $\omega$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xiv) $g^p$-continuous [120] if $f^l(A)$ is $g^p$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xv) $\alpha gr$-continuous [121] if $f^l(A)$ is $\alpha gr$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xvi) $g^s$-continuous [122] if $f^l(A)$ is $g^s$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xvii) $g^s$-continuous [124] if $f^l(A)$ is $g^s$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xviii) $g^\#\alpha$-continuous [91] if $f^l(A)$ is $g^\#\alpha$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

(xix) $\alpha gs$-continuous [99] if $f^*(A)$ is $\alpha gs$-closed set in $(X, \tau)$ for every closed set $A$ of $(Y, \sigma)$.

**Definition 2.1.3:** A function $f: (X, \tau) \to (Y, \sigma)$ is called a

i) g-closed [73] (resp. g-open [112]) if $f(G)$ is g-closed (resp. g-open) in $(Y, \sigma)$ for every closed (resp. open) set $G$ in $(X, \tau)$.
ii) semi-closed [84] if \( f(G) \) is semi-closed in \((Y, \sigma)\) for every closed set \( G \) in \((X, \tau)\).

iii) sg-open [22] if \( f(G) \) is sg-open in \((Y, \sigma)\) for every open set \( G \) in \((X, \tau)\).

iv) \( \alpha g\)-closed [26] (resp. \( \alpha g\)-open [26]) if \( f(G) \) is \( \alpha g\)-closed (resp. \( \alpha g\)-open) in \((Y, \sigma)\) for every closed (resp. open) set \( G \) in \((X, \tau)\).

v) gp-closed [89] if \( f(G) \) is gp-closed in \((Y, \sigma)\) for every closed set \( G \) in \((X, \tau)\).

vi) gpr-closed [47] (resp. gpr-open [47]) if \( f(G) \) is gpr-closed (resp. gpr-open) in \((Y, \sigma)\) for every closed (resp. open) set \( G \) in \((X, \tau)\).

vii) \( \alpha gr\)-closed [121] (resp. \( \alpha gr\)-open [121]) if \( f(G) \) is \( \alpha gr\)-closed (resp. \( \alpha gr\)-open) in \((Y, \sigma)\) for every closed (resp. open) set \( G \) in \((X, \tau)\).

viii) \( \omega \)-closed [106] (resp. \( \omega \)-open [106]) if \( f(G) \) is \( \omega \)-closed (resp. \( \omega \)-open) in \((Y, \sigma)\) for every closed (resp. open) set \( G \) in \((X, \tau)\).

ix) \( \alpha gs\)-closed [99] (resp. \( \alpha gs\)-open [99]) if \( f(G) \) is \( \alpha gs\)-closed (resp. \( \alpha gs\)-open) in \((Y, \sigma)\) for every closed (resp. open) set \( G \) in \((X, \tau)\).

**Definition 2.1.4:** A function \( f: (X, \tau) \to (Y, \sigma) \) is called a

i) irresolute [20] if \( f^{-1}(G) \) is semi-open set in \((X, \tau)\) for each semi-open set \( G \) of \((Y, \sigma)\).

ii) gc-irresolute [112] if \( f^{-1}(G) \) is g-closed set in \((X, \tau)\) for g-closed set \( G \) of \((Y, \sigma)\).

iii) \( \alpha g\)-irresolute [25] if \( f^{-1}(G) \) is \( \alpha g\)-closed set in \((X, \tau)\) for every \( \alpha g\)-closed set \( G \) of \((Y, \sigma)\).
iv) gs- irresolute [112] if \( f^{-1}(G) \) is gs-closed set in \((X, \tau)\) for every gs-closed set \( G \) of \((Y, \sigma)\).

v) g*- irresolute [119] if \( f^{-1}(G) \) is g*-closed set in \((X, \tau)\) for every g*-closed set \( G \) of \((Y, \sigma)\).

vi) \( \omega \)- irresolute [106] if \( f^{-1}(G) \) is \( \omega \)-closed set in \((X, \tau)\) for \( \omega \)-closed set \( G \) of \((Y, \sigma)\).

vii) \( \alpha^g s \)- irresolute [99] if \( f^{-1}(G) \) is \( \alpha^g s \)-closed set in \((X, \tau)\) for every \( \alpha^g s \)-closed set \( G \) of \((Y, \sigma)\).

**Definition 2.1.5:** A bijective function \( f: (X, \tau) \to (Y, \sigma) \) is called a

i) generalized homeomorphism (g- homeomorphism) [69] if \( f \) is both g-continuous and g-open,

ii) gc- homeomorphism [69] if both \( f \) and \( f^{-1} \) are gc-irresolute functions,

iii) \( \alpha \)-generalized-homeomorphism (\( \alpha g \)- homeomorphism) [23] if \( f \) is both \( \alpha g \)-continuous and \( \alpha g \)-open,

iv) \( \alpha^* \)- homeomorphism [23] if \( f \) and \( f^{-1} \) are \( \alpha \)-irresolute functions,

v) \( g^\# \alpha \)- homeomorphism [91] if \( f \) is both \( g^\# \alpha \)-open and \( g^\# \alpha \)-continuous maps.
§ 2.2 $\alpha g^s$-Continuous Functions in Topological Spaces

In this section, we introduce the concept of a new class of continuous functions called $\alpha g^s$-continuous functions and investigate some of their properties. We also introduce the class of $\alpha g^s$-irresolute functions analogous to irresolute functions. Moreover, we introduce the concepts of $\alpha g^s$-closed functions and $\alpha g^s$-open functions and obtain some of their properties.

**Definition 2.2.1:** A function $f: X \rightarrow Y$ is said to be $\alpha g^s$-semi-continuous (briefly $\alpha g^s$-continuous) if the inverse image of every closed set in $Y$ is $\alpha g^s$-closed in $X$.

**Theorem 2.2.2:** A function $f: X \rightarrow Y$ is $\alpha g^s$-continuous if and only if the inverse image of every open set in $Y$ is $\alpha g^s$-open in $X$.

**Proof:** Let $F$ be an open set in $Y$. Then $F^c$ is closed in $Y$. Since $f$ is $\alpha g^s$-continuous, $f^{-1}(F^c)$ is $\alpha g^s$-closed in $X$. But $f^{-1}(F^c) = X - f^{-1}(F)$ which is $\alpha g^s$-closed set in $X$. Therefore $f^{-1}(F)$ is $\alpha g^s$-open set in $X$.

Conversely, assume that the inverse image of every open set in $Y$ is $\alpha g^s$-open in $X$. Let $V$ be a closed set in $Y$. Then $V^c$ is open in $Y$. By hypothesis, $f^{-1}(V^c) = X - f^{-1}(V)$ is $\alpha g^s$-closed set in $X$. So $f^{-1}(V)$ is $\alpha g^s$-closed in $X$. Thus $f$ is $\alpha g^s$-continuous function.

**Theorem 2.2.3:** If a function $f: X \rightarrow Y$ is continuous, then $f$ is $\alpha g^s$-continuous.

**Proof:** Let $V$ be an open set in $Y$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. And therefore $f^{-1}(V)$ is $\alpha g^s$-open in $X$. Hence $f$ is $\alpha g^s$-continuous.
The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.4:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}$, \{b, c\}\}. Define a function $f: X \to Y$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then $f$ is $\alpha g^{*}$s -continuous but not continuous, since for the open set $\{b, c\}$ in $Y$, $f^{-1}(\{b, c\}) = \{a, b\}$ is not open in $X$ but it is $\alpha g^{*}$s-open in $X$.

**Theorem 2.2.5:** Every $\alpha g^{*}$s-continuous function is $\alpha g r$ -continuous function but not conversely.

**Proof:** Let $f: X \to Y$ be a $\alpha g^{*}$s-continuous function. Let $V$ be a closed set in $Y$. Then $f^{-1}(V)$ is $\alpha g^{*}$s-closed in $X$ as $f$ is $\alpha g^{*}$s-continuous. By Theorem 1.2.6, $f^{-1}(V)$ is $\alpha g r$-closed in $X$. Hence $f$ is $\alpha g^{*}$s-continuous function.

**Example 2.2.6:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}$, \{b, c\}\}. Then the identity function $f: X \to Y$ is $\alpha g r$ -continuous but not a $\alpha g^{*}$s-continuous, since for the closed set $\{a\}$ in $Y$, $f^{-1}(\{a\}) = \{a\}$ is not a $\alpha g^{*}$s-closed but it is $\alpha g r$-open in $X$.

**Theorem 2.2.7:** Every $\alpha$-continuous function is $\alpha g^{*}$s -continuous function but not conversely.

**Proof:** Let $f: X \to Y$ be an $\alpha$-continuous function. Let $F$ be a closed set in $Y$. Then $f^{-1}(V)$ is $\alpha$-closed in $X$. And therefore $f^{-1}(V)$ is $\alpha g^{*}$s-closed in $X$. Hence $f$ is $\alpha g^{*}$s-continuous function.

**Example 2.2.8:** Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}$, \{a\}, \{b\}, \{a, b\}\}. Then the identity function $f: X \to Y$ is $\alpha g^{*}$s -continuous
but not $\alpha$-continuous, since for the closed set $\{b, c\}$ in $Y$, $f^{-1}(\{b, c\}) = \{b, c\}$ is not an $\alpha$-closed in $X$ but it is $\alpha g^* s$-closed in $X$.

**Theorem 2.2.9:** Every $\alpha g^* s$ -continuous function is $g^* s$ -continuous function but not conversely.

**Proof:** Let $f: X \to Y$ be a $\alpha g^* s$-continuous function. Let $F$ be a closed set in $Y$. Then $f^{-1}(F)$ is $\alpha g^* s$-closed in $X$. And therefore $f^{-1}(F)$ is $g^* s$-closed in $X$. Therefore $f$ is $g^* s$-continuous function.

**Example 2.2.10:** Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Define a function $f: X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then the function $f$ is $g^* s$ -continuous but not $\alpha g^* s$-continuous, since for the closed set $\{a\}$ in $Y$, $f^{-1}(\{a\}) = \{b\}$ is not a $\alpha g^* s$-closed in $X$ but it is $g^* s$-closed in $X$.

**Theorem 2.2.11:** Every $\alpha g^* s$ -continuous function is $\alpha g s$ -continuous function but not conversely.

**Proof:** Let $f: X \to Y$ be a $\alpha g^* s$-continuous function. Let $F$ be a closed set in $Y$. Then $f^{-1}(F)$ is $\alpha g^* s$-closed in $X$. And therefore $f^{-1}(F)$ is $\alpha g s$-closed in $X$. Therefore $f$ is $\alpha g s$-continuous function.

**Example 2.2.12:** Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f: X \to Y$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then the function $f$ is $\alpha g s$ -continuous but not $\alpha g^* s$-continuous, since for the closed set $\{a, c\}$ in $Y$, $f^{-1}(\{a, c\}) = \{a, b\}$ is not $\alpha g^* s$-closed in $X$ but it is $\alpha g s$-closed in $X$. 

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Theorem 2.2.13: Every $\alpha g^s$-continuous function is $\alpha g$-continuous function but not conversely.
Proof: The proof follows from Theorem 1.2.10.

Example 2.2.14: In Example 2.2.12, the function $f$ is $\alpha g$-continuous but not $\alpha g^s$-continuous, since for the closed set $\{a, c\}$ in $Y$, $f^1(\{a, c\}) = \{a, b\}$ is not a $\alpha g^s$-closed in $X$ but it is $\alpha g$-closed in $X$.

Theorem 2.2.15: Every $\alpha g^s$-continuous function is $gs$-continuous function but not conversely.
Proof: Let $f: X \to Y$ be a $\alpha g^s$-continuous function and $F$ be a closed set in $Y$. Then $f^{-1}(F)$ is $\alpha g^s$-closed in $X$. And therefore $f^{-1}(F)$ is $gs$-closed in $X$. Hence $f$ is $gs$-continuous function.

Example 2.2.16: Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, b\}\}$. Define a function $f: X \to Y$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then the function $f$ is $gs$-continuous but not $\alpha g^s$-continuous, since for the closed set $\{c\}$ in $Y$, $f^{-1}(\{c\}) = \{b\}$ is not $\alpha g^s$-closed in $X$ but it is $gs$-closed in $X$.

Theorem 2.2.17: Every $\alpha g^s$-continuous function is $gsp$-continuous function but not conversely.
Proof: Let $f: X \to Y$ be a $\alpha g^s$-continuous function. Let $K$ be a closed set in $Y$. Then $f^{-1}(K)$ is $\alpha g^s$-closed in $X$. And therefore $f^{-1}(K)$ is $gsp$-closed in $X$. Therefore $f$ is $gsp$-continuous function.
**Example 2.2.18:** In the Example 2.2.16, the function \( f \) is gsp -continuous but not \( \alpha g^*s \)-continuous, since for the closed set \( \{c\} \) in \( Y \), \( f^{-1}(\{c\}) = \{b\} \) is not \( \alpha g^*s \)-closed in \( X \) but it is gsp-closed in \( X \).

**Theorem 2.2.19:** Every \( \alpha g^*s \) -continuous function is gp -continuous function but not conversely.

**Proof:** Let \( f: X \to Y \) be a \( \alpha g^*s \)-continuous function. Let \( K \) be a closed set in \( Y \). Then \( f^{-1}(K) \) is \( \alpha g^*s \)-closed in \( X \). And therefore \( f^{-1}(K) \) is gp-closed in \( X \). Therefore \( f \) is gp-continuous function.

**Example 2.2.20:** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \). Define a function \( f: X \to Y \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then the function \( f \) is gp -continuous but not \( \alpha g^*s \)-continuous, since for the closed set \( \{b, c\} \) in \( Y \), \( f^{-1}(\{b, c\}) = \{a, c\} \) is not a \( \alpha g^*s \)-closed in \( X \) but it is gp-closed in \( X \).

**Theorem 2.2.21:** Every \( \alpha g^*s \) -continuous function is sg -continuous function but not conversely.

**Proof:** Let \( f: X \to Y \) be a \( \alpha g^*s \)-continuous function. Let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is \( \alpha g^*s \)-closed in \( X \). And therefore \( f^{-1}(F) \) is sg-closed in \( X \). Thus \( f \) is sg-continuous function.

**Example 2.2.22:** In Example 2.2.16, the function \( f \) is sg -continuous but not \( \alpha g^*s \)-continuous, since for the closed set \( \{c\} \) in \( Y \), \( f^{-1}(\{c\}) = \{b\} \) is not a \( \alpha g^*s \)-closed in \( X \) but it is sg-closed in \( X \).

**Theorem 2.2.23:** Every \( \alpha g^*s \) -continuous function is g*p -continuous function but not conversely.
Proof: Let \( f: X \rightarrow Y \) be a \( \alpha g \)s-continuous function. Let \( K \) be a closed set in \( Y \). Then \( f^{-1}(K) \) is \( \alpha g \)s-closed in \( X \). And therefore \( f^{-1}(K) \) is \( g^*p \)-closed in \( X \). Hence \( f \) is \( g^*p \)-continuous function.

Example 2.2.24: In Example 2.2.20, the function \( f \) is \( g^*p \) -continuous but not \( \alpha g \)s-continuous, since for the closed set \( \{b, c\} \) in \( Y \), \( f^{-1}(\{b, c\}) = \{a, c\} \) is not a \( \alpha g \)s-closed in \( X \) but it is \( g^*p \)-closed in \( X \).

Theorem 2.2.25: Every \( \alpha g \)s -continuous function is \( g^# \alpha \) -continuous function but not conversely.

Proof: Let \( f: X \rightarrow Y \) be a \( \alpha g \)s-continuous function. Let \( K \) be a closed set in \( Y \). Then \( f^{-1}(K) \) is \( \alpha g \)s-closed in \( X \). Therefore \( f^{-1}(K) \) is \( g^# \alpha \)-closed in \( X \). Hence \( f \) is \( g^# \alpha \)-continuous function.

Example 2.2.26: Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Define a function \( f: X \rightarrow Y \) by \( f(a) = a \), \( f(b) = c \) and \( f(c) = b \). Then the function \( f \) is \( g^# \alpha \)-continuous but not \( \alpha g \)s-continuous, since for the closed set \( \{a, c\} \) in \( Y \), \( f^{-1}(\{a, c\}) = \{a, b\} \) is not a \( \alpha g \)s-closed in \( X \) but it is \( g^# \alpha \)-closed in \( X \).

Theorem 2.2.27: Every \( \alpha g \)s -continuous function is \( g^#s \) -continuous function but not conversely.

Proof: Let \( f: X \rightarrow Y \) be a \( \alpha g \)s-continuous function. Let \( K \) be a closed set in \( Y \). Then \( f^{-1}(K) \) is \( \alpha g \)s-closed in \( X \). And therefore \( f^{-1}(K) \) is \( g^#s \)-closed in \( X \). Hence \( f \) is \( g^#s \)-continuous function.

Example 2.2.28: Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a, b\}\} \). Define a function \( f: X \rightarrow Y \) by \( f(a) = c \), \( f(b) = b \) and \( f(c) = a \).
Then the function \( f \) is \( g^s \)-continuous but not a \( \alpha g^s \)-continuous, since for the closed set \( \{c\} \) in \( Y \), \( \Gamma^f(\{c\}) = \{a\} \) is not a \( \alpha g^s \)-closed in \( X \) but it is \( g^s \)-closed in \( X \).

**Remark 2.2.29:** The concepts of \( \alpha g^s \)-continuous functions are independent from \( g \)-continuous functions and \( g^* \)-continuous functions as seen from the following examples.

**Example 2.2.30:** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a, c\}\} \). Then the identity function \( f: X \rightarrow Y \) is \( \alpha g^s \)-continuous but not \( g \)-continuous and \( g^* \)-continuous, since for the closed set \( \{b\} \) in \( Y \), \( f^{-1}(\{b\}) = \{b\} \) is not \( g \)-closed set and \( g^* \)-closed set in \( X \) but it is \( \alpha g^s \)-closed in \( X \).

**Example 2.2.31:** In Example 2.2.16, define a function \( f: X \rightarrow Y \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). Then the function \( f \) is both \( g \)-continuous and \( g^* \)-continuous but not a \( \alpha g^s \)-continuous, since for the closed set \( \{b, c\} \) in \( Y \), \( f^{-1}(\{b, c\}) = \{a, c\} \) is not \( \alpha g^s \)-closed in \( X \).

**Remark 2.2.32:** The composition of two \( \alpha g^s \)-continuous functions need not be a \( \alpha g^s \)-continuous function as seen from the following example.

**Example 2.2.33:** Let \( X = Y = Z = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a, c\}\} \). Define a function \( f: X \rightarrow Y \) by \( f(a) = f(c) = c \) and \( f(b) = b \) and let \( g: Y \rightarrow Z \) be the identity function. Then \( f \) and \( g \) are \( \alpha g^s \)-continuous functions but their composition function \( g f: X \rightarrow Z \) is not an \( \alpha g^s \)-continuous, since for the closed set \( \{b\} \) in \( Z \), \( (g f)^{-1}(\{b\}) = f^{-1}(g^{-1}(\{b\})) = f^{-1}(\{b\}) = \{b\} \) is not \( \alpha g^s \)-closed in \( X \).
**Theorem 2.2.34:** If \( f: X \to Y \) and \( g: Y \to Z \) are two \( \alpha_g\)-continuous functions and \( Y \) is \( \alpha_g T_{1/2} \)-space, then \( gof: X \to Z \) is \( \alpha_g\)-continuous function.

**Proof:** Let \( F \) be a closed set in \( Z \). Then \( g^{-1}(F) \) is \( \alpha_g\)-closed set in \( Y \) since \( g \) is \( \alpha_g\)-continuous function. As \( Y \) is \( \alpha_g T^* \)-space, \( g^{-1}(F) \) is closed set in \( Y \). Again since \( f \) is \( \alpha_g\)-continuous, \( f^{-1}(g^{-1}(F)) = (gof)^{-1}(F) \) is \( \alpha_g\)-closed in \( X \). Hence \( gof: X \to Z \) is \( \alpha_g\)-continuous.

**Theorem 2.2.35:** If \( f: X \to Y \) is \( \alpha_g\)-continuous and \( g: Y \to Z \) is continuous, then their composition \( gof: X \to Z \) is \( \alpha_g\)-continuous function.

**Proof:** Let \( F \) be a closed set in \( Z \). Since \( g \) is continuous function, \( g^{-1}(F) \) is closed set in \( Y \). Again since \( f \) is \( \alpha_g\)-continuous, \( f^{-1}(g^{-1}(F)) = (gof)^{-1}(F) \) is \( \alpha_g\)-closed in \( X \). Hence \( gof: X \to Z \) is \( \alpha_g\)-continuous.

**Theorem 2.2.36:** Every strongly continuous function is \( \alpha_g\)-continuous.

**Proof:** Let \( f: X \to Y \) be a strongly continuous. Let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is both open and closed in \( X \). And therefore \( f^{-1}(F) \) is \( \alpha_g\)-closed in \( X \). Hence \( f \) is \( \alpha_g\)-continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.37:** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, c\}\} \). \( \sigma = \{Y, \phi, \{a\}, \{a, b\}\} \). Define a function \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then \( f \) is \( \alpha_g\)-continuous function but not strongly continuous, since for the closed set \( \{a, c\} \) in \( Y \), \( f^{-1}(\{a, c\}) = \{a, b\} \) is not both open and closed in \( X \).
Remark 2.2.38: From the above results we get the following diagram.

![Diagram](image)

**Theorem 2.2.39:** Every perfectly continuous function is $\alpha g^s$-continuous.

**Proof:** Let $f: X \to Y$ be a perfectly continuous function. Let $G$ be an open set in $Y$. Then $f^{-1}(G)$ is both open and closed in $X$. So $f^{-1}(G)$ is open in $X$. And therefore $f^{-1}(G)$ is $\alpha g^s$-open in $X$. Hence $f$ is $\alpha g^s$-continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.40:** In Example 2.2.37, the function $f$ is $\alpha g^s$-continuous function but not perfectly continuous, since for the open set $\{a, c\}$ in $Y$, $f^{-1}(\{a, c\}) = \{a, b\}$ is not both open and closed in $X$. 

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Definition 2.2.41: A function $f: X \to Y$ is called $\alpha g^*\text{-semi- irresolute}$ function (briefly $\alpha g^*s\text{-irresolute}$) if the inverse image of every $\alpha g^*s\text{-closed}$ set in $Y$ is $\alpha g^*s\text{-closed}$ in $X$.

Theorem 2.2.42: A function $f: X \to Y$ is $\alpha g^*s\text{-irresolute}$ if and only if the inverse image of every $\alpha g^*s\text{-open}$ set in $Y$ is $\alpha g^*s\text{-open}$ in $X$.

Proof: Suppose that a function $f: X \to Y$ is $\alpha g^*s\text{-irresolute}$ function. Let $B$ be a $\alpha g^*s\text{-open}$ set in $Y$. Then $B^c$ is $\alpha g^*s\text{-closed}$ in $Y$. Since $f$ is $\alpha g^*s\text{-irresolute}$ function, $f^{-1}(B^c)$ is $\alpha g^*s\text{-closed}$ set in $X$. Therefore $f^{-1}(B)$ is $\alpha g^*s\text{-open}$ in $X$.

Conversely, assume that the inverse image of every $\alpha g^*s\text{-open}$ set in $Y$ is $\alpha g^*s\text{-open}$ in $X$. Let $K$ be a $\alpha g^*s\text{-closed}$ in $Y$. Then $K^c$ is $\alpha g^*s\text{-open}$ in $Y$. By hypothesis, $f^{-1}(K^c)$ is $\alpha g^*s\text{-open}$ in $Y$. That is $f^{-1}(K)$ is $\alpha g^*s\text{-closed}$ in $X$. Hence $f$ is $\alpha g^*s\text{-irresolute}$ function.

Theorem 2.2.43: If a function $f: X \to Y$ is $\alpha g^*s\text{-irresolute}$, then $f$ is $\alpha g^*s\text{-continuous}$.

Proof: Let $F$ be any closed set in $Y$. Then $F$ is $\alpha g^*s\text{-closed}$ set in $Y$. As $f$ is $\alpha g^*s\text{-irresolute}$, $f^{-1}(F)$ is $\alpha g^*s\text{-closed}$ set in $X$. Therefore $f$ is $\alpha g^*s\text{-continuous}$ function.

The converse of the above theorem need not be true as seen from the following example.

Example 2.2.44: Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Define a function $f: X \to Y$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then $f$ is not a $\alpha g^*s\text{-irresolute}$, since for the $\alpha g^*s\text{-closed}$ set $\{c\}$ in $Y$, $f^{-1}(\{c\})$ is not $\alpha g^*s\text{-open}$ in $X$. 

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\( f^{-1}(\{c\}) = \{a\} \) is not a \( \alpha g^s \)-closed set of \( X \). However \( f \) is \( \alpha g^s \)-continuous function.

**Theorem 2.2.45:** If \( f: X \rightarrow Y \) is \( \alpha g^s \)-irresolute and \( g: Y \rightarrow Z \) is \( \alpha g^s \)-continuous, then their composition \( gof: X \rightarrow Z \) is \( \alpha g^s \)-continuous function.

**Proof:** Let \( V \) be an open set in \( Z \). Since \( g \) is \( \alpha g^s \)-continuous, \( g_-(V) \) is \( \alpha g^s \)-open in \( Y \). Again since \( f \) is \( \alpha g^s \)-irresolute, \( f^{-1}(g_-(V)) \) is \( \alpha g^s \)-open in \( X \). But \( f^{-1}(g_-(V)) = (gof)_-(V) \) is \( \alpha g^s \)-open in \( X \). Thus \( gof \) is \( \alpha g^s \)-continuous function.

**Theorem 2.2.46:** If \( f: X \rightarrow Y \), \( g: Y \rightarrow Z \) be two \( \alpha g^s \)-irresolute functions, then their composition \( gof: X \rightarrow Z \) is a \( \alpha g^s \)-irresolute function.

**Proof:** Let \( V \) be a \( \alpha g^s \)-open in \( Z \). Then \( f^{-1}(V) \) is \( \alpha g^s \)-open set in \( Y \) since \( g \) is \( \alpha g^s \)-irresolute. Again since \( f \) is \( \alpha g^s \)-irresolute, \( f^{-1}(g^{-1}(V)) \) is \( \alpha g^s \)-open in \( X \). But \( f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \) is \( \alpha g^s \)-open in \( X \). Hence \( gof \) is \( \alpha g^s \)-irresolute function.

**Remark 2.2.47:** The notions of irresolute functions and \( \alpha g^s \) -irresolute functions are independent of each other as seen from the following examples

**Example 2.2.48:** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Define a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(c) = a \) and \( f(b) = b \). Then the function \( f \) is \( \alpha g^s \)-irresolute but not irresolute.

**Example 2.2.49:** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \). Define a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). Then the function \( f \) is irresolute but not \( \alpha g^s \)-irresolute.
**Theorem 2.2.50:** Let $f: X \to Y$ be onto $\alpha g^*s$-irresolute and closed function. If $X$ is $\alpha g s T^*_{1/2}$-space, then $Y$ is also $\alpha g s T^*_{1/2}$-space.

**Proof:** Let $F$ be a $\alpha g^*s$-closed set in $Y$. Then $f^{-1}(F)$ is $\alpha g^*s$-closed in $X$ as $f$ is $\alpha g^*s$-irresolute function. Since $X$ is $\alpha g s T^*_{1/2}$-space, $f^{-1}(F)$ is closed in $X$. Again since $f$ is closed function, $f(f^{-1}(F)) = F$ is closed in $Y$. Thus $Y$ is $\alpha g s T^*_{1/2}$-space.

**Theorem 2.2.51:** If $f$ is $\alpha g^*s$-continuous and $g$ is $\alpha g^*s$-irresolute and $Y$ is $\alpha g s T^*_{1/2}$-space, then $gof: X \to Z$ is $\alpha g^*s$-irresolute function.

**Proof:** Let $F$ be a $\alpha g^*s$-closed set in $Z$. Then $g^{-1}(F)$ is $\alpha g^*s$-closed in $Y$ as $g$ is $\alpha g^*s$-irresolute function. Since $Y$ is $\alpha g s T^*_{1/2}$-space, $g^{-1}(F)$ is closed in $Y$. Again since $f$ is $\alpha g^*s$-continuous, $f^{-1}(g^{-1}(F))$ is $\alpha g^*s$-closed in $X$. But $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ is $\alpha g^*s$-closed set in $X$. Hence $gof$ is $\alpha g^*s$-irresolute function.

**Definition 2.2.52:** A function $f: X \to Y$ is called $\alpha g^*s$-semi-open (briefly $\alpha g^*s$-open) function if the image of every open set in $X$ is $\alpha g^*s$-open in $Y$.

**Definition 2.2.53:** A function $f: X \to Y$ is called $\alpha g^*s$-semi-closed (briefly $\alpha g^*s$-closed) function if the image of every open set in $X$ is $\alpha g^*s$-open in $Y$.

**Theorem 2.2.54:** Every open function is $\alpha g^*s$-open function.

**Proof:** Let $f: X \to Y$ be an open function. Let $G$ be an open set in $X$. Then $f(G)$ is open in $Y$. And therefore $f(G)$ is $\alpha g^*s$-open in $Y$. Hence $f$ is $\alpha g^*s$-open function.

The converse of the above theorem need not be true as seen from the following example.
Example 2.2.55: Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, c\}\} \). Then the identity function \( f: X \rightarrow Y \) is not an open function, since for the open set \( \{a, b\} \) in \( X \), \( f(\{a, b\}) = \{a, b\} \) is not an open in \( Y \). However \( f \) is \( \alpha g^*s\)-open.

Theorem 2.2.56: If \( f: X \rightarrow Y \) is \( \alpha g^*s\)-open function and \( Y \) is a \( \alpha g^*T^{1/2} \)-space, then \( f \) is an open function.

Proof: Let \( G \) be an open set in \( X \). Then \( f(G) \) is \( \alpha g^*s\)-open in \( Y \), since \( f \) is \( \alpha g^*s\)-open function. And then \( f(G) \) is open in \( Y \) as \( Y \) is \( \alpha g^*T^{1/2} \)-space. Hence \( f \) is open function.

Theorem 2.2.57: Every \( \alpha g^*s\)-open function is \( \alpha gs\)-open function.

Proof: Let \( f: X \rightarrow Y \) be a \( \alpha g^*s\)-open function. Let \( G \) be an open set in \( X \). Then \( f(G) \) is \( \alpha g^*s\)-open in \( Y \) as \( f \) is \( \alpha g^*s\)-open function. Therefore \( f(G) \) is \( \alpha gs\)-open in \( Y \). Hence \( f \) is \( \alpha gs\)-open function.

The converse of the above theorem need not be true as seen from the following example.

Example 2.2.58: Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, c\}\} \). Define a function \( f: X \rightarrow Y \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then the function \( f \) is \( \alpha gs\)-open but not \( \alpha g^*s\)-open, since for the open set \( \{b\} \) in \( X \), \( f(\{b\}) = \{c\} \) is not \( \alpha g^*s\)-open in \( Y \) but it is \( \alpha gs\)-open set in \( Y \).

Theorem 2.2.59: Every \( \alpha \)-open function is \( \alpha g^*s\)-open function.

Proof: Let \( f: X \rightarrow Y \) be an \( \alpha \)-open function. Let \( G \) be an open set in \( X \). Then \( f(G) \) is \( \alpha \)-open in \( Y \). And therefore \( f(G) \) is \( \alpha g^*s\)-open in \( Y \). Hence \( f \) is \( \alpha g^*s\)-open function.
The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.60:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Define a function $f: X \to Y$ by $f(a) = f(b) = a$ and $f(c) = b$. Then the function $f$ is $\alpha g^*s$-open function but not an $\alpha$-open function, since for the open set $\{a\}$ in $X$, $f(\{a\}) = \{a\}$ is not $\alpha$-open in $Y$ but it is $\alpha g^*s$-open set in $Y$.

**Theorem 2.2.61:** Every $\alpha g^*s$-open function is $\alpha gr$-open function.

**Proof:** Let $f: X \to Y$ be a $\alpha g^*s$ -open function. Let $G$ be an open set in $X$. Then $f(G)$ is $\alpha g^*s$-open in $Y$. And therefore $f(G)$ is $\alpha gr$-open in $Y$. Hence $f$ is $\alpha gr$-open.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.62:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then the identity function $f: X \to Y$ is $\alpha gr$-open function but not $\alpha g^*s$-open, since for the open set $\{b\}$ in $X$, $f(\{b\}) = \{b\}$ is not $\alpha g^*s$-open in $Y$ but it is $\alpha gr$-open set in $Y$.

**Theorem 2.2.63:** Every closed function is $\alpha g^*s$ -closed function but not conversely.

**Proof:** Let $f: X \to Y$ be a closed function. Let $K$ be a closed set in $X$. Then $f(K)$ is closed in $Y$. So $f(G)$ is $\alpha g^*s$-closed in $Y$. Hence $f$ is $\alpha g^*s$-closed function.
Example 2.2.64: In the Example 2.2.55, the function $f$ is $\alpha g^*$-closed but not a closed function, since for the closed set $\{c\}$ in $X$, $f(\{c\}) = \{c\}$ is not a closed in $Y$ but it is $\alpha g^*$-closed set in $Y$.

Theorem 2.2.65: Every $\alpha g^*$-closed function is $\alpha g s$ -closed function.

Proof: Let $f: X \to Y$ be a $\alpha g^*$-closed function. Let $G$ be a closed set in $X$. Then $f(G)$ is $\alpha g^*$-closed in $Y$. So $f(G)$ is $\alpha g s$-closed in $Y$. Hence $f$ is $\alpha g s$-closed function.

The converse of the above theorem need not be true as seen from the following example.

Example 2.2.66: In Example 2.2.58, the function $f$ is $\alpha g s$-closed function but not $\alpha g^*$-closed function, since for the closed set $\{a, c\}$ in $X$, $f(\{a, c\}) = \{a, b\}$ is not $\alpha g^*$-closed in $Y$ but it is $\alpha g s$-closed set in $Y$.

Theorem 2.2.67: Every $\alpha$-closed function is $\alpha g^*$ -closed function.

Proof: Let $f: X \to Y$ be an $\alpha$-closed function. Let $G$ be a closed set in $X$. Then $f(G)$ is $\alpha$-closed in $Y$. So $f(G)$ is $\alpha g^*$-closed in $Y$. Hence $f$ is $\alpha g^*$-closed function.

The converse of the above theorem need not be true as seen from the following example.

Example 2.2.68: In Example 2.2.60, the function $f$ is $\alpha g^*$-closed function but not an $\alpha$-closed function as the open set $\{b, c\}$ in $X$, $f(\{b, c\}) = \{b, c\}$ is not $\alpha$-closed in $Y$ but it is $\alpha g^*$-closed set in $Y$.

Theorem 2.2.69: Every $\alpha g^*$-closed function is $\alpha g r$-closed function.
**Proof:** Let \( f: X \rightarrow Y \) be a \( \alpha g^*s \)-closed function. Let \( G \) be a closed set in \( X \). Then \( f(G) \) is \( \alpha g^*s \)-closed in \( Y \). Therefore \( f(G) \) is \( \alpha gr \)-closed in \( Y \). Hence \( f \) is \( \alpha gr \)-closed function.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.2.70:** In Example 2.2.60, the function \( f \) is \( \alpha gr \)-closed but not an \( \alpha g^*s \)-closed function, since for the closed set \( \{a, c\} \) in \( X \), \( f(\{a, c\}) = \{a, c\} \) is not \( \alpha g^*s \)-closed in \( Y \) but it is \( \alpha gr \)-closed set in \( Y \).

**Theorem 2.2.71:** A function \( f: X \rightarrow Y \) is \( \alpha g^*s \) -closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \) there is a \( \alpha g^*s \)-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof:** Suppose \( f \) is \( \alpha g^*s \)-closed function. Let \( S \) be a subset of \( Y \) and \( U \) be an open set of \( X \) such that \( f^{-1}(S) \subseteq U \). Then \( V = Y - f(X- U) \) is \( \alpha g^*s \)-open set containing \( S \) such that \( f^{-1}(V) \subseteq U \).

Conversely, suppose that \( F \) is a closed set in \( X \). Then \( f^{-1}(Y-f(F)) = X - F \) is open. By hypothesis, there is a \( \alpha g^*s \)-open set \( V \) of \( Y \) such that \( Y - f(F) \subseteq V \) and \( f^{-1}(V) \subseteq X - F \). Therefore \( F \subseteq X - f^{-1}(V) \). Hence \( Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V \), which implies \( f(F) = Y - V \). Since \( Y-V \) is \( \alpha g^*s \)-closed, \( f(F) \) is \( \alpha g^*s \)-closed. Thus \( f \) is \( \alpha g^*s \)-closed function.

**Remark 2.2.72:** The composition of two \( \alpha g^*s \)-closed functions need not be a \( \alpha g^*s \)-closed function as seen from the following example.

**Example 2.2.73:** Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{a, c\}, X\} \), \( \sigma = \{\emptyset, \{a\}, Y\} \) and \( \gamma = \{\emptyset, \{b\}, \{a, b\}, Z\} \). Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) as identity.
functions. Then \( f \) and \( g \) are both \( \alpha g^*\)-closed functions but their composition \( \text{gof}: X \to Z \) is not a \( \alpha g^*\)-closed function, since for the closed set \( \{b\} \) in \( X \), 
\[
(\text{gof})(\{b\}) = g(f(\{b\})) = g\{b\} = \{b\}
\]
is not \( \alpha g^*\)-closed in \( Z \).

**Theorem 2.2.74**: If \( f: X \to Y \) is closed and \( g: Y \to Z \) is \( \alpha g^*\)-closed, then \( \text{gof}: X \to Z \) is \( \alpha g^*\)-closed function.

**Proof**: Let \( V \) be any closed set in \( X \). Since \( f \) is closed, \( f(V) \) is closed in \( Y \). Again since \( g \) is \( \alpha g^*\)-closed function, \( g(f(V)) \) is \( \alpha g^*\)-closed set in \( Z \). But 
\[
g(f(V)) = (\text{gof})(V)
\]
is \( \alpha g^*\)-closed set in \( Z \). Therefore \( \text{gof}: X \to Z \) is \( \alpha g^*\)-closed function.

**Theorem 2.2.75**: If \( f: X \to Y \) is \( \alpha g^*\)-closed and \( Y = \text{ags}^{T_{1/2}} \)-space, then \( f \) is closed function.

**Proof**: Let \( f: X \to Y \) be a \( \alpha g^*\)-closed function. Let \( G \) be a closed set in \( X \). Then \( f(G) \) is \( \alpha g^*\)-closed set in \( Y \). As \( Y = \text{ags}^{T_{1/2}} \)-space, \( f(G) \) is closed in \( Y \). Hence \( f \) is closed function.

**Theorem 2.2.76**: If \( f: X \to Y \) and \( g: Y \to Z \) are two \( \alpha g^*\)-closed functions and \( Y = \text{ags}^{T_{1/2}} \)-space, then \( \text{gof}: X \to Z \) is \( \alpha g^*\)-closed function.

**Proof**: Let \( F \) be a closed set in \( X \). Then \( f(F) \) is \( \alpha g^*\)-closed in \( Y \) as \( f \) is \( \alpha g^*\)-closed function. Since \( Y = \text{ags}^{T_{1/2}} \)-space, \( f(F) \) is closed set in \( Y \). Again since \( g \) is \( \alpha g^*\)-closed, \( g(f(F)) \) is \( \alpha g^*\)-closed set in \( Z \). But 
\[
g(f(F)) = (\text{gof})(F)
\]
is \( \alpha g^*\)-closed set in \( Z \). Hence \( \text{gof}: X \to Z \) is \( \alpha g^*\)-closed function.

**Theorem 2.2.77**: Let \( f: X \to Y \) and \( g: Y \to Z \) be two functions and \( \text{gof}: X \to Z \) is \( \alpha g^*\)-closed. Then the following statements are true.

i) If \( f \) is continuous and surjective, then \( g \) is \( \alpha g^*\)-closed.

ii) If \( g \) is \( \alpha g^*\)-irresolute and injective, then \( f \) is \( \alpha g^*\)-closed.
Proof: i) Let $F$ be a closed in $Y$. Then $f^{-1}(F)$ is closed in $X$ as $f$ is continuous. Since $gof$ is $\alpha g^s$-closed and $f$ is surjective, $(gof)(f^{-1}(F))$ is $\alpha g^s$-closed set in $Z$. That is $(gof)(f^{-1}(F)) = g(f^{-1}(f^{-1}(F))) = g(F)$ is $\alpha g^s$-closed set in $Z$. Hence $g$ is $\alpha g^s$-closed function.

ii) Let $H$ be a closed set in $X$. Since $gof$ is closed $(gof)(H)$ is an $\alpha g^s$-closed set in $Z$. Since $g$ is $\alpha g^s$- irresolute and injective, $g^{-1}((gof)(H)) = g^{-1}(g(f(H))) = f(H)$ is $\alpha g^s$-closed in $Y$. Thus $f$ is $\alpha g^s$-closed function.

Theorem 2.2.78: Let a function $f: X \to Y$ be bijection. Then the following statements are equivalent.

i) $f^{-1}: Y \to X$ is $\alpha g^s$-continuous

ii) $f$ is $\alpha g^s$-open

iii) $f$ is $\alpha g^s$-closed

Proof: (i) $\Rightarrow$ (ii): - Let $G$ be an open set in $X$. By (i), $(f^{-1})^{-1}(G) = f(G)$ is $\alpha g^s$-open in $Y$ and so $f$ is $\alpha g^s$-open.

(ii) $\Rightarrow$ (iii): - Let $F$ be a closed set of $X$. Then $X - F$ is open set of $X$. By hypothesis (ii) $f(X - F)$ is $\alpha g^s$-open in $Y$. That is $f(X - F) = Y - f(F)$ is $\alpha g^s$-open in $Y$. Therefore $f(F)$ is $\alpha g^s$-closed in $Y$. Hence $f$ is $\alpha g^s$-closed function.

(iii) $\Rightarrow$ (i): - Let $F$ be a closed set of $X$. Then by (iii), $f(F)$ is $\alpha g^s$-closed set in $Y$. But $f(F) = (f^{-1})^{-1}(F)$ is $\alpha g^s$-closed set in $Y$. Therefore $f^{-1}: Y \to X$ is $\alpha g^s$-continuous function.

Theorem 2.2.79: A function $f: X \to Y$ is $\alpha g^s$-open if and only if for any subset $S$ of $Y$ and for any closed set $F$ containing $f^{-1}(S)$, there exists $\alpha g^s$-closed set $K$ of $Y$ containing $S$ such that $f^{-1}(K) \subseteq F$.  

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Proof: Suppose $f: X \to Y$ is $\alpha g^s$-open function. Let $S$ be a subset of $Y$ and $F$ be a closed set of $X$ containing $f^{-1}(S)$. Then $K = Y - f(X - F)$ is a $\alpha g^s$-closed set containing $S$ such that $f^{-1}(K) \subseteq F$.

Conversely, suppose that $U$ is an open set of $X$. Then $f^{-1}(Y - f(U)) \subseteq X - f^{-1}[f(U)] \subseteq X - U$ and $X - U$ is closed. By hypothesis, there is a $\alpha g^s$-closed set $K$ of $Y$ such that $Y - f(U) \subseteq K$ and $f^{-1}(K) \subseteq X - V$. Therefore $U \subseteq X - f^{-1}(K)$. Hence $Y - K \subseteq f(U) \subseteq f[X - f^{-1}(K)] \subseteq Y - K$, which implies $f(U) \subseteq Y - K$. Since $Y - K$ is $\alpha g^s$-open, $f(U)$ is $\alpha g^s$-open and thus $f$ is $\alpha g^s$-open function.
§ 2.3 Strongly αg*s-Continuous Functions and Other Related Functions

In this section we introduce some stronger forms of continuous functions namely, strongly αg*s-continuous, perfectly αg*s-continuous and completely αg*s-continuous functions in topological spaces and studied some of their properties.

**Definition 2.3.1:** A function \( f: X \rightarrow Y \) is called strongly \( \alpha g*s \)-continuous if the inverse image of every \( \alpha g*s \)-open set in \( Y \) is open in \( X \).

**Theorem 2.3.2:** A function \( f: X \rightarrow Y \) is strongly \( \alpha g*s \)-continuous if and only if the inverse image of every \( \alpha g*s \)-closed set in \( Y \) is closed in \( X \).

**Proof:** Suppose \( f: X \rightarrow Y \) is strongly \( \alpha g*s \)-continuous. Let \( G \) be a \( \alpha g*s \)-closed set in \( Y \). Then \( Y - G \) is \( \alpha g*s \)-open set in \( Y \). Since \( f \) is strongly \( \alpha g*s \)-continuous, \( f^{-1}(Y- G) \) is open in \( X \). But \( f^{-1}(Y- G) = X - f^{-1}(G) \). Thus \( f^{-1}(G) \) is closed in \( X \).

Conversely, suppose that the inverse image of every \( \alpha g*s \)-closed set in \( Y \) is closed in \( X \). Let \( U \) be a \( \alpha g*s \)-open set in \( Y \). Then \( Y - U \) is \( \alpha g*s \)-closed in \( Y \). By hypothesis, \( f^{-1}(Y - U) \) is closed set in \( X \). But \( f^{-1}(Y - U) = X - f^{-1}(U) \) is closed in \( X \). Therefore \( f^{-1}(U) \) is open in \( X \). Hence \( f \) is strongly \( \alpha g*s \) -continuous function.

**Theorem 2.3.3:** Every strongly \( \alpha g*s \)-continuous function is a continuous function but not conversely.
Proof: Let $f: X \to Y$ be a strongly $\alpha g^*s$-continuous function. Let $G$ be an open set in $Y$. Then $G$ is $\alpha g^*s$-open set in $Y$. Since $f$ is strongly $\alpha g^*s$-continuous, $f^{-1}(G)$ is open in $X$. Hence $f$ is continuous function.

Example 2.3.4: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f: X \to Y$ be the identity function. Then $f$ is not $\alpha g^*s$-continuous, since for the $\alpha g^*s$-open set $\{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not open in $X$. However $f$ is continuous function.

Theorem 2.3.5: Every strongly continuous function is strongly $\alpha g^*s$-continuous.

Proof: Let $f: X \to Y$ be a strongly continuous function. Let $G$ be a $\alpha g^*s$-open set in $Y$. Then $f^{-1}(G)$ is both open and closed in $X$. Therefore $f^{-1}(G)$ is open in $X$. Hence $f$ is strongly $\alpha g^*s$-continuous function.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3.6: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Then the identity function $f: X \to Y$ is strongly $\alpha g^*s$-continuous but not strongly continuous, since for the subset $\{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ is open in $X$ but not a closed set in $X$.

Theorem 2.3.7: If $f: X \to Y$ is continuous and $Y$ is $\alpha g^*_tT^*_1/2$-space, then $f$ is strongly $\alpha g^*s$-continuous.

Proof: Let $f: X \to Y$ be a continuous function. Let $G$ be a $\alpha g^*s$-open set in $Y$. Then $G$ is open in $Y$ as $Y$ is $\alpha g^*_tT^*_1/2$-space. Since $f$ is continuous, $f^{-1}(G)$ is open in $X$. Hence $f$ is strongly $\alpha g^*s$-continuous function.
**Theorem 2.3.8:** If \( f: X \to Y \) and \( g: Y \to Z \) are strongly \( \alpha g^s \)-continuous functions, then their composition \( g \circ f: X \to Z \) is strongly \( \alpha g^s \)-continuous.

**Proof:** Let \( G \) be a \( \alpha g^s \)-open set in \( Z \). Since \( g \) is strongly \( \alpha g^s \)-continuous, \( g^{-1}(G) \) is open in \( Y \). Therefore \( g^{-1}(G) \) is \( \alpha g^s \)-open in \( Y \). Again since \( f \) is strongly \( \alpha g^s \)-continuous, \( f^{-1}(g^{-1}(G)) \) is open in \( X \). Therefore \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \) which is open in \( X \). Hence \( g \circ f \) is strongly \( \alpha g^s \)-continuous.

**Theorem 2.3.9:** If \( f: X \to Y \) is continuous and \( g: Y \to Z \) is strongly \( \alpha g^s \)-continuous, then \( g \circ f: X \to Z \) is strongly \( \alpha g^s \)-continuous function.

**Proof:** Let \( G \) be a \( \alpha g^s \)-open set in \( Z \). Since \( g \) is strongly \( \alpha g^s \)-continuous, \( g^{-1}(G) \) is open in \( Y \). Then \( f^{-1}(g^{-1}(G)) \) is open in \( X \) as \( f \) is continuous. Therefore \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \) is open in \( X \). Hence \( g \circ f \) is strongly \( \alpha g^s \)-continuous function.

**Theorem 2.3.10:** If \( f: X \to Y \) is \( \alpha g^s \)-continuous and \( g: Y \to Z \) is strongly \( \alpha g^s \)-continuous, then their composition \( g \circ f: X \to Z \) is \( \alpha g^s \)-irresolute.

**Proof:** Let \( G \) be a \( \alpha g^s \)-open set in \( Z \). Then \( g^{-1}(G) \) is open in \( Y \) as \( g \) is strongly \( \alpha g^s \)-continuous. Since \( f \) is \( \alpha g^s \)-continuous, \( f^{-1}(g^{-1}(G)) \) is \( \alpha g^s \)-open in \( X \). But \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \) is \( \alpha g^s \)-open in \( X \). Hence \( g \circ f \) is \( \alpha g^s \)-irresolute.

**Theorem 2.3.11:** If \( f: X \to Y \) is strongly \( \alpha g^s \)-continuous and \( g: Y \to Z \) is continuous, then \( g \circ f: X \to Z \) is continuous function.

**Proof:** Let \( G \) be an open set in \( Z \). Then \( g^{-1}(G) \) is open in \( Y \) since \( g \) is continuous. So \( g^{-1}(G) \) is \( \alpha g^s \)-open in \( Y \). Again since \( f \) is strongly \( \alpha g^s \)-continuous, \( f^{-1}(g^{-1}(G)) \) is open in \( X \). Therefore \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \) is open in \( X \). Hence \( g \circ f \) is continuous function.
Definition 2.3.12: A function \( f: X \to Y \) is said to be perfectly \( \alpha g^*s \)-continuous if the inverse image of every \( \alpha g^*s \)-open set in \( Y \) is both open and closed in \( X \).

Theorem 2.3.13: A function \( f: X \to Y \) is perfectly \( \alpha g^*s \)-continuous if and only if the inverse image of every \( \alpha g^*s \)-closed set in \( Y \) is both open and closed in \( X \).

Theorem 2.3.14: A function \( f: X \to Y \) is perfectly \( \alpha g^*s \)-continuous then \( f \) is strongly \( \alpha g^*s \)-continuous function.

Proof: Let \( f: X \to Y \) be a perfectly \( \alpha g^*s \)-continuous function. Let \( G \) be a \( \alpha g^*s \)-open set in \( Y \). Then \( f^{-1}(G) \) is both open and closed in \( X \). Therefore \( f^{-1}(G) \) is open in \( X \). Hence \( f \) is strongly \( \alpha g^*s \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3.15: In Example 2.3.6, the function \( f \) is strongly \( \alpha g^*s \)-continuous but not perfectly \( \alpha g^*s \)-continuous, since for the \( \alpha g^*s \)-open set \( \{a, b\} \) in \( Y \), \( f^{-1}(\{a, b\}) = \{a, b\} \) is open but not closed in \( X \).

Theorem 2.3.16: Every perfectly \( \alpha g^*s \)-continuous function is continuous function.

Proof: Let \( f: X \to Y \) be a perfectly \( \alpha g^*s \)-continuous function. Let \( G \) be an open set in \( Y \). Then \( G \) is \( \alpha g^*s \)-open in \( Y \). Since \( f \) is perfectly \( \alpha g^*s \)-continuous, \( f^{-1}(G) \) is both open and closed in \( X \). Therefore \( f^{-1}(G) \) is open in \( X \). Hence \( f \) is continuous function.

The converse of the above theorem need not be true as seen from the following example.
Example 2.3.17: In Example 2.3.4, the function $f$ is continuous but not perfectly $\alpha g^s$-continuous, since for the $\alpha g^s$-open set $\{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not both open and closed in $X$.

Theorem 2.3.18: Every perfectly $\alpha g^s$-continuous function is perfectly continuous.

Proof: Let $f: X \to Y$ be a perfectly $\alpha g^s$-continuous function. Let $G$ be an open set in $Y$. So $G$ is $\alpha g^s$-open in $Y$. Since $f$ is perfectly $\alpha g^s$-continuous, $f^{-1}(G)$ is both open and closed in $X$. Hence $f$ is perfectly continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3.19: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}\}$. Let $f: X \to Y$ be the identity function. Then $f$ is perfectly continuous but not perfectly $\alpha g^s$-continuous, since for the $\alpha g^s$-open set $\{a, b\}$ in $Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ is neither open nor closed in $X$.

Theorem 2.3.20: If a function $f: X \to Y$ is perfectly continuous and $Y$ is $\alpha g T^{*1/2}$-space, then $f$ is perfectly $\alpha g^s$-continuous.

Proof: Let $G$ be a $\alpha g^s$-open set in $Y$. Then $G$ is open in $Y$ as $Y$ is $\alpha g T^{*1/2}$-space. Since $f$ is perfectly continuous, $f^{-1}(G)$ is both open and closed in $X$. Therefore $f$ is perfectly $\alpha g^s$-continuous function.

Theorem 2.3.21: Let $X$ be a discrete topological space, $Y$ be any topological space and $f: X \to Y$ be a function. Then the following are equivalent.

i) $f$ is perfectly $\alpha g^s$-continuous

ii) $f$ is strongly $\alpha g^s$-continuous.
Proof: (i) \implies (ii): Follows from the Theorem 2.3.14.

(ii) \implies (i): Let G be a \(\alpha g^s\)-open set in Y. By hypothesis, \(f^{-1}(G)\) is open in X. Since X is discrete topological space, \(f^{-1}(G)\) is closed in X. Therefore \(f^{-1}(G)\) is both open and closed in X. Hence f is perfectly \(\alpha g^s\)-continuous function.

Theorem 2.3.22: If \(f: X \to Y\) and \(g: Y \to Z\) be two perfectly \(\alpha g^s\)-continuous functions, then \(gof: X \to Z\) is perfectly \(\alpha g^s\)-continuous.

Proof: Let G be a \(\alpha g^s\)-open set in Z. Then \(g^{-1}(G)\) is both open and closed in Y as g is perfectly \(\alpha g^s\)-continuous. So \(g^{-1}(G)\) is \(\alpha g^s\)-open in Y. Since f is perfectly \(\alpha g^s\)-continuous, \(f^{-1}(g^{-1}(G))\) is both open and closed in X. That is \(f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)\) is both open and closed in X. Hence gof is perfectly \(\alpha g^s\)-continuous.

Theorem 2.3.23: If \(f: X \to Y\) is perfectly \(\alpha g^s\)-continuous and \(g: Y \to Z\) is \(\alpha g^s\)-irresolute, then \(gof: X \to Z\) is perfectly \(\alpha g^s\)-continuous.

Proof: Let G be a \(\alpha g^s\)-open set in Z. Then \(g^{-1}(G)\) is \(\alpha g^s\)-open in Y as g is \(\alpha g^s\)-irresolute. Since f is perfectly \(\alpha g^s\)-continuous, \(f^{-1}(g^{-1}(G))\) is both open and closed in X. But \(f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)\) is both open and closed in X. Hence gof is perfectly \(\alpha g^s\)-continuous.

Theorem 2.3.24: If \(f: X \to Y\) is \(\alpha g^s\)-continuous and \(g: Y \to Z\) strongly continuous then \(gof: X \to Z\) is \(\alpha g^s\)-continuous.

Proof: Let G be an open set in Z. Then \(g^{-1}(G)\) is both open and closed in Y as g is strongly continuous. So \(g^{-1}(G)\) is open in Y. Since f is \(\alpha g^s\)-continuous, \(f^{-1}(g^{-1}(G))\) is open in X. That is \(f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)\) is open in X. Hence gof is \(\alpha g^s\)-continuous.
**Remark 2.3.25:** From the above results we have the following diagram.

\[
\begin{array}{ccc}
\text{strongly continuity} & \text{strongly } \alpha g^*\text{-continuity} & \text{continuity} \\
\downarrow & & \downarrow \\
\text{perfectly continuity} & \text{perfectly } \alpha g^*\text{-continuity} & \\
\end{array}
\]

**Theorem 2.3.26:** If \( f: X \to Y \) is \( \alpha g^*\)-irresolute and \( g: Y \to Z \) is perfectly \( \alpha g^*\)-continuous, then \( gof: X \to Z \) is \( \alpha g^*\)-irresolute function.

**Proof:** Let \( G \) be a \( \alpha g^*\)-open set in \( Z \). Since \( g \) is perfectly \( \alpha g^*\)-continuous, \( g^{-1}(G) \) is both open and closed in \( Y \). So \( g^{-1}(G) \) is \( \alpha g^*\)-open in \( Y \). Again since \( f \) is \( \alpha g^*\)-irresolute, \( f^{-1}(g^{-1}(G)) \) is \( \alpha g^*\)-open in \( X \). That is \( f^{-1}(g^{-1}(G)) = (gof)^{-1}(G) \) is \( \alpha g^*\)-open in \( X \). Hence \( gof \) is \( \alpha g^*\)-irresolute.

**Definition 2.3.27:** A function \( f: X \to Y \) is said to be completely \( \alpha g^*\)-continuous if the inverse image of every \( \alpha g^*\)-open set in \( Y \) is regular-open in \( X \).

**Theorem 2.3.28:** A function \( f: X \to Y \) is completely \( \alpha g^*\)-continuous if and only if the inverse image of every \( \alpha g^*\)-closed set in \( Y \) is regular closed in \( X \).

**Proof:** Suppose a function \( f: X \to Y \) is completely \( \alpha g^*\)-continuous. Let \( F \) be a \( \alpha g^*\)-closed set in \( Y \). Then \( Y - F \) is \( \alpha g^*\)-open in \( Y \). Since \( f \) is completely \( \alpha g^*\)-continuous, \( f^{-1}(Y - F) \) is regular open in \( X \). That is \( f^{-1}(Y - F) = X - f^{-1}(F) \) is regular open in \( X \). Hence \( f^{-1}(F) \) is regular closed in \( X \).

Conversely, suppose that the inverse image of every \( \alpha g^*\)-closed set in \( Y \) is regular closed in \( X \). Let \( K \) be a \( \alpha g^*\)-open set in \( Y \). Then \( Y - K \) is
\( \alpha g^s \)-closed in \( Y \). By hypothesis, \( f^{-1}(Y-K) \) is regular closed in \( X \). That is \( f^{-1}(Y-K) = X - f^{-1}(K) \) is regular closed set in \( X \). Therefore \( f^{-1}(K) \) is regular open in \( X \). Hence \( f \) is completely \( \alpha g^s \)-continuous function.

**Theorem 2.3.29:** If a function \( f : X \rightarrow Y \) is completely \( \alpha g^s \)-continuous then \( f \) is continuous.

**Proof:** Let \( G \) be an open set in \( Y \). Then \( G \) is \( \alpha g^s \)-open set in \( Y \). Since \( f \) is completely \( \alpha g^s \)-continuous, \( f^{-1}(G) \) is regular open in \( X \). So \( f^{-1}(G) \) is open in \( X \). Hence \( f \) is continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.3.30:** In Example 2.3.6, the function \( f \) is continuous but not completely \( \alpha g^s \)-continuous, since for the \( \alpha g^s \)-open set \( \{a, b\} \) of \( Y \), \( f^{-1}(\{a, b\}) = \{a, b\} \) is not regular open in \( X \).

**Theorem 2.3.31:** Every completely \( \alpha g^s \)-continuous function completely continuous.

**Proof:** Let \( f : X \rightarrow Y \) be a completely \( \alpha g^s \)-continuous function. Let \( G \) be an open set in \( Y \). Then \( G \) is \( \alpha g^s \)-open in \( Y \). Since \( f \) is completely \( \alpha g^s \)-continuous, \( f^{-1}(G) \) is regular-open in \( X \). Hence \( f \) is completely continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.3.32:** Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let \( f : X \rightarrow Y \) be the identity function. Then \( f \) is completely
continuous but not completely $\alpha g^s$-continuous, since for the $\alpha g^s$-open set \{a, b\} in Y, $f^{-1}(\{a, b\}) = \{a, b\}$ is not regular open in X.

**Theorem 2.3.33**: If a function $f: X \to Y$ is completely $\alpha g^s$-continuous then $f$ is strongly $\alpha g^s$-continuous.

**Proof**: Let $f: X \to Y$ be completely $\alpha g^s$-continuous. Let $G$ be $\alpha g^s$-open set in Y. Since $f$ is completely $\alpha g^s$-continuous, $f^{-1}(G)$ is regular open in X. Therefore $f^{-1}(G)$ is open in X. Hence $f$ is strongly $\alpha g^s$-continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.3.34**: In Example 2.3.6, the function $f$ is strongly $\alpha g^s$-continuous but not completely $\alpha g^s$-continuous, since for the $\alpha g^s$-open set \{a, b\} in Y, $f^{-1}(\{a, b\}) = \{a, b\}$ is not regular open in X.

**Theorem 2.3.35**: If a function $f: X \to Y$ is completely continuous and Y is $\alpha g^s T^{*\frac{1}{2}}$-space, then $f$ is completely $\alpha g^s$-continuous.

**Proof**: Let $G$ be a $\alpha g^s$-open set in Y. Then G is an open in Y as Y is $\alpha g^s T^{*\frac{1}{2}}$-space. Since $f$ is completely continuous, $f^{-1}(G)$ is regular open in X. Therefore $f$ is completely $\alpha g^s$-continuous function.

**Theorem 2.3.36**: If $f: X \to Y$ is completely continuous and $g: Y \to Z$ completely $\alpha g^s$-continuous then $gof: X \to Z$ is completely $\alpha g^s$-continuous.

**Proof**: Let $G$ be a $\alpha g^s$-open set in Z. Then $g^{-1}(G)$ is regular-open in Y as g is completely $\alpha g^s$-continuous. So $g^{-1}(G)$ is open in Y. Since $f$ is completely continuous, $f^{-1}(g^{-1}(G))$ is regular-open in X. That is $f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)$ is regular open in X. Hence $gof$ is completely $\alpha g^s$-continuous.
Theorem 2.3.37: If $f: X \to Y$ is completely $\alpha g^*s$-continuous and $g: Y \to Z$ is $\alpha g^*s$-irresolute, then $gof: X \to Z$ is completely $\alpha g^*s$-continuous.

Proof: Let $G$ be a $\alpha g^*s$-open set in $Z$. Since $g$ is $\alpha g^*s$-irresolute, $g^{-1}(G)$ is $\alpha g^*s$-open in $Y$. Since $f$ is completely $\alpha g^*s$-continuous, $f^{-1}(g^{-1}(G))$ is regular-open in $X$. That is $f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)$ is regular open in $X$. Hence $gof$ is completely $\alpha g^*s$-continuous.

Theorem 2.3.38: If $f: X \to Y$ is completely $\alpha g^*s$-continuous and $g: Y \to Z$ strongly $\alpha g^*s$-continuous, then $gof: X \to Z$ is completely $\alpha g^*s$-continuous function.

Proof: Let $G$ be a $\alpha g^*s$-open set in $Z$. Since $g$ is strongly $\alpha g^*s$-continuous, $g^{-1}(G)$ is open in $Y$. So $g^{-1}(G)$ is $\alpha g^*s$-open in $Y$. Again since $f$ is completely $\alpha g^*s$-continuous, $f^{-1}(g^{-1}(G))$ is regular-open in $X$. But $f^{-1}(g^{-1}(G)) = (gof)^{-1}(G)$ is regular open in $X$. Hence $gof$ is completely $\alpha g^*s$-continuous.

Remark 2.3.39: From the above observations we get the following diagram.
§ 2.4 αg*s-Homeomorphisms in Topological Spaces

In this section we introduce the concept of αg*s-homeomorphisms in topological spaces and obtain some of their properties.

**Definition 2.4.1:** A bijective function \( f: (X, \tau) \to (Y, \sigma) \) is called αg*s-homeomorphism if \( f \) and \( f^{-1} \) are αg*s-continuous functions.

**Example 2.4.2:** Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a, b\}\} \). Define a function \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then the function \( f \) is bijective, αg*s-continuous and \( f^{-1} \) is also αg*s-continuous. Therefore \( f \) is a αg*s-homeomorphism.

**Theorem 2.4.3:** Every homeomorphism is a αg*s-homeomorphism.

**Proof:** Let \( (X, \tau) \to (Y, \sigma) \) be a homeomorphism. Then \( f \) and \( f^{-1} \) are continuous functions. Since every continuous function is αg*s-continuous Therefore \( f \) and \( f^{-1} \) are αg*s-continuous functions. Hence \( f \) is αg*s-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.4.4:** In Example 2.4.2, the function \( f \) is αg*s-homeomorphism but not a homeomorphism.

**Theorem 2.4.5:** If \( f: (X, \tau) \to (Y, \sigma) \) is αg*s-homeomorphism then \( f \) is αg-homeomorphism.

**Proof:** Let \( f: (X, \tau) \to (Y, \sigma) \) is αg*s-homeomorphism. Then \( f \) and \( f^{-1} \) are αg*s-continuous functions. Since every αg*s-continuous is αg-continuous.
Therefore $f$ and $f^1$ are $\alpha g$-continuous functions. Hence $f$ is $\alpha g$-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.4.6:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then the function $f$ is $\alpha g$-homeomorphism but not a $\alpha g^s$-homeomorphism, since for the open set $\{b, c\}$ in $(X, \tau)$, $f(\{b, c\}) = \{a, b\}$ is not $\alpha g^s$-open in $(Y, \sigma)$ but it is $\alpha g$-open in $(Y, \sigma)$.

**Theorem 2.4.7:** Every $\alpha^*$-homeomorphism is $\alpha g^s$-homeomorphism but not conversely.

**Proof:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\alpha^*$-homeomorphism. Then $f$ and $f^1$ are $\alpha$-continuous functions. Since every $\alpha$-continuous function is $\alpha g^s$-continuous function, if follows that, $f$ and $f^1$ are $\alpha g^s$-continuous functions. Hence $f$ is $\alpha g^s$-homeomorphism.

**Example 2.4.8:** Let $X = \{a, b, c\} = Y$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\alpha g^s$-homeomorphism but not $\alpha^*$-homeomorphism, since for the open set $\{a, b\}$ in $(Y, \sigma)$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not $\alpha$-open in $(X, \tau)$ but it is $\alpha g^s$-open in $(X, \tau)$.

**Remark 2.4.9:** The concepts of $\alpha g^s$-homeomorphisms and $g$-homeomorphisms are independent of each other as seen from the following examples.
Example 2.4.10: In Example 2.4.8, the function $f$ is $\alpha g^*s$-homeomorphism but not a $g$-homeomorphism, since for the open set $\{a, c\}$ in $X$, $f(\{a, c\}) = \{a, c\}$ is not $g$-open in $Y$ but it is $\alpha g^*s$-open in $Y$.

Example 2.4.11: Let $X = \{a, b, c\} = Y$, $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then the function $f$ is $g$-homeomorphism but not a $\alpha g^*s$-homeomorphism, since for the open set $\{a\}$ of $X$, $f(\{a\}) = \{c\}$ is not a $\alpha g^*s$-open in $Y$ but it is $g$-open in $Y$.

Theorem 2.4.12: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijective and $\alpha g^*s$-continuous function. Then the following statements are equivalent.

a) $f$ is $\alpha g^*s$-open

b) $f$ is $\alpha g^*s$-homeomorphism

c) $f$ is $\alpha g^*s$-closed

Proof: The proof follows from Theorem 2.2.78.

Remark 2.4.13: From the above results we have the following diagram.

\[
\begin{array}{ccc}
\text{homeomorphism} & \rightarrow & \text{g-homeomorphism} \\
\downarrow & & \downarrow \text{g-homeomorphism} \\
\text{g-homeomorphism} & \rightarrow & \text{g-homeomorphism} \\
\downarrow & \alpha g^*s\text{-homeomorphism} & \downarrow \\
\alpha^*\text{-homeomorphism} & \alpha g^*s\text{-homeomorphism} & \alpha g^*s\text{-homeomorphism} \\
\end{array}
\]

Remark 2.4.14: The composition of two $\alpha g^*s$-homeomorphisms need not be a $\alpha g^*s$-homeomorphism as seen from the following example.
Example 2.4.15: Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, Y\}$ and $\eta = \{\emptyset, \{a\}, \{a, b\}, Z\}$. Define functions $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is defined as $g(a) = b$, $g(b) = c$ and $g(c) = a$. Then $f$ and $g$ are both $\alpha g^s$-homeomorphisms but the composition $gof$ is not a $\alpha g^s$-homeomorphism, since for the open set $\{a, b\}$ in $(Z, \eta)$, $(gof)^{-1}(\{a, b\}) = f^{-1}(g^{-1}(\{a, b\})) = f^{-1}(\{a, c\}) = \{b, c\}$ is not an $\alpha g^s$-open in $(X, \tau)$.

Remark 2.4.16: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma)$ be $\alpha g^s$-homeomorphism and $(Y, \sigma)$ is $\alpha g T^*_1/2$-space, then $gof$ is $\alpha g^s$-homeomorphism.