CHAPTER 6

g*-CLOSED SETS IN BITOPOLOGICAL SPACES**

§ 6.1 Introduction

The concepts of g*-preclosed (briefly g*p-closed) sets, Tp*, *Tp and aTp*-space were introduced and studied by Veerakumar [120] in topological spaces in 2002. Also g*p-continuity and g*p-irresolute functions were introduced and studied in [120]. The class of g*p-closed sets is properly lies between the class of preclosed sets and the class of gp-closed sets.

This chapter contains five sections. In section 2, we introduce a new class of closed sets called, g*p-closed sets and g*p-open sets in bitopological spaces. Among many other results it is observed that every Tj - preclosed set is (Xj, Xj) - g*p-closed set and (xj5 Xj) - gp-closed set is (Xj, Xj) - g*p-closed set but not conversely.

In Section 3, we introduce the new spaces such as, (x, Xj)-Tp*-spaces, (Xj, Xj)-*Tp - spaces and (x, Xj)- aTp*-spaces as an application and study some of their properties. It is observed that (x, Xj)-Tp*-spaces are independent from (x, Xj)-T1/2-spaces and every (x, Xj)-Tp*-space is (x, Xj)-T*1/2-space but not conversely.

In Section 4, we introduce a new class of continuous functions, called

g*p-continuous functions in bitopological spaces and is denoted by D*P(τ₁, τ₂)-σ_k-continuity in bitopological spaces. During this process, some of their properties are obtained. It is found that every D*(τ₁, τ₂)-σ_k-continuous function is D*P(τ₁, τ₂)-σ_k-continuous but not conversely.

In the last section of this chapter, we introduce the concepts of g*p-bi-continuity, g*p-strongly-bi-continuity and pairwise g*p-irresolute functions in bitopological spaces and study some of their properties.

Throughout this chapter (X, τ₁, τ₂), (Y, σ₁, σ₂) and (Z, η₁, η₂) denote non-empty bitopological spaces on which no separation axioms are assumed unless otherwise mentioned and the fixed integers i, j, e, m, n ∈ {1, 2}.

§ 6.2 g*p-Closed Sets in Bitopological Spaces

In this section we introduce g*p-closed sets and g*p-open sets in bitopological spaces and study some of their properties.

Definition 6.2.1: A subset A of a bitopological space (X, τ₁, τ₂) is said to be (τ₁, τ₂)-g*-preclosed (briefly (τ₁, τ₂)-g*p-closed) set if τ_j-pcl(A) ⊆ U whenever A ⊆ U and U ∈ GO(X, τ_i).

We denote the family of all (τ₁, τ₂) – g*-preclosed sets in a bitopological space (X, τ₁, τ₂) by D*P(τ₁, τ₂).

Remark 6.2.2: If τ₁ = τ₂ in Definition 6.2.1, then (τ₁, τ₂) – g*p-closed set reduces to a g*p-closed set [120] in single topological spaces.
Theorem 6.2.3: Every \( \tau_j \) - preclosed (resp. \( \tau_j \) - closed, \( \tau_j \)-\( \alpha \)-closed) set is \((\tau_i, \tau_j)\) - \(g^p\)-closed.

Proof: The proof follows from definitions.

The converse of the above theorem need not be true as seen from the following example.

Example 6.2.4: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{a, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}\} \). Then the subset \( \{a, b\} \) is \((\tau_1, \tau_2)\) - \(g^p\)-closed set but not a \(\tau_2\)-preclosed set in \((X, \tau_1, \tau_2)\).

Example 6.2.5: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}\} \). Then the subset \( \{a, c\} \) is \((\tau_1, \tau_2)\) - \(g^p\)-closed set but not a \(\tau_2\)-closed set in \((X, \tau_1, \tau_2)\).

Example 6.2.6: Let \((X, \tau_1, \tau_2)\) be bitopological space where \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\} \). Then the subset \( \{a, c\} \) is \((\tau_1, \tau_2)\)- \(g^p\)-closed set but not a \(\tau_2\)-closed set in \((X, \tau_1, \tau_2)\).

Theorem 6.2.7: If \( A \) is both \( \tau_i \) - \(g\)-open and \((\tau_i, \tau_j)\) - \(g^p\)-closed, then \( A \) is \( \tau_j \) - preclosed set.

Proof: If \( A \in GO(X, \tau_i) \), then by hypothesis, \( \tau_j\text{-pcl}(A) \subseteq A \). But \( A \subseteq \tau_j\text{-pcl}(A) \) always. Therefore \( \tau_j\text{-pcl}(A) = A \). Hence \( A \) is \( \tau_j \) - preclosed set.

Theorem 6.2.8: In a bitopological space \((X, \tau_1, \tau_2)\), every \((\tau_i, \tau_j)\) - \(g^p\)-closed set is (i) \((\tau_i, \tau_j)\) - \(gp\)-closed and (ii) \((\tau_i, \tau_j)\) - \(gpr\)-closed set.

Proof: The proof follows from the definitions.

The converse of the above theorem need not be true as seen from the following examples.
Example 6.2.9: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Then the subset $\{a, c\}$ is $(\tau_1, \tau_2)$-gp-closed set but not a $(\tau_1, \tau_2)$-g*p-closed set.

Example 6.2.10: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{c\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then the subset $\{a, c\}$ is $(\tau_1, \tau_2)$-gpr-closed set but not a $(\tau_1, \tau_2)$-g*p-closed set.

Theorem 6.2.11: Every $(\tau_1, \tau_2)$-gαs-closed set is $(\tau_1, \tau_2)$-g*p-closed but not conversely.

Proof: Let $A$ be a $(\tau_1, \tau_2)$-gαs-closed set in $(X, \tau_1, \tau_2)$ and let $G$ be $\tau_1$-g-open set and so $\tau_1$-gs-open such that $A \subseteq G$. Then $\tau_1$-pcl$(A) \subseteq G$ as $A$ is $(\tau_1, \tau_2)$-gp-closed. But $\tau_1$-pcl$(A) \subseteq \tau_1$-pcl$(A)$ is always true. Therefore $\tau_1$-pcl$(A) \subseteq \tau_1$-pcl$(A)$. Hence $A$ is $(\tau_1, \tau_2)$-g*p-closed in $(X, \tau_1, \tau_2)$.

Example 6.2.12: In the Example 5.2.10, the subset $\{a, b\}$ is $(\tau_1, \tau_2)$-g*p-closed set but not a $(\tau_1, \tau_2)$-gαs-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

Theorem 6.2.13: Every $(\tau_1, \tau_2)$-g-* closed set is $(\tau_1, \tau_2)$-g*p-closed set but not conversely.

Proof: Let $A$ be a $(\tau_1, \tau_2)$-g*-closed set in $(X, \tau_1, \tau_2)$. Let $G$ be a $\tau_1$-g-open set such that $A \subseteq G$. Then $\tau_1$-cl $(A) \subseteq G$. But $\tau_1$-pcl $(A) \subseteq \tau_1$-cl $(A) \subseteq G$ which implies that $\tau_1$-pcl $(A) \subseteq G$. Therefore $A$ is a $(\tau_1, \tau_2)$-g*p-closed set in $(X, \tau_1, \tau_2)$. 

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Example 6.2.14: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then the set $\{a, b\}$ is $(\tau_1, \tau_2)$-g*-p-closed set but not a $(\tau_1, \tau_2)$-g*-closed set in $(X, \tau_1, \tau_2)$.

Remark 6.2.15: The following examples show that $(\tau_i, \tau_j)$-g-closed sets and $(\tau_i, \tau_j)$-g*p-closed sets are independent of each other.

Example 6.2.16: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}\}$. Then the subset $\{a\}$ is $(\tau_1, \tau_2)$-g*p-closed but not a $(\tau_1, \tau_2)$-g-closed set.

Example 6.2.17: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then the subset $\{a, c\}$ is $(\tau_1, \tau_2)$-g-closed set but not a $(\tau_1, \tau_2)$-g*p-closed set.

Remark 6.2.18: The following examples show that $(\tau_i, \tau_j)$-rg-closed sets and $(\tau_i, \tau_j)$-g*p-closed sets are independent of each other.

Example 6.2.19: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Then the subset $\{b, c\}$ is $(\tau_1, \tau_2)$-g*p-closed set but not a $(\tau_1, \tau_2)$-rg-closed set.

Example 6.2.20: In Example 6.2.17, the subset $A=\{a, c\}$ is $(\tau_1, \tau_2)$-rg-closed set but not a $(\tau_1, \tau_2)$-g*p-closed set.

Remark 6.2.21: $(\tau_i, \tau_j)$-g*p-closed sets and $\tau_j$-g-closed sets are independent of each other as seen from the following examples.

Example 6.2.22: In Example 6.2.17, the set $\{a, c\}$ is $\tau_2$-g-closed set but not a $(\tau_1, \tau_2)$-g*p-closed set.
Example 6.2.23: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \) and \( \tau_2 = \{X, \phi, \{b, c\}\} \). Then the subset \( \{b, c\} \) is \((\tau_1, \tau_2) - g^p\)-closed set but not a \( \tau_2 - g \)-closed set.

Remark 6.2.24: \((\tau_i, \tau_j) - g^p\)-closed sets and \((\tau_i, \tau_j) - \omega\)-closed sets are independent of each other as shown from the following examples.

Example 6.2.25: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{c\}, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}\} \). Then the subset \( A = \{c\} \) is \((\tau_1, \tau_2)- g^p\)-closed set but not a \((\tau_1, \tau_2) - \omega\)-closed set.

Example 6.2.26: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \). Then the set \( \{a, b\} \) is \((\tau_1, \tau_2)- \omega\)-closed set but not a \((\tau_1, \tau_2)- g^p\)-closed set.

Remark 6.2.27: \((\tau_i, \tau_j) - g^p\)-closed sets and \((\tau_i, \tau_j) - w^g\)-closed sets are independent of each other as seen from the following examples.

Example 6.2.28: In Example 6.2.16, the set \( \{a, b\} \) is \((\tau_1, \tau_2)- w^g\)-closed set but not a \((\tau_1, \tau_2)- g^p\)-closed set in \((X, \tau_1, \tau_2)\).

Example 6.2.29: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{b\}, \{b, c\}\} \) and \( \tau_2 = \{X, \phi, \{a, b\}\} \). Then the set \( \{b, c\} \) is \((\tau_1, \tau_2)- g^p\)-closed set but not a \((\tau_1, \tau_2)- w^g\)-closed set in \((X, \tau_1, \tau_2)\).

Remark 6.2.30: If \( A, B \in D^*P(\tau_i, \tau_j) \), then \( A \cup B \notin D^*P(\tau_i, \tau_j) \) as shown from the following example.

Example 6.2.31: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \) and \( \tau_2 = \{X, \phi, \{a, b\}\} \). Then the subsets \( \{a\} \) and \( \{b\} \) are \((\tau_1, \tau_2)- g^p\)-closed sets but their union \( A \cup B = \{a, b\} \) is not a \((\tau_1, \tau_2)- g^p\)-closed set in \((X, \tau_1, \tau_2)\).
Remark 6.2.32: The intersection of two \((\tau_i, \tau_j)\)-g*p-closed sets need not be \((\tau_i, \tau_j)\)-g*p-closed set as seen from the following example.

Example 6.2.33: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}, \{a, c\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\). Then the subsets \(\{a, b\}\) and \(\{a, c\}\) are \((\tau_1, \tau_2)\)-g*p-closed sets but their intersection \(A \cap B = \{a\}\) is not a \((\tau_1, \tau_2)\)-g*p-closed set in \((X, \tau_1, \tau_2)\).

Remark 6.2.34: From the above results we have the following diagram.

Remark 6.2.35: \(D^*P(\tau_1, \tau_2)\) is generally not equal to \(D^*P(\tau_2, \tau_1)\) as seen from the following example.

Example 6.2.36: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{a, c\}\}\). Then the subset \(\{a, b\} \in D^*P(\tau_2, \tau_1)\) but \(\{a, b\} \notin D^*P(\tau_1, \tau_2)\).
Theorem 6.2.37: If $\tau_1 \subseteq \tau_2$ in $(X, \tau_1, \tau_2)$, then $D^*P(\tau_2, \tau_1) \subseteq D^*P(\tau_1, \tau_2)$.

Proof: Let $A$ be a $(\tau_2, \tau_1)$ - $g^*p$-closed set and $U$ be a $\tau_1$-g-open set containing $A$. Since $\tau_1 \subseteq \tau_2$, it follows that $\tau_2$-pcl(A) $\subseteq \tau_1$-pcl(A) and $GO(X, \tau_1) \subseteq GO(X, \tau_2)$. Since $A \in D^*P(\tau_2, \tau_1)$, $\tau_1$-pcl(A) $\subseteq U$, $\tau_2$-pcl(A) $\subseteq U$, $U$ is $\tau_1$-g-open. Thus $A$ is $(\tau_1, \tau_2)$ - $g^*$-closed. Hence $D^*P(\tau_2, \tau_1) \subseteq D^*P(\tau_1, \tau_2)$.

The converse of the above theorem need not be true as seen from the following example.

Example 6.2.38: In Example 6.2.4, $D^*P(\tau_2, \tau_1) \subseteq D^*P(\tau_1, \tau_2)$ but $\tau_1$ is not contained in $\tau_2$.

Theorem 6.2.39: For each point $x$ of $(X, \tau_1, \tau_2)$, a singleton set $\{x\}$ is $\tau_1$-g-closed set or $\{x\}^c$ is $(\tau_1, \tau_1)$ - $g^*p$-closed set.

Proof: Suppose $\{x\}$ is not $\tau_1$ - g-closed. Then $\{x\}^c$ is not $\tau_1$ - g-open. Therefore $\tau_1$ - g-open set containing $\{x\}$ is $X$ only. Also $\tau_1$-pcl($\{x\}^c)$ $\subseteq X$. Hence $\{x\}^c$ is $(\tau_1, \tau_1)$ - $g^*p$-closed.

Theorem 6.2.40: If a set $A$ is $(\tau_1, \tau_2)$ - $g^*p$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_1$-pcl(A)-A contains no non-empty $\tau_1$-g-closed set.

Proof: Let $A$ be a $(\tau_1, \tau_2)$ - $g^*p$-closed set and $F$ be $\tau_1$-g-closed set contained in $\tau_1$-pcl(A)-A. Since $A \in D^*P(\tau_1, \tau_2)$, we have $\tau_1$-pcl(A) $\in F^c$. Consequently $F \subseteq \tau_1$-pcl(A) $\cap (X- \tau_1$-pcl(A)) = $\phi$.

The converse of the above theorem need not be true as seen from the following example.
Example 6.2.41: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). If \( A = \{b\} \), then \( \tau_2.pcl(A) - A = \{c\} \) does not contain any non-empty \( \tau_1 \)-g-closed set. But \( A \) is not a \( (\tau_1, \tau_2) \)-g*p-closed set.

Theorem 6.2.42: If \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed set in \( (X, \tau_1, \tau_2) \), then \( A \) is \( \tau_j \)-preclosed if and only if \( \tau_j.pcl(A) - A \) is \( \tau_j \)-g-closed set.

Proof: Necessity: If \( A \) is \( \tau_j \) - preclosed, then \( \tau_j.pcl(A) = A \). That is \( \tau_j.pcl (A) - A = \emptyset \) and hence \( \tau_j.pcl (A) - A \) is \( \tau_j \)-g-closed set.

Sufficiency: If \( \tau_j.pcl(A) - A \) is \( \tau_j \)-g-closed, by Theorem 6.2.40, \( \tau_j.pcl(A) - A = \emptyset \), since \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed. That is \( \tau_j.pcl (A) = A \). Therefore \( A \) is \( \tau_j \)-preclosed.

Theorem 6.2.43: If \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed set, then \( \tau_j.pcl(\{x\}) \cap A \neq \emptyset \), for each \( x \in \tau_j.pcl (A) \).

Proof: If \( \tau_j.pcl(\{x\}) \cap A = \emptyset \), for each \( x \in \tau_j.pcl(A) \), then \( A \subseteq (\tau_j.pcl(\{x\}))^c \).

Since \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed, we have \( \tau_j.pcl(A) \subseteq (\tau_j.pcl(\{x\}))^c \). This shows that \( x \notin \tau_j.pcl (A) \). This contradicts the assumption.

Theorem 6.2.44: If \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed set and \( A \subseteq B \subseteq \tau_j.pcl(A) \), then \( B \) is \( (\tau_n, \tau_j) \) - g*p-closed set.

Proof: Let \( B \subseteq G \), where \( G \) is \( \tau_j \)-g-open. Then \( A \subseteq B \) implies \( A \subseteq G \). As \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed set, \( \tau_j.pcl(A) \subseteq G \). Now \( B \subseteq \tau_j.pcl(A) \) which implies \( \tau_j.pcl(B) \subseteq \tau_j.pcl(\tau_j.pcl(A)) = \tau_j.pcl(A) \). Thus \( \tau_j.pcl(B) \subseteq G \). And therefore \( B \) is \( (\tau_n, \tau_j) \) - g*p-closed set.

Theorem 6.2.45: Let \( A \subseteq Y \subseteq X \) and suppose that \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed in \( (X, \tau_1, \tau_2) \). Then \( A \) is \( (\tau_n, \tau_j) \) - g*p-closed relative to \( Y \).
**Proof:** Let $S$ be any $\tau_i$-$g$-open set in $Y$ such that $A \subseteq S$. Then $S = U \cap Y$ for some $U \in \text{GO}(X, \tau_i)$. Thus $A \subseteq U \cap Y$ and so $A \subseteq U$. Since $A$ is $(\tau_n, \tau_j)$-$g^*p$-closed in $X$, $\tau_p.pcl(A) \subseteq U$ and therefore $Y \cap \tau_p.pcl(A) \subseteq Y \cap U$. That is $\tau_p.pcl_y(A) \subseteq S$, since $\tau_p.pcl_y(A) = Y \cap \tau_p.pcl(A)$. Hence $A$ is $(\tau_n, \tau_j)$-$g^*p$-closed relative to $Y$.

**Theorem 6.2.46:** In a bitopological space $(X, \tau_1, \tau_2)$, $\text{GO}(X, \tau_i) \subseteq \{F \subseteq X: F^c \in \tau_j\}$ if and only if every subset of $X$ is a $(\tau_n, \tau_j)$-$g^*p$-closed set.

**Proof:** Suppose that $\text{GO}(X, \tau_i) \subseteq \{F \subseteq X: F^c \in \tau_j\}$. Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $U \in \text{GO}(X, \tau_i)$ such that $A \subseteq U$. Then $\tau_p.pcl(A) \subseteq \tau_p.pcl(U) = U$. Hence $A$ is $(\tau_n, \tau_j)$-$g^*p$-closed set.

Conversely, suppose that every subset of $(X, \tau_1, \tau_2)$ is $(\tau_n, \tau_j)$-$g^*p$-closed set. Let $U \in \text{GO}(X, \tau_i)$. Since $U$ is $(\tau_n, \tau_j)$-$g^*p$-closed, we have $\tau_p.pcl(U) \subseteq U$. Therefore $U \in \{F \subseteq X: F^c \in \tau_j\}$ and we have $\text{GO}(X, \tau_i) \subseteq \{F \subseteq X: F^c \in \tau_j\}$.

Now we introduce the following.

**Definition 6.2.47:** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_n, \tau_j)$-$g^*p$-open set if $A^c$ is $(\tau_n, \tau_j)$-$g^*p$-closed in $(X,\tau_1, \tau_2)$.

**Theorem 6.2.48:** Every $\tau_j$-preopen (resp. $\tau_j$-open, $\tau_j$-$\alpha$-open and $(\tau_n, \tau_j)$-$g^*$-open) set is $(\tau_n, \tau_j)$-$g^*p$-open set but not conversely.

**Proof:** Proof follows from the Theorems 6.2.3 and 6.2.11.
Theorem 6.2.49: Every $(X_j, T_j) - g*p$-open set is (i) $(X_j, T_j) - g*p$-open and (ii) $(T_j, X_j) - g*p$r-open set but not conversely.

Proof: Proof follows from the Theorem 6.2.8.

Theorem 6.2.50: A subset $A$ of $(X, T_1, T_2)$ is $(X_j, T_j) - g*p$-open set if and only if $F \subseteq T_j$-pint $(A)$ whenever $F$ is $T_1$-$g$-closed set and $F \subseteq A$.

Proof: Suppose that $F$ is $T_1$-$g$-closed set, $F \subseteq A$ and $F \subseteq T_j$-pint $(A)$. Let $G$ be $T_1$-$g$-open and $A^c \subseteq G$. Then $G^c \subseteq A$ and $G^c$ is $T_1$-$g$-closed. Thus $G^c \subseteq T_j$-pint $(A)$ and $(T_j$-pint $(A))^c \subseteq G$. It follows that $T_j$-pcl $(A^c) \subseteq G$ and hence $A^c$ is $(T_n, T_j) - g*p$-closed. Hence $A$ is $(T_n, T_j) - g*p$-open.

Conversely, suppose that $A$ is $(T_n, T_j) - g*p$-open, $F \subseteq A$ and $F$ is $T_j$-$g$-closed. Then $F^c$ is $T_1$-$g$-open and $A^c \subseteq F^c$. Therefore $(T_j$-pcl $(A^c)) \subseteq F^c$ and hence $(T_j$-pint $(A))^c \subseteq F^c$. Thus $F \subseteq T_j$-pint $(A)$.

Theorem 6.2.51: If a subset $A$ of $(X, T_1, T_2)$ is $(T_n, T_j) - g*p$-closed, then $T_j$-pcl $(A) - A$ is $(T_n, T_j) - g*p$-open.

Proof: Let $A$ be a $T_1$-$g$-closed set such that $F \subseteq T_j$-pcl $(A) - A$. It follows that $F = \phi$. Therefore $F \subseteq (T_j$-pint $(T_j$-pint $(A) - A)$. Thus $T_j$-pcl $(A) - A$ is $(T_n, T_j) - g*p$-open.
§ 6.3 Applications of $(\tau_1, \tau_2) - g^p$-Closed Sets

In this section, we introduce new spaces namely, $(\tau_1, \tau_2) - T_p^*$-spaces, $(\tau_1, \tau_2) - T_p^*$-spaces and $(\tau_1, \tau_2) - T_p^*$-spaces as an application in bitopological spaces and obtain some of their properties.

**Definition 6.3.1:** A bitopological space $(X, \tau_1, \tau_2)$ is said to be a $(\tau_1, \tau_2) - T_p^*$-space if every $(\tau_1, \tau_2) - g^p$-closed set is $\tau_2$-closed.

**Remark 6.3.2:** If $\tau_1 = \tau_2$ in the Definition 6.3.1, we obtain the definition of $T_p^*$-space [120].

**Theorem 6.3.3:** If $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2) - T_p^*$-space, then it is $(\tau_1, \tau_2) - T_{1/2}$-space but not conversely.

**Proof:** Let $(X, \tau_1, \tau_2)$ be a $(\tau_1, \tau_2) - T_p^*$-space and $A$ be a $(\tau_1, \tau_2) - g^p$-closed set $(X, \tau_1, \tau_2)$. By Theorem 6.2.11, $A$ is $(\tau_1, \tau_2) - g^p$-closed set. Since $(X, \tau_1, \tau_2)$ is a $(\tau_1, \tau_2) - T_p^*$-space, $A$ is $\tau_2$-closed. Hence $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2) - T_{1/2}$-space.

**Example 6.3.4:** Let $X, \tau_1$ and $\tau_2$ be as in Example 6.2.14, Then $(X, \tau_1, \tau_2)$ is a $(\tau_1, \tau_2) - T_{1/2}$-space but not a $(\tau_1, \tau_2) - T_p^*$-space.

**Theorem 6.3.5:** If a bitopological space $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2) - T_p^*$-space, then $\{x\}$ is $\tau_2$-open or $\tau_2$-g-closed for each $x \in X$.

**Proof:** Suppose that $\{x\}$ is not $\tau_2$-g-closed set of $(X, \tau_1, \tau_2)$. Then $\{x\}^c$ is $(\tau_1, \tau_2) - g^p$-closed set of $(X, \tau_1, \tau_2)$ by Theorem 6.2.39. Since $(X, \tau_1, \tau_2)$ is a $(\tau_1, \tau_2) - T_p^*$-space, $\{x\}^c$ is $\tau_2$-closed. Therefore $\{x\}$ is $\tau_2$-open.

The converse of the above theorem need not be true as shown from the following example.
Example 6.3.6: In Example 6.2.14, the space \((X, \tau_1, \tau_2)\) is not a \((\tau_1, \tau_2) - T_p^*\) space. However any singleton set of \((X, \tau_1, \tau_2)\) is \(\tau_2\)-open set or \(\tau_1\)-g-closed.

Remark 6.3.7: \((X, \tau_1)\)-space is not generally \(T_p^*\)-space even if \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_p^*\) - space as shown in the following Example 6.3.8. Also \((X, \tau_1, \tau_2)\) is not generally \((\tau_1, \tau_2) - T_p^*\) - space even if both \((X, \tau_1)\) and \((X, \tau_2)\) are \(T_p^*\)-spaces. This is shown in Example 6.3.9.

Example 6.3.8: In the Example 6.2.25, the space \((X, \tau_1)\) is not a \(T_p^*\)-space but \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_p^*\)-space.

Example 6.3.9: Let \(X = \{a, b, c\}, \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}\). Then both \((X, \tau_1)\) and \((X, \tau_2)\) are \(T_p^*\)-spaces but \((X, \tau_1, \tau_2)\) is not a \((\tau_1, \tau_2) - T_p^*\)-space.

Remark 6.3.10: The following examples show that the concept of \((\tau_i, \tau_j) - T_p^*\) - spaces and \((\tau_i, \tau_j) - T_{1/2}\) -spaces are independent of each other.

Example 6.3.11: Let \(X, \tau_1\) and \(\tau_2\) be as in Example 6.2.25. Then \((X, \tau_1, \tau_2)\) is a \((\tau_1, \tau_2) - T_p^*\) - space but not a \((\tau_1, \tau_2) - T_{1/2}\) - space.

Example 6.3.12: Let \(X=\{a, b, c\}, \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}\). Then \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{1/2}\) - space but not a \((\tau_1, \tau_2) - T_p^*\)-space.

Theorem 6.3.13: If \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j) - T_p^*\)-space, then \(X\) is \((\tau_i, \tau_j) - T_{1/2}\) -space but not conversely.

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Proof: Let \((X, \tau_1, \tau_2)\) be a \((\tau_n, \tau_j)\)-\(T_p\)-space and \(A\) be a \((\tau_n, \tau_j)\)-\(g^*s\)-closed set \((X, \tau_1, \tau_2)\). By Theorem 6.2.11, \(A\) is \((\tau_n, \tau_j)\)-\(g^p\)-closed set. Since \((X, \tau_1, \tau_2)\) is \((\tau_n, \tau_j)\)-\(T_p\)-space, \(A\) is \(\tau_j\)-closed. Hence \((X, \tau_1, \tau_2)\) is \((\tau_n, \tau_j)\)-\(\alpha_g T^*_1/2\)-space.

Example 6.3.14: Let \(X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}\). Then the space \((X, \tau_1, \tau_2)\) is a \((\tau_1, \tau_2)\)-\(\alpha_g T^*_1/2\)-space but not a \((\tau_1, \tau_2)\)-\(T_p\)-space.

Definition 6.3.15: A bitopological space \((X, \tau_1, \tau_2)\) is said to be strongly pairwise \(T_p\)-space if it is both \((\tau_1, \tau_2)\)-\(T_p\)-space and \((\tau_2, \tau_1)\)-\(T_p\)-space.

Theorem 6.3.16: If \((X, \tau_1, \tau_2)\) is strongly pairwise \(T_p\)-space then it is strongly pairwise \(T^*_1/2\)-space but not conversely.

Proof: The proof follows from Theorem 6.3.3.

Example 6.3.17: Let \(X, \tau_1\) and \(\tau_2\) be as in Example 6.2.14, Then \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\)-\(T^*_1/2\)-space and also a \((\tau_2, \tau_1)\)-\(T^*_1/2\)-space and therefore it is strongly pairwise \(T^*_1/2\)-space. But \((X, \tau_1, \tau_2)\) is not strongly pairwise \(T_p\)-space, since it is not a \((\tau_1, \tau_2)\)-\(T_p\)-space.

We now introduce the following.

Definition 6.3.18: A bitopological space \((X, \tau_1, \tau_2)\) is said to be a \((\tau_n, \tau_j)\)-\(*T_p\)-space if every \((\tau_n, \tau_j)\)-\(gp\)-closed set is \((\tau_n, \tau_j)\)-\(g^p\)-closed set.

Remark 6.3.19: The concepts of \((\tau_n, \tau_j)\)-\(*T_p\)-spaces and \((\tau_n, \tau_j)\)-\(T_p\)-spaces are independent of each other as seen from the following examples.
Example 6.3.20: In the Example 6.3.12, the space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{p}^{*}\) - space but not a \((\tau_1, \tau_2) - T_{p}^{*}\)-space.

Example 6.3.21: Let \(X=\{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}\). Then the space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{p}^{*}\) - space but not a \((\tau_1, \tau_2) - T_{p}^{*}\)-space.

Remark 6.3.22: The following examples show that the concept of \((\tau_n, \tau_j) - T_{p}^{*}\)-spaces and \((\tau_n, \tau) - T_{1/2}^{*}\)-spaces are independent of each other.

Example 6.3.23: In the Example 6.2.25, the space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{1/2}^{*}\) - space but not a \((\tau_1, \tau_2) - T_{p}^{*}\)-space.

Example 6.3.24: In the Example 6.2.16, the space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{p}^{*}\) - space but not a \((\tau_1, \tau_2) - T_{1/2}^{*}\)-space.

Theorem 6.3.25: If a bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - T_{p}^{*}\)-space, then \(\{x\}\) is \(\tau_j\)-closed or \(\tau_j\)-g*p-open for each \(x \in X\).

Proof: Suppose that \(\{x\}\) is not \(\tau_j\)-closed set of \((X, \tau_1, \tau_2)\). Then \(\{x\}\) is \((\tau_n, \tau_j) - gp\)-closed set of \((X, \tau_1, \tau_2)\) by Theorem 6.2.39. Since \((X, \tau_1, \tau_2)\) is a \((\tau_n, \tau_j) - T_{p}^{*}\)-space, \(\{x\}\) is \(\tau_j\)-g*p-closed set. Therefore \(\{x\}\) is \(\tau_j\)-g*p-open set.

The converse of the above theorem need not be true as shown from the following example

Example 6.3.26: Let \(X=\{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{a, c\}\}\). Then any singleton set of \((X, \tau_1, \tau_2)\) is \(\tau_2\)-g*p-open or \(\tau_1\)-closed, but \((X, \tau_1, \tau_2)\) is not a \((\tau_1, \tau_2) - T_{p}^{*}\)-space.
**Definition 6.3.27:** A bitopological space $(X, \tau_i, \tau_j)$ is said to be a $(\tau_i, \tau_j)$-\(\alpha T_p^*\)-space if every $(\tau_i, \tau_j) - g^*p$-closed set is $\tau_j$-preclosed set.

**Theorem 6.3.28:** Every $(\tau_i, \tau_j) - T_p^*$-space is $(\tau_i, \tau_j) - \alpha T_p^*$-space but not conversely.

**Proof:** Follows from the definitions.

**Example 6.3.29:** In the Example 6.2.10, the space $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2)$-\(\alpha T_p^*\)-space but not a $(\tau_1, \tau_2) - T_p^*$-space.

**Remark 6.3.30:** From the above results we have the following diagram.

```
(T_i, T_j)- T_{1/2}^*\text{-space} \quad \downarrow \quad (T_i, T_j)- T_{2/1}^*\text{-space}
```

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(T_i, T_j)- T_{1/2}^*\text{-space} \quad \downarrow \quad (T_i, T_j)- T_{2/1}^*\text{-space}
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```
(T_i, T_j)- T_{1/2}^*\text{-space} \quad \downarrow \quad (T_i, T_j)- T_{2/1}^*\text{-space}
```

where $A \rightarrow B$ represents $A$ implies $B$ but not conversely and $A \leftrightarrow B$ represents $A$ and $B$ are independent of each other.

**Remark 6.3.31:** The following examples show that the concepts of $(\tau_i, \tau_j)$-\(\alpha T_p^*\)-spaces and $(\tau_i, \tau_j) - *T_p^*$-spaces are independent of each other.

**Example 6.3.32:** In Example 6.2.31, the space $(X, \tau_1, \tau_2)$ is a $(\tau_1, \tau_2) - \alpha T_p^*$-space but not a $(\tau_1, \tau_2) - *T_p^*$-space.
Example 6.3.33: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{c\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then the space $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2)$ - $*_p$ - space but not a $(\tau_1, \tau_2)$ - $\alpha_Tp$ - space.

§ 6.4. $g*p$-Continuous Functions in Bitopological Spaces

In this section we introduce $g*p$-continuous functions in bitopological spaces and obtain some of their properties.

Definition 6.4.1: A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $D*P (\tau_i, \tau_j)$- $\sigma_k$-continuous ($g*p$-continuous) if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$- $g*p$-closed set in $(X, \tau_1, \tau_2)$.

Remark 6.4.2: If $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$ in Definition 6.4.1, then $g*p$-continuous functions of bitopological spaces coincides with $g*p$-continuity [120] of topological spaces.

Theorem 6.4.3: If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_j$ - $\sigma_k$-continuous, then $f$ is $D*P (\tau_i, \tau_j)$- $\sigma_k$-continuous.

Proof: Let $V$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Then $f^{-1}(V)$ is $\tau_j$-closed set. By Theorem 6.2.3, $f^{-1}(V)$ is $(\tau_i, \tau_j)$- $g*p$-closed set in $(X, \tau_1, \tau_2)$. Then $f$ is $D*P(\tau_i, \tau_j)$- $\sigma_k$-continuous.

The converse of the above theorem need not be true as seen from the following example.
Example 6.4.4: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{b\}\}$, $\tau_2 = \{X, \phi, \{b, c\}\}$ and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}\}$, $\sigma_2 = \{Y, \phi, \{q\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = p$, $f(b) = f(c) = q$. Then $f$ is $D^*P(\tau_1, \tau_2)$-$\sigma_2$-continuous but $f$ is not $\tau_1$-$\sigma_2$-continuous.

**Theorem 6.4.5:** If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\alpha^*(\tau_1, \tau_j)$-$\sigma_k$-continuous, then $f$ is $D^*P(\tau_i, \tau_j)$ - $\sigma_k$-continuous but not conversely.

**Proof:** Let $V$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Then $f^1(V)$ is $(\tau_i, \tau_j)$-$\alpha g^* s$-closed set in $(X, \tau_1, \tau_2)$ as $f$ is $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous. By Theorem 6.2.11, $f^1(V)$ is $(\tau_i, \tau_j)$-$g^* p$-closed set in $(X, \tau_1, \tau_2)$. Hence $f$ is $D^*P(\tau_i, \tau_j)$-$\sigma_k$-continuous function.

Example 6.4.6: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{c\}, \{a, c\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous but not $\alpha^*(\tau_i, \tau_2)$-$\sigma_2$-continuous, since for the $\sigma_2$- closed set $\{a, b\}$ in $(Y, \sigma_1, \sigma_2)$, $f^1(\{a, b\}) = \{a, b\}$ is not a $(\tau_i, \tau_j)$-$\alpha g^* s$-closed set in $(X, \tau_1, \tau_2)$ but it is $(\tau_i, \tau_2)$-$g^* p$-closed set in $(X, \tau_1, \tau_2)$.

**Theorem 6.4.7:** If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D^*P(\tau_i, \tau_j)$ - $\sigma_k$-continuous, then $f$ is $\zeta(\tau_i, \tau_j)$ - $\sigma_k$-continuous.

**Proof:** Let $V$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Then $f^1(V)$ is $(\tau_i, \tau_j)$- $g^* p$-closed set in $(X, \tau_1, \tau_2)$ as $f$ is $D^*P(\tau_i, \tau_j)$-$\sigma_k$-continuous. By Theorem 6.2.8 (ii), $f^1(V)$ is $(\tau_i, \tau_j)$- $gpr$-closed set in $(X, \tau_1, \tau_2)$. Hence $f$ is $\zeta(\tau_i, \tau_j)$ - $\sigma_k$-continuous function.

The converse of the above theorem need not be true as seen from the following example.
Example 6.4.8: Let $X = Y = \{a, b, c\}$, $\tau_i = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{a, b\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then the function $f$ is $\zeta(\tau_1, \tau_2)$-$\sigma_2$-continuous but not $D^*P(\tau_i, \tau_2)$-$\sigma_2$-continuous.

Theorem 6.4.9: If a function $f: (X, \tau_i, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D^*P(X, \tau_i)$-$\sigma_k$-continuous, then $f$ is $(\tau_i, \tau_j)$-$\sigma_k$-continuous.

Proof: Let $K$ be a $\sigma_k$-closed set in $(Y, \sigma_i, \sigma_2)$. Then $f^{-1}(K)$ is $(\tau_i, \tau_j)$-$g^*p$-closed set in $(X, \tau_1, \tau_2)$ as $f$ is $D^*P(\tau_i, \tau_j)$-$\sigma_k$-continuous. By Theorem 6.2.8 (i), $f^{-1}(V)$ is $(\tau_i, \tau_j)$-$gp$-closed set in $(X, \tau_1, \tau_2)$. Hence $f$ is $(\tau_i, \tau_j)$-$gp$-$\sigma_k$-continuous function.
The converse of the above theorem need not be true as seen from the following example.

Example 6.4.10: Let $X = Y = \{a, b, c\}$, $\tau_i = \{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{a, b\}\}$ and $\sigma_2 = \{Y, \phi, \{a\}\}$. Then the identity function $f: (X, \tau_i, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_2)$-$gp$-$\sigma_1$-continuous but not $D^*P(\tau_i, \tau_2)$-$\sigma_1$-continuous.

Theorem 6.4.11: If $f: (X, \tau_i, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D^*(\tau_i, \tau_j)$-$\sigma_k$-continuous, then $f$ is $D^*P(\tau_i, \tau_j)$-$\sigma_k$-continuous but not conversely.

Proof: Let $V$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Then $f^{-1}(V)$ is $(\tau_i, \tau_j)$-$g^*$-closed set in $(X, \tau_1, \tau_2)$. By Theorem 6.2.13, $f^{-1}(V)$ is $(\tau_i, \tau_j)$-$g^*p$-closed set in $(X, \tau_1, \tau_2)$. Therefore $f$ is $D^*P(\tau_i, \tau_j)$-$\sigma_k$-continuous.

Example 6.4.12: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Define a function
f: (X, τ_1, τ_2) \rightarrow (Y, σ_1, σ_2) by f(a) = b, f(b) = c and f(c) = a. Then the function f is D^*(τ_1, τ_2)-σ_2-continuous but not D*(τ_1, τ_2)-σ_2-continuous.

**Remark 6.4.13:** D(τ_1, τ_2)-σ_k-continuous and D^*P(τ_1, τ_2)-σ_k-continuous functions are independent of each other as seen from the following examples.

**Example 6.4.14:** Let X = Y = \{a, b, c\}, τ_1 = \{X, φ, \{a\}, \{b, c\}\}, τ_2 = \{X, φ, \{a\}\} and σ_1 = \{Y, φ, \{a\}, \{a, b\}\}, σ_2 = \{Y, φ, \{a\}, \{a, c\}\}. Define a function f: (X, τ_1, τ_2) \rightarrow (Y, σ_1, σ_2) by f(a) = b, f(b) = a and f(c) = c. Then f is D(τ_1, τ_2)-σ_2-continuous but not a D^*P(τ_1, τ_2)-σ_2-continuous, since for the σ_2-closed set \{b, c\} in (Y, σ_1, σ_2), f^{-1}(\{b, c\}) = \{a, c\} is not a (τ_1, τ_2)-g^*-p-closed set (X, τ_1, τ_2).

**Example 6.4.15:** Let X = Y = \{a, b, c\}, τ_1 = \{X, φ, \{a\}, \{b\}, \{a, b\}\}, τ_2 = \{X, φ, \{a\}\} and σ_1 = \{Y, φ, \{a\}, \{b\}, \{a, b\}\}, σ_2 = \{Y, φ, \{a\}, \{a, b\}\}. Then the identity function f: (X, τ_1, τ_2) \rightarrow (Y, σ_1, σ_2) is D^*P(τ_1, τ_2)-σ_1-continuous but not a D(τ_1, τ_2)-σ_1-continuous.

**Remark 6.4.16:** The concepts of C(τ_1, τ_2)-σ_k-continuous functions and D^*P(τ_1, τ_2)-σ_k-continuous functions are independent of each other as seen from the following examples.

**Example 6.4.17:** Let X = Y = \{a, b, c\}, τ_1 = \{X, φ, \{a\}, \{b\}, \{a, b\}\}, τ_2 = \{X, φ, \{a\}\} and σ_1 = \{Y, φ, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}, σ_2 = \{Y, φ, \{a\}, \{a, b\}\}. Define a function f: (X, τ_1, τ_2) \rightarrow (Y, σ_1, σ_2) by f(a) = a, f(b) = c and f(c) = b. Then f is C(τ_1, τ_2)-σ_2-continuous but not D^*P(τ_1, τ_2)-σ_2-continuous.
Example 6.4.18: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{c\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{a, b\}\}$. Then the identity function $f$ on $X$ is $D*P(\tau_1, \tau_2)$-\(\sigma_2\)-continuous but not $C(\tau_1, \tau_2)$-\(\sigma_2\)-continuous.

Remark 6.4.19: $D_\ell(\tau_1, \tau_2)$-\(\sigma_k\)-continuous and $D*P(\tau_n, \tau_j)$-\(\sigma_k\)-continuous functions are independent of each other as seen from the following examples.

Example 6.4.20: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D*P(\tau_1, \tau_2)$-\(\sigma_2\)-continuous but not $D_\ell(\tau_1, \tau_2)$-\(\sigma_2\)-continuous function.

Example 6.4.21: In Example 6.4.14, define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then the function $f$ is $D_\ell(\tau_1, \tau_2)$-\(\sigma_2\)-continuous but not $D*P(\tau_1, \tau_2)$-\(\sigma_2\)-continuous.

Remark 6.4.22: The following examples show that $W(\tau_n, \tau_j)$-\(\sigma_k\)-continuous and $D*P(\tau_n, \tau_j)$-\(\sigma_k\)-continuous functions are independent of each other.

Example 6.4.23: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$, $\tau_2 = \{X, \phi, \{a\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then $f$ is $W(\tau_1, \tau_2)$-\(\sigma_2\)-continuous but not $D*P(\tau_1, \tau_2)$-\(\sigma_2\)-continuous.

Example 6.4.24: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{b\}, \{b, c\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the identity
function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( D^*P(\tau_1, \tau_2) \)-\( \sigma_2 \)-continuous but not \( W(\tau_1, \tau_2) \)-\( \sigma_2 \)-continuous.

**Theorem 6.4.25:** If \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( D^*P(\tau_1, \tau_2) \)-\( \sigma_k \)-continuous and \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\)-\( T^*_p \)-space, then \( f \) is \( \tau_j \)-\( \sigma_k \)-continuous.

**Proof:** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \( D^*P(\tau_1, \tau_2) \)-\( \sigma_k \)-continuous function and let \( V \) be a \( \sigma_k \)-closed set in \((Y, \sigma_1, \sigma_2)\). Then \( f^{-1}(V) \) is \( (\tau_1, \tau_2) \)-\( g^*p \)-closed set in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\)-\( T^*_p \)-space, \( f^{-1}(V) \) is \( \tau_j \)-closed set in \((X, \tau_1, \tau_2)\). Therefore \( f \) is \( \tau_j \)-\( \sigma_k \)-continuous function.

**Remark 6.4.26:** From the above results we have the following implications.
§ 6.5 Some Stronger forms of $g^p$-Continuous Functions in Bitopological Spaces

In this section, we introduce $g^p$-bi-continuous functions, $g^p$-strongly-bi-continuous functions in bitopological spaces. We also study their relations with some existing functions in bitopological spaces. Further we introduce and study the pairwise $g^p$- irresolute functions in bitopological spaces and obtain some of their properties.

**Definition 6.5.1:** A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

i) $g^p$-bi-continuous if $f$ is $D^P(\tau_1, \tau_2)-\sigma_2$-continuous and $D^P(\tau_2, \tau_1)-\sigma_1$-continuous.

ii) $g^p$-strongly-bi-continuous (briefly $g^p$-s-bi-continuous) if $f$ is $g^p$-bi-continuous, $D^P(\tau_1, \tau_2)-\sigma_1$-continuous and $D^P(\tau_2, \tau_1)-\sigma_2$-continuous.

**Theorem 6.5.2:** Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

i) If $f$ is bi-continuous then $f$ is $g^p$-bi-continuous.

ii) If $f$ is s-bi-continuous then $f$ is $g^p$-s-bi-continuous.

**Proof:** i) Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bi-continuous function. Then $f$ is $\tau_1-\sigma_1$-continuous and $\tau_2-\sigma_2$-continuous. By Theorem 6.4.3, $f$ is $D^P(\tau_1, \tau_2)-\sigma_2$-continuous and $D^P(\tau_2, \tau_1)-\sigma_1$-continuous. Thus $f$ is $g^p$-bi-continuous.

(ii) Similar to (i), using Theorem 6.4.3.

The converse of the above theorem need not be true as seen from the following example.
Example 6.5.3: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \phi, \{b, c\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi, \{p\}\}$ and $\sigma_2=\mathcal{P}(Y)$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y,\sigma_1, \sigma_2)$ by $f(a) = p$ and $f(b) = f(c) = q$. Then $f$ is $g^*p$-s-bi-continuous but not s-bi-continuous. This function is also $g^*p$-bi-continuous but not bi-continuous.

Theorem 6.5.4: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y,\sigma_1, \sigma_2)$ be a function.

i) If $f$ is $g^*p$-bi-continuous, then $f$ is $gpr$-bi-continuous.

ii) If $f$ is $g^*p$-s-bi-continuous, then $f$ is $gpr$-s-bi-continuous.

Proof: i) Let $f: (X, \tau_1, \tau_2) \rightarrow (Y,\sigma_1, \sigma_2)$ be a $g^*p$-bi-continuous function. Then $f$ is $D^*P(\tau_1, \tau_2)$-$\sigma_2$-continuous and $D^*P(\tau_2, \tau_1)$-$\sigma_1$-continuous. By Theorem 6.4.7, $f$ is $\zeta(\tau_1, \tau_2)$-$\sigma_2$-continuous and $\zeta(\tau_2, \tau_1)$-$\sigma_1$-continuous. Thus $f$ is $gpr$-bi-continuous.

(ii) Similar to (i), using Theorem 6.4.7.

The converse of the above theorem need not be true as seen from the following example.

Example 6.5.5: Let $X = Y= \{a, b, c\}$, $\tau_1 = \{X,\emptyset, \{a\}\}$, $\{b\}$, $\{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$, $\{a, c\}\}$ and $\sigma_1=\{Y, \phi, \{a\}, \{b\}$, $\{a, c\}$, $\{a, c\}$ and $\sigma_2=\{Y, \phi, \{b\}$, $\{c\}$, $\{a, c\}$, $\{b, c\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y,\sigma_1, \sigma_2)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then $f$ is $gpr$-bi-continuous but not a $g^*p$-bi-continuous, since for the $\sigma_2$-closed set $\{a, c\}$ in $(Y,\sigma_1, \sigma_2)$, $f'(\{a, c\}) = \{a, b\}$ is not $(\tau_1, \tau_2)$-$g^*p$-closed set but it is $(\tau_1, \tau_2)$-$gpr$-closed set in $(X, \tau_1, \tau_2)$. This function is also $gpr$-s-bi-continuous but not $g^*p$-s-bi-continuous.
Theorem 6.5.6: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

i) If $f$ is $g^*$-bi-continuous then $f$ is $g^*$-p-bi-continuous.

ii) If $f$ is $g^*$-s-bi-continuous then $f$ is $g^*$-p-s-bi-continuous.

Proof: i) Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $g^*$-bi-continuous. Then $f$ is $D^*(\tau_1, \tau_2)-\sigma_2$-continuous and $D^*(\tau_2, \tau_1)-\sigma_1$-continuous. By Theorem 6.4.11, $f$ is $D^*P(\tau_1, \tau_2)-\sigma_2$-continuous and $D^*P(\tau_2, \tau_1)-\sigma_1$-continuous. Thus $f$ is $g^*$-p-bi-continuous.

(ii) Similar to (i), using Theorem 6.4.11.

The converse of the above theorem need not be true as seen from the following example.

Example 6.5.7: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \emptyset, \{a\}\}$, $\sigma_2 = \{Y, \emptyset, \{a\}, \{a, c\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $g^*$-p-bi-continuous but not $g^*$-bi-continuous, since for the $\sigma_2$-closed set $\{b\}$ in $(Y, \sigma_1, \sigma_2)$, $f^{-1}(\{b\}) = \{b\}$ is not $(\tau_1, \tau_2)$-$g^*$-closed set but it is $(\tau_1, \tau_2)$-$g^*$-p-closed set in $(X, \tau_1, \tau_2)$. This function is also $g^*$-p-s-bi-continuous but not $g^*$-s-bi-continuous.

Theorem 6.5.8: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.

i) If $f$ is $\alpha g^*$ -bi-continuous then $f$ is $g^*$-p-bi-continuous.

ii) If $f$ is $\alpha g^*$-s-bi-continuous then $f$ is $g^*$-p-s-bi-continuous.

Proof: i) Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $\alpha g^*$-s-bi-continuous. Then $f$ is $\alpha^*(\tau_1, \tau_2)-\sigma_2$-continuous and $\alpha^*(\tau_2, \tau_1)-\sigma_1$-continuous. By Theorem 6.4.5, $f$ is $D^*P(\tau_1, \tau_2)-\sigma_2$-continuous and $D^*P(\tau_2, \tau_1)-\sigma_1$-continuous. Thus $f$ is $g^*$-p-bi-continuous.

(ii) Similar to (i), using Theorem 6.4.5.
The converse of the above theorem need not be true as seen from the following example.

**Example 6.5.9:** Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{b\}, \{b, c\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}\} \) and \( \sigma_2 = \{Y, \phi, \{b\}, \{a, c\}\} \). Define a function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = a \), \( f(b) = b \) and \( f(c) = c \). Then the function \( f \) is \( g^*p\)-bi-continuous (resp. \( g^*p\)-s-bi-continuous) but not \( \alpha g^*s\)-bi-continuous (resp. \( \alpha g^*s\)-s-bi-continuous) function.

**Remark 6.5.10:** \( g^*p\)-bi-continuous functions and \( g\)-bi-continuous functions are independent of each other as seen from the following examples.

**Example 6.5.11:** In Example 6.5.5, the function \( f \) is \( g\)-bi-continuous but not \( g^*p\)-bi-continuous function.

**Example 6.5.12:** Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a, c\}\} \), \( \tau_2 = \{X, \phi, \{b\}, \{b, c\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}\} \), \( \sigma_2 = \{Y, \phi, \{b\}, \{a, c\}\} \). Then the identity function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( g^*p\)-bi-continuous \( f \) but it is not \( g\)-bi-continuous function.

**Remark 6.5.13:** \( g^*p\)-s-bi-continuous functions and \( g\)-s-bi-continuous functions are independent of each other as seen from the following examples.

**Example 6.5.14:** In Example 6.5.5, the function \( f \) is \( g\)-s-bi-continuous but not a \( g^*p\)-s-bi-continuous function.

**Example 6.5.15:** In Example 6.5.12, the function \( f \) is \( g^*p\)-s-bi-continuous but it is not \( g\)-s-bi-continuous function.
Remark 6.5.16: From the above results we get the following implications.

\[ \text{g-bi-continuous} \leftrightarrow \text{g*-bi-continuous} \rightarrow \text{gpr-bi-continuous} \]

Definition 6.5.17: A function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called pairwise g*p- irresolute function if \( f^{-1}(A) \in D^*P(\tau_i, \tau_j) \) in \( (X, \tau_1, \tau_2) \) for every \( A \in D^*P(\sigma_k, \sigma_l) \) in \( (Y, \sigma_1, \sigma_2) \).

Remark 6.5.18: If \( \tau_1 = \tau_2 = \tau \) and \( \sigma_1 = \sigma_2 = \sigma \) simultaneously, then \( f \) becomes a g*p-irresolute function [120].

Theorem 6.5.19: If a function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise g*p - irresolute then \( f \) is \( D^*P(\tau_i, \tau_j) - \sigma_k \)-continuous.

Proof: Let \( F \) be any \( \sigma_k \)- closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( F \) is \( \in D^*P(\sigma_k, \sigma_2) \) in \( (Y, \sigma_1, \sigma_2) \). Since \( f \) is pairwise g*p-irresolute, \( f^{-1}(F) \in D^*P(\tau_i, \tau_j) \) in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D^*P(\tau_i, \tau_j) - \sigma_k \)-continuous function.
The converse of the above theorem need not be true as seen from the following example.

**Example 6.5.20:** Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\} \), \( \sigma_2 = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\} \). Define a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). Then \( f \) is \( D^*P(\tau_1, \tau_2) \)-\( \sigma_2 \)-continuous but not a pairwise \( g^*p \)-irresolute.

**Theorem 6.5.21:** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise \( g^*p \)-irresolute if and only if for every \( (\sigma_k, \sigma_\ell) \)-\( g^*p \)-open set \( A \) in \( (Y, \sigma_1, \sigma_2) \), the inverse image \( f^{-1}(A) \) is \( (\tau_i, \tau_j) \)-\( g^*p \)-open in \( (X, \tau_1, \tau_2) \).

**Theorem 6.5.22:** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) are pairwise \( g^*p \)-irresolute functions, then their composition \( gof \) is also pairwise \( g^*p \)-irresolute.

**Proof:** Let \( A \in (\eta_\ell, \eta_n) \) in \( (Z, \eta_1, \eta_2) \). Since \( g \) is pairwise \( g^*p \)-irresolute, \( g^{-1}(A) \in D^*P(\sigma_k, \sigma_\ell) \) in \( (Y, \sigma_1, \sigma_2) \). By hypothesis, \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \in D^*P(\tau_i, \tau_j) \) in \( (X, \tau_1, \tau_2) \). Thus \( gof \) is pairwise \( g^*p \)-irresolute function.

**Theorem 6.5.23:** If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be any two functions. Then

i) \( gof : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is \( D^*P(\tau_i, \tau_j) \)-\( \eta_n \)-continuous if \( g \) is \( \sigma_\ell \)-\( \eta_n \)-continuous and \( f \) is \( D^*(\tau_i, \tau_j) \)-\( \sigma_k \)-continuous.

ii) \( gof : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is \( D^*P(\tau_i, \tau_j) \)-\( \eta_n \)-continuous if \( f \) is pairwise \( g^*p \)-irresolute and \( g \) is \( D^*(\sigma_k, \sigma_\ell) \)-\( \eta_n \)-continuous.

**Proof:** i) Let \( A \) be any \( \eta_n \)-open set in \( (Z, \eta_1, \eta_2) \). Since \( g \) is \( \sigma_\ell \)-\( \eta_n \)-continuous, we have \( g^{-1}(A) \) is \( \sigma_\ell \)-open set in \( (Y, \sigma_1, \sigma_2) \). Again since \( f \) is \( D^*P(\tau_i, \tau_j) \)-\( \sigma_k \)-
continuous, \( f^{-1}(g^{-1}(A)) \) is \((\tau_i, \tau_j)-\text{g*p-closed}\) in \((X, \tau_i, \tau_j)\). But \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \). Hence \( \text{gof} \) is \( \text{D*P}(\tau_i, \tau_j)-\eta_n\)-continuous.

(ii) Let \( A \) be any \( \eta_n\)-open set in \((Z, \eta_1, \eta_2)\). Since \( g \) is \( \text{D}(\sigma_k, \sigma_2)-\eta_n\)-continuous, we have \( g^{-1}(A) \) is \((\sigma_k, \sigma_2)-\text{g*p-closed}\) set in \((Y, \sigma_1, \sigma_2)\). Again since \( f \) is pairwise \( \text{g*p-irresolute}, \) \( f^{-1}(g^{-1}(A)) \) is \((\tau_i, \tau_j)-\text{g*p-closed}\) in \((X, \tau_i, \tau_j)\). But \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \). Hence \( \text{gof} \) is \( \text{D*P}(\tau_i, \tau_j)-\eta_n\)-continuous.

**Theorem 6.5.24:** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \( \text{D*P}(i, j)-\sigma_k\)-continuous function.

(i) If \((X, \tau_1, \tau_2)\) is a \( T_{p^*}\)-space, then \( f \) is \( \tau_j-\sigma_k\)-continuous.

(ii) If \((X, \tau_1, \tau_2)\) is a \( \alpha T_{p^*}\)-space, then \( f \) is \( \tau_j-\sigma_k\)-precontinuous.

**Proof:** (i) Let \((X, \tau_1, \tau_2)\) is a \( T_{p^*}\)-space and \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \( \text{D*P}(i, j)-\sigma_k\)-continuous. Let \( V \) be a \( \sigma_k\)-closed set. Then \( f^{-1}(V) \) is \((i, j)-\text{g*p-closed}\) in \((X, \tau_1, \tau_2)\). Since \((X, \tau_1, \tau_2)\) is a \( T_{p^*}\)-space, \( f^{-1}(V) \) is \( \tau_j\)-closed. Thus \( f \) is \( \tau_j-\sigma_k\)-continuous.

(ii) Similar to (i).