CHAPTER 5

αg*s- CLOSED SETS IN BITOPOLITICAL SPACES

§ 5.1 Introduction

In 1963, Kelly [51] defined a bitopological space \((X, \tau_1, \tau_2)\) to be a set \(X\) equipped with two topologies \(\tau_1, \tau_2\) on \(X\) and initiated a systematic study of bitopological spaces. After the work of him Fukutake [40] extended the concept of generalized closed sets and \(T_{1/2}\)-spaces of Levine [57] to bitopological spaces and obtained some of their properties. Various authors, like Arya and Nour [7], Popa [95], Reilly [101], Sampath Kumar [104], Devi [23], Arockiarani [4], Gnanambal [47], Sundaram [112], Nagaveni [80], Fukutake et al [42] and El-Tantawy and Abu-Donia [118] have turned their attention to the various concepts of topology by considering bitopological spaces instead of topological spaces. In 2002, Fukutake, Sundaram and Sheik John [42] introduced and studied the concept of \(\omega\)-closed sets, \(\omega\)-open sets and \(\omega\)-continuity in bitopological spaces. Recently Rajamani and Vishwanathan [100] introduced and investigated the concept of \(\alpha\)-generalized semi-closed (briefly \(\alpha g s\)-closed) sets in bitopological spaces.

This chapter contains five sections. In section 2, we introduce, the concept of \(\alpha g^*s\)-closed sets and \(\alpha g^*s\)-open sets in bitopological spaces. Among many other results it is observed that every \((\tau_i, \tau_j)\) - \(\alpha g^*s\)-closed set is \((\tau_i, \tau_j)\) - \(\alpha g s\)-closed set but not conversely.
In Section 3, we introduce a new space namely, $(\tau_i, \tau_j)_{\text{ag}_T^{*1/2}}$-spaces as an application and study some of their properties. It is observed that $(\tau_i, \tau_j)_{\text{ag}_T^{*1/2}}$-spaces are independent of $(\tau_i, \tau_j)_{T^{1/2}}$-spaces and $(\tau_i, \tau_j)_{T^{*1/2}}$-spaces.

In Section 4, we introduce a new class of functions, called $\alpha^s$-continuous functions in bitopological spaces and is denoted by $\alpha^s(\tau_i, \tau_j)$-$\sigma_k$-continuity. During this process, some of their properties are obtained. It is found that every $\tau_j$-$\sigma_k$-continuous function is $\alpha^s(\tau_i, \tau_j)$-$\sigma_k$-continuous and every $\alpha^s(\tau_i, \tau_j)$-$\sigma_k$-continuous function is $\alpha(\tau_i, \tau_j)$-$\sigma_k$-continuous function but not conversely.

In the last section of this chapter, we introduce the concept of $\alpha^s$-bi-continuity, $\alpha^s$-strongly-bi-continuity and pairwise $\alpha^s$-irresolute functions in bitopological spaces and study some of their properties.

Throughout this thesis $(X, \tau_1, \tau_2)$, $(Y, \sigma_1, \sigma_2)$ and $(Z, \eta_1, \eta_2)$ denote non-empty bitopological spaces on which no separation axioms are assumed unless otherwise mentioned and the fixed integers $i, j, k, e, m, n \in \{1, 2\}$.

Here we present some of the definitions, which are used in our study.

**Definition 5.1.1:** Let $i, j \in \{1, 2\}$ be fixed integers. In a bitopological space $(X, \tau_i, \tau_j)$, a subset $A$ of $(X, \tau_i, \tau_j)$ is called

(i) $(\tau_i, \tau_j)$-$g$-closed [40] if $\tau_j$-$\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$

(ii) $(\tau_i, \tau_j)$-$gp$-closed [118] if $\tau_j$-$\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$
(iii) \((\tau_i, \tau_j)\)-gs-closed [118 ] if \(\tau_j\)-scl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(\tau_i\).

(iv) \((\tau_i, \tau_j)\)-rg-closed [4] if \(\tau_j\)-cl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(\tau_i\).

(v) \((\tau_i, \tau_j)\)-gpr-closed [47] if \(\tau_j\)-pcl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(\tau_i\).

(vi) \((\tau_i, \tau_j)\)-wg-closed set [80] if \(\tau_j\)-cl (\(\tau_i\) int(A)) \(\subseteq U\) whenever \(A \subseteq U\) and \(U \in \tau_i\).

(vii) \((\tau_i, \tau_j)\)-\(\omega\)-closed set [42] if \(\tau_j\)-cl(A)) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open set in \(\tau_i\).

(viii) \((\tau_i, \tau_j)\)-g*-closed [107] if \(\tau_j\)-cl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-g-open set.

(ix) \((\tau_i, \tau_j)\)-ags-closed [100] if \(\tau_j\)-acl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open set in \(\tau_i\).

The family of all \((\tau_i, \tau_j)\)-g-closed (resp. \((\tau_i, \tau_j)\)-rg-closed, \((\tau_i, \tau_j)\)-gpr-closed, \((\tau_i, \tau_j)\)-wg-closed, \((\tau_i, \tau_j)\)-\(\omega\)-closed and \((\tau_i, \tau_j)\)-g*-closed) subsets of a bitopological space \((X, \tau_1, \tau_2)\) is denoted by \(D(X, \tau_1, \tau_2)\) (resp. \(D_i(\tau_i, \tau_j), \zeta(\tau_i, \tau_j), W(\tau_i, \tau_j), C(\tau_i, \tau_j)\) and \(D^*(\tau_i, \tau_j)\))

**Definition 5.1.2:** A bitopological space \((X, \tau_1, \tau_2)\) is called

(i) \((\tau_i, \tau_j)\)-\(T_{1/2}\) -space [40] if every \((\tau_i, \tau_j)\)-g-closed set is \(\tau_j\)-closed.

(ii) \((\tau_i, \tau_j)\)-\(T_{*1/2}\) -space [107] if every \((\tau_i, \tau_j)\)-g*-closed set is \(\tau_j\)-closed.
(iii) strongly pairwise $T^{*1/2}$-space \[107\] if it is both $(\tau_1, \tau_2)$- $T^{*1/2}$-space and $(\tau_2, \tau_1)$- $T^{*1/2}$-space.

**Definition 5.1.3:** A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called

(i) $\tau_i$-$\sigma_k$-continuous \[70\] if $f^{-1}(V) \in \tau_j$ for every $V \in \sigma_k$.

(ii) $D(\tau_i, \tau_j)$-$\sigma_k$-continuous \[70\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-g-closed in $(X, \tau_1, \tau_2)$.

(iii) $(\tau_i, \tau_j)$-gp-$\sigma_k$-continuous \[118\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-gp-closed in $(X, \tau_1, \tau_2)$.

(iv) $(\tau_i, \tau_j)$-gs-$\sigma_k$-continuous \[118\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-gs-closed in $(X, \tau_1, \tau_2)$.

(v) $D_*(\tau_i, \tau_j)$-$\sigma_k$-continuous \[4\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-rg-closed in $(X, \tau_1, \tau_2)$.

(vi) $\zeta(\tau_i, \tau_j)$-$\sigma_k$-continuous \[47\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-gpr-closed in $(X, \tau_1, \tau_2)$.

(vii) $W(\tau_i, \tau_j)$-$\sigma_k$-continuous \[80\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-wg-closed in $(X, \tau_1, \tau_2)$.

(viii) $C(\tau_i, \tau_j)$-$\sigma_k$-continuous \[42\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-$\omega$-closed in $(X, \tau_1, \tau_2)$.

(ix) $D^*(\tau_i, \tau_j)$-$\sigma_k$-continuous \[107\] if the inverse image of every $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ is $(\tau_i, \tau_j)$-$g^*$-closed in $(X, \tau_1, \tau_2)$. 

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(x) \( \alpha(\tau_1, \tau_2)\)-\( \sigma_k\)-continuous [100] if the inverse image of every \( \sigma_k\)-closed set in \((Y, \sigma_1, \sigma_2)\) is \((\tau_1, \tau_2)\)-\( \alpha_{gs}\)-closed in \((X, \tau_1, \tau_2)\).

**Definition 5.1.4:** A function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called

(i) bi-continuous [70] if \( f \) is \( \tau_1\)-\( \sigma_1\)-continuous and \( \tau_2\)-\( \sigma_2\)-continuous.

(ii) generalized bi-continuous (g-bi-continuous) [70] if \( f \) is \( D(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( D(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

(iii) rg-bi-continuous [4] if \( f \) is \( D_r(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( D_r(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

(iv) gpr-bi-continuous [47] if \( f \) is \( \zeta(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( \zeta(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

(v) \( \omega\)-bi-continuous [42] if \( f \) is \( C(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( C(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

(vi) \( g^*\)-bi-continuous [107] if \( f \) is \( D^*(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( D^*(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

(vii) \( \alpha_{gs}\)-bi-continuous [100] if \( f \) is \( \alpha(\tau_1, \tau_2)\)-\( \sigma_2\)-continuous and \( \alpha(\tau_2, \tau_1)\)-\( \sigma_1\)-continuous.

**Definition 5.1.5:** A function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called

(i) strongly-bi-continuous(s-bi-continuous) [70] if \( f \) is bi-continuous, \( \tau_1\)-\( \sigma_2\)-continuous and \( \tau_2\)-\( \sigma_1\)-continuous.
(ii) generalized strongly-bi-continuous (g-s-bi-continuous) [70] if $f$ is g-bi-continuous, $D(\tau_1, \tau_2)$-$\sigma_1$-continuous and $D(\tau_2, \tau_1)$-$\sigma_2$-continuous.

(iii) rg-s-bi-continuous [4] if $f$ is rg-bi-continuous, $D_1(\tau_1, \tau_2)$-$\sigma_1$-continuous and $D_2(\tau_2, \tau_1)$-$\sigma_2$-continuous.

(iv) gpr-s-bi-continuous [47] if $f$ is gpr-bi-continuous, $\zeta(\tau_1, \tau_2)$-$\sigma_1$-continuous and $\zeta(\tau_2, \tau_1)$-$\sigma_2$-continuous.

(v) $\omega$-s-bi-continuous [42] if $f$ is $\omega$-bi-continuous, $C(\tau_1, \tau_2)$-$\sigma_1$-continuous and $C(\tau_2, \tau_1)$-$\sigma_2$-continuous.

(vi) $g^*$-s-bi-continuous [106] if $f$ is $D^*(\tau_1, \tau_2)$-$\sigma_1$-continuous and $D^*(\tau_2, \tau_1)$-$\sigma_2$-continuous.

(vii) $\alpha$gs-s-bi-continuous [100] if $f$ is $\alpha$gs-bi-continuous, $\alpha(\tau_1, \tau_2)$-$\sigma_1$-continuous and $\alpha(\tau_2, \tau_1)$-$\sigma_2$-continuous.
§ 5.2 \( \alpha g^s\)-Closed Sets in Bitopological Spaces

In this section we introduce \( \alpha g^s\)-closed sets and \( \alpha g^s\)-open sets in bitopological spaces and study some of their basic properties.

**Definition 5.2.1:** A subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-semi-closed (briefly \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-closed) set if \( \tau_2^{-\text{acl}}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \in \text{GSO}(X, \tau_1) \).

We denote the family of all \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-closed sets in a bitopological space \( (X, \tau_1, \tau_2) \) by \( \alpha^*(\tau_1, \tau_2) \).

**Remark 5.2.2:** If \( \tau_1 = \tau_2 \) in Definition 5.2.1, then \( (\tau_1, \tau_1)\)-\( \alpha g^s\)-closed set reduces to a \( \alpha g^s\)-closed set in a single topological space.

**Theorem 5.2.3:** (1) Every \( \tau_1 \)-closed set is \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-closed set.

(2) Every \( \tau_1 \)-\( \alpha\)-closed set is \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-closed set.

**Proof:** (1) Let \( A \) be a \( \tau_1 \)-closed set in \( (X, \tau_1, \tau_2) \) and let \( G \) be \( \tau_1 \)-gs-open set such that \( A \subseteq G \). Then \( \tau_1^{-\text{cl}}(A) = A \subseteq G \) as \( A \) is \( \tau_1 \)-closed. But \( \tau_1^{-\text{acl}}(A) \subseteq \tau_1^{-\text{cl}}(A) \) is always true. Therefore \( \tau_1^{-\text{acl}}(A) \subseteq G \) Hence \( A \) is \( (\tau_1, \tau_1)\)-\( \alpha g^s\)-closed in \( (X, \tau_1, \tau_2) \).

(2) Proof follows from the definitions.

The converse of the above theorem need not be true as seen from the following example.

**Example 5.2.4:** Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{a, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a, b\}\} \). Then the subset \( \{b, c\} \) is \( (\tau_1, \tau_2)\)-\( \alpha g^s\)-closed set but not \( \tau_2 \)-closed and \( \tau_2 \)-\( \alpha\)-closed set in \( (X, \tau_1, \tau_2) \).
Theorem 5.2.5: Every \((\tau_i, \tau_j)\)-\(\alpha g^*s\)-closed set is \((\tau_i, \tau_j)\) - gs-closed set but not conversely.

Proof: Let \(A\) be a \((\tau_i, \tau_j)\)-\(\alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\) and let \(G\) be \(\tau_i\) - semi-open set and so \(\tau_i\) -gs-open such that \(A \subseteq G\). Then \(\tau_j\text{-}\text{acl}(A) \subseteq G\) as \(A\) is \((\tau_i, \tau_j)\)-\(\alpha g^*s\)-closed. But \(\tau_j\text{-scl}(A) \subseteq \tau_j\text{-}\text{acl}(A)\) is always true. Therefore \(\tau_j\text{-scl}(A) \subseteq G\). Hence \(A\) is \((\tau_i, \tau_j)\) - gs-closed in \((X, \tau_1, \tau_2)\).

Example 5.2.6: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then the subset \(\{b\}\) is \((\tau_1, \tau_2)\)- gs-closed set but not a \((\tau_1, \tau_2)\)- \(\alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\).

Theorem 5.2.7: Every \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed set is \((\tau_i, \tau_j)\) - \(\alpha g^*s\)-closed but not conversely.

Proof: Let \(A\) be a \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\) and let \(G\) be \(\tau_i\) - semi-open set and so \(\tau_i\) -gs-open such that \(A \subseteq G\). Then \(\tau_j\text{-}\text{acl}(A) \subseteq G\) as \(A\) is \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed. Hence \(A\) is \((\tau_i, \tau_j)\) - \(\alpha g^*s\)-closed in \((X, \tau_1, \tau_2)\).

Example 5.2.8: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \phi, \{c\}, \{a, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then the subset \(\{b\}\) is \((\tau_1, \tau_2)\)- \(\alpha g^*s\)-closed set but not a \((\tau_1, \tau_2)\)- \(\alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\).

Theorem 5.2.9: Every \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed set is \((\tau_i, \tau_j)\) - gp-closed but not conversely.

Proof: Let \(A\) be a \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\) and let \(G\) be \(\tau_i\) -open set and so it is \(\tau_i\) -gs-open such that \(A \subseteq G\). Then \(\tau_j\text{-}\text{acl}(A) \subseteq G\) as \(A\) is \((\tau_i, \tau_j)\)- \(\alpha g^*s\)-closed. But \(\tau_j\text{-pcl}(A) \subseteq \tau_j\text{-}\text{acl}(A)\) is always true. Therefore \(\tau_j\text{-pcl}(A) \subseteq G\). Hence \(A\) is \((\tau_i, \tau_j)\) - gp-closed in \((X, \tau_1, \tau_2)\).
**Example 5.2.10:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset $\{a, b\}$ is $(\tau_1, \tau_2)$-gp-closed set but not a $(\tau_1, \tau_2)$-ag*s-closed set in $(X, \tau_1, \tau_2)$.

**Theorem 5.2.11:** Every $(\tau_i, \tau_j)$-ag*s-closed set is $(\tau_i, \tau_j)$ - gpr-closed but not conversely.

**Proof:** Let $A$ be a $(\tau_i, \tau_j)$- ag*s-closed set in $(X, \tau_1, \tau_2)$ and let $G$ be $\tau_i$-regular-open set and so $\tau_i$-gs-open such that $A \subseteq G$. Then $\tau_j$-acl$(A) \subseteq G$ as $A$ is $(\tau_i, \tau_j)$-ag*s-closed. But $\tau_j$-pcl$(A) \subseteq \tau_j$-acl$(A)$ is always true. Therefore $\tau_j$-pcl$(A) \subseteq G$. Hence $A$ is $(\tau_i, \tau_j)$ - gpr-closed in $(X, \tau_i, \tau_j)$.

**Example 5.2.12:** In Example 5.2.10, the subset $\{a, b\}$ is $(\tau_1, \tau_2)$- gpr-closed set but not a $(\tau_1, \tau_2)$- ag*s-closed set in $(X, \tau_1, \tau_2)$.

**Theorem 5.2.13:** If $A$ is both $\tau_i$ - gs-open and $(\tau_i, \tau_j)$ - ag*s-closed, then $A$ is $\tau_j$ - $\alpha$-closed set.

**Proof:** If $A \in$ GSO$(X, \tau_i)$, then by hypothesis, $\tau_j$-acl$(A) \subseteq A$. But $A \subseteq \tau_j$-acl$(A)$ is always true. Therefore $\tau_j$-acl$(A) = A$. Hence $A$ is $\tau_j$ - $\alpha$-closed set.

**Remark 5.2.14:** The concepts of the $(\tau_i, \tau_j)$ - g-closed sets and $(\tau_i, \tau_j)$ - ag*s-closed sets are independent of each other as seen from the following example.

**Example 5.2.15:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then the subset $\{b\} \in \alpha^*(\tau_1, \tau_2)$ but $\{b\} \notin D(\tau_1, \tau_2)$ and $\{a, c\} \in D(\tau_1, \tau_2)$ but $\{a, c\} \notin \alpha^*(\tau_1, \tau_2)$.

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Remark 5.2.16: The following example shows that \((\tau_i, \tau_j) - G^*-closed\) sets and \((\tau_i, \tau_j) - \alpha g^*s-closed\) sets are independent of each other.

Example 5.2.17: In the Example 5.2.15, the subset \(\{b\} \in \alpha^*(\tau_i, \tau_j)\) but \(\{b\} \notin D^*(\tau_i, \tau_j)\) and \(\{a, c\} \in D^*(\tau_i, \tau_j)\) but \(\{a, c\} \notin \alpha^*(\tau_i, \tau_j)\).

Theorem 5.2.18: If \(A, B \in \alpha^*(\tau_i, \tau_j)\), then \(A \cup B \in \alpha^*(\tau_i, \tau_j)\).

Proof: Let \(G\) be a \(\tau_i\)-gs-open set containing \(A \cup B\). Since \(A, B \in \alpha^*(\tau_i, \tau_j)\), we have \(\tau_j\-\text{acl}(A) \subseteq G\) and \(\tau_j\-\text{acl}(B) \subseteq G\). Therefore \(\tau_j\-\text{acl}(A \cup B) \subseteq G\). Hence \(A \cup B \in \alpha^*(\tau_i, \tau_j)\).

Remark 5.2.19: From the above results we have the following diagram.

\[
\begin{array}{ccc}
(i, j) \text{-gs-closed} & \longrightarrow & (i, j) \text{-g-closed} \\
\tau_j \text{-closed} & \longrightarrow & (i, j) \text{-} \alpha g^*s \text{-closed} \\
(i, j) \text{-gpr-closed} & \longrightarrow & (i, j) \text{-gp-closed}
\end{array}
\]

where \(A \rightarrow B\) represents \(A\) implies \(B\) but not conversely and \(A \leftrightarrow B\) represents \(A\) and \(B\) are independent of each other.

Remark 5.2.20: The intersection of two \((\tau_i, \tau_j) - \alpha g^*s-closed\) sets need not be \((\tau_i, \tau_j) - \alpha g^*s-closed\) set as seen from the following example.

Example 5.2.21: Let \(X = \{a, b, c\}, \tau_1 = \{X, \phi, \{c\}, \{a, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then the subsets \(\{a, b\}\) and \(\{a, c\}\) are \((\tau_i, \tau_j) - \alpha g^*s-\)
closed sets but their intersection $A \cap B = \{a\}$ is not a $(\tau_i, \tau_j) - \alpha g^s$-closed set in $(X, \tau_1, \tau_2)$.

**Remark 5.2.22:** $\alpha^*(\tau_1, \tau_2)$ is generally not equal to $\alpha^*(\tau_2, \tau_1)$ as seen from the following example.

**Example 5.2.23:** In Example 5.2.8, the subset $\{a, c\} \in \alpha^*(\tau_2, \tau_1)$ but $\{a, c\} \notin \alpha^*(\tau_1, \tau_2)$.

**Theorem 5.2.24:** If $\tau_1 \subseteq \tau_2$ in $(X, \tau_1, \tau_2)$, then $\alpha^*(\tau_2, \tau_1) \subseteq \alpha^*(\tau_1, \tau_2)$.

**Proof:** Let $A$ be a $(\tau_2, \tau_1) - \alpha g^s$-closed set and $U$ be a $\tau_1$-gs-open set containing $A$. Since $\tau_1 \subseteq \tau_2$, it follows that $\tau_2$-acl$(A) \subseteq \tau_1$-acl$(A)$ and $GSO(X, \tau_1) \subseteq GSO(X, \tau_2)$. Since $A \in \alpha^*(\tau_2, \tau_1)$, $\tau_1$-acl$(A) \subseteq U$, $\tau_2$-acl$(A) \subseteq U$, $U$ is $\tau_1$-gs-open. Thus $A$ is $(\tau_1, \tau_2) - \alpha g^s$-closed. Hence $\alpha^*(\tau_2, \tau_1) \subseteq \alpha^*(\tau_1, \tau_2)$.

The converse of the above theorem need not be true as seen from the following example.

**Example 5.2.25:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}\}$. Then $\alpha^*(\tau_2, \tau_1) \subseteq \alpha^*(\tau_1, \tau_2)$ but $\tau_1$ is not contained in $\tau_2$.

**Theorem 5.2.26:** Let $(X, \tau_1, \tau_2)$ be a bitopological space. For each point $x$ of $(X, \tau_1, \tau_2)$, a singleton set $\{x\}$ is $\tau_i$-gs-closed set or $\{x\}^c$ is $(\tau_i, \tau_j) - \alpha g^s$-closed set.

**Proof:** Suppose $\{x\}$ is not $\tau_i$-gs-closed. Then $\{x\}^c$ is not $\tau_i$-gs-open. Therefore $\tau_i$-gs-open set containing $\{x\}$ is $X$ only. Also $\tau_i$-acl$(\{x\}^c) \subseteq X$. Hence $\{x\}^c$ is $(\tau_i, \tau_j) - \alpha g^s$-closed set in $(X, \tau_1, \tau_2)$.
Theorem 5.2.27: If a set $A$ is $(\tau_n, \tau_j) - \alpha g^s$-closed set in $(X, \tau_1, \tau_2)$, then $\tau_j \text{acl}(A) - A$ contains no non-empty $\tau_i - gs$-closed set.

**Proof:** Let $A$ be a $(\tau_n, \tau_j) - \alpha g^s$-closed set and $F$ be $\tau_i - gs$-closed set contained in $\tau_j \text{acl}(A) - A$. Since $A \in \alpha^*(\tau_n, \tau_j)$, we have $\tau_j \text{acl}(A) \in F^c$. Consequently $F \subseteq \tau_j \text{acl}(A) \cap (X - (\tau_j \text{acl}(A))) = \phi$.

The converse of the above theorem need not be true as seen from the following example.

**Example 5.2.28:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. If $A = \{a\}$, then $\tau_2 \text{acl}(A) - A = \{c\}$ does not contain any non-empty $\tau_1 - gs$-closed set. But $A$ is not a $(\tau_n, \tau_2) - \alpha g^s$-closed set.

Theorem 5.2.29: If $A$ is $(\tau_n, \tau_j) - \alpha g^s$ -closed set and $\tau_j \text{acl}(A) - A$ is $\tau_i - gs$-closed set in $(X, \tau_1, \tau_2)$, then $A$ is $\tau_j - \alpha$-closed set.

**Proof:** Let $\tau_j \text{acl}(A) - A$ is $\tau_i - gs$-closed, By Theorem 5.2.25, $\tau_j \text{acl}(A) - A = \phi$, since $A$ is $(\tau_n, \tau_j) - \alpha g^s$ -closed. That is $\tau_j \text{acl}(A) = A$. Therefore $A$ is $\tau_j - \alpha$-closed.

Theorem 5.2.30: If $A$ is $(\tau_n, \tau_j) - \alpha g^s$-closed set, then $\tau_j \text{acl}(\{x\}) \cap A \neq \phi$, for each $x \in \tau_j \text{acl}(A)$.

**Proof:** If $\tau_j \text{acl}(\{x\}) \cap A = \phi$, for each $x \in \tau_j \text{acl}(A)$. Then $A \subseteq (\tau_j \text{acl}(\{x\}))^c$. Since $A$ is $(\tau_n, \tau_j) - \alpha g^s$-closed, we have $\tau_j \text{acl}(A) \subseteq (\tau_j \text{acl}(\{x\}))^c$. This shows that $x \notin \tau_j \text{acl}(A)$. This contradicts the assumption.

The converse of the above theorem need not be true as seen from the following example.
Example 5.2.31: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. For the subset $A = \{a, b\}$ is not $(\tau_1, \tau_2)$-$\alpha g^*s$-closed set but $\tau_1.acl(\{b\}) \cap A \neq \phi$, for each $x \in \tau_2.acl(A)$.

Theorem 5.2.32: If $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set and $A \subseteq B \subseteq \tau_j.acl(A)$, then $B$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set.

Proof: Let $B \subseteq G$, where $G$ is $\tau_1$-gs-open. Then $A \subseteq B$ implies $A \subseteq G$. As $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set, $\tau_j.acl(A) \subseteq G$. Now $B \subseteq \tau_j.acl(A)$ which implies $\tau_j.acl(B) \subseteq \tau_j.acl(\tau_j.acl(A)) = \tau_j.acl(A)$. Thus $\tau_j.acl(B) \subseteq G$. Hence $B$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set.

Theorem 5.2.33: Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed in $(X, \tau_1, \tau_2)$. Then $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed relative to $Y$.

Proof: Let $U$ be any $\tau_j$-gs-open set in $Y$ such that $A \subseteq U$. Then $U = G \cap Y$ for some $G \in \text{GSO}(X, \tau_i)$. Thus $A \subseteq G \cap Y$ and so $A \subseteq G$. Since $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set in $X$, $\tau_j.acl(A) \subseteq G$ and therefore $Y \cap \tau_j.acl(A) \subseteq Y \cap G$. That is $\tau_j.acl_Y(A) \subseteq U$, since $\tau_j.acl_Y(A) = Y \cap \tau_j.acl(A)$. Hence $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed relative to $Y$.

Theorem 5.2.34: In a bitopological space $(X, \tau_1, \tau_2)$, $\text{GSO}(X, \tau_i) \subseteq \{F \subseteq X: F^c \in \tau_j\}$ if and only if every subset of $X$ is a $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set.

Proof: Suppose that $\text{GSO}(X, \tau_i) \subseteq \{F \subseteq X: F^c \in \tau_j\}$. Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $U \in \text{GSO}(X, \tau_i)$ such that $A \subseteq U$. Then $\tau_j.acl(A) \subseteq \tau_j.acl(U) = U$. Hence $A$ is $(\tau_n, \tau_j)$-$\alpha g^*s$-closed set.
Conversely, suppose that every subset of \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_j) - \alpha g^*s\)-closed set. Let \(U \in \text{GSO}(X, \tau_i)\). Since \(U\) is \((\tau_1, \tau_j) - \alpha g^*s\)-closed, we have \(\tau_j, \alpha cl (U) \subseteq U\). Therefore \(U \in \{F \subseteq X : F^c \in \tau_1\}\) and we have \(\text{GSO}(X, \tau_i) \subseteq \{F \subseteq X : F^c \in \tau_1\}\).

We now introduce the following.

**Definition 5.2.35**: A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be \((\tau_1, \tau_j) - \alpha g^*s\)-open set if \(A^c\) is \((\tau_1, \tau_j) - \alpha g^*s\)-closed set in \((X, \tau_1, \tau_2)\).

**Theorem 5.2.36**: In a bitopological space \((X, \tau_1, \tau_2)\)

1. Every \(\tau_j\)-open set is \((\tau_1, \tau_j) - \alpha g^*s\)-open set.
2. Every \((\tau_1, \tau_j) - \alpha g^*s\)-open set is \((\tau_1, \tau_j) - \text{gs}\)-open set.
3. Every \((\tau_1, \tau_j) - \alpha g^*s\)-open set is \((\tau_1, \tau_j) - \alpha g^-s\)-open set.
4. Every \((\tau_1, \tau_j) - \alpha g^*s\)-open set is \((\tau_1, \tau_j) - \text{gpr}\)-open set.
5. Every \((\tau_1, \tau_j) - \alpha g^*s\)-open set is \((\tau_1, \tau_j) - \text{gpr}\)-open set.

**Proof**: The proof follows from the Theorems 5.2.3, 5.2.5, 5.2.7, 5.2.9 and 5.2.11 respectively.

**Theorem 5.2.37**: If \(A\) and \(B\) are \((\tau_1, \tau_j) - \alpha g^*s\)-open sets then \(A \cap B\) is also \((\tau_1, \tau_j) - \alpha g^*s\)-open set.

**Proof**: The proof follows from the Theorem 5.2.18.

**Theorem 5.2.38**: A subset \(A\) of \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_j) - \alpha g^*s\)-open set if and only if \(F \subseteq \tau_j - \alpha \text{int}(A)\) whenever \(F\) is \(\tau_1 - \text{gs}\)-closed set and \(F \subseteq A\).

**Proof**: Suppose that \(F\) is \(\tau_1 - \text{gs}\)-closed set, \(F \subseteq A\) and \(F \subseteq \tau_j - \alpha \text{int}(A)\). Let \(G\) be \(\tau_1 - \text{gs}\)-open and \(A^c \subseteq G\). Then \(G^c \subseteq A\) and \(G^c\) is \(\tau_1 - \text{gs}\)-closed. Thus \(G^c\)
\[ \subset \tau_2\text{-clint}(A) \text{ and } (\tau_2\text{-clint}(A))^c \subset G. \] It follows that \( \tau_2\text{-cl}(A^c) \subset G \) and hence \( A^c \) is \( (\tau_i, \tau_j) \text{-} \alpha\text{gs}\text{-}s\text{-closed}. \) Hence \( A \) is \( (\tau_i, \tau_j) \text{-} \alpha\text{gs}\text{-}s\text{-open}. \)

Conversely, suppose that \( A \) is \( (\tau_i, \tau_j)\alpha\text{gs}\text{-}s\text{-open}, \) \( F \subset A \) and \( F \) is \( \tau_j\text{-gs}\text{-closed}. \) Then \( F^c \) is \( \tau_j\text{-gs}\text{-open} \) and \( A^c \subset F^c. \) Therefore \( (\tau_j\text{-cl}(A^c)) \subset F^c \) and hence \( (\tau_j\text{-clint}(A))^c \subset F^c. \) Thus \( F \subset \tau_j\text{-clint}(A). \)

**Theorem 5.2.39:** If a subset \( A \) of \((X, \tau_1, \tau_2)\) is \( (\tau_i, \tau_j) \text{-} \alpha\text{gs}\text{-}s\text{-closed}, \) then \( \tau_j\text{-cl}(A) - A \) is \( (\tau_i, \tau_j) \text{-} \alpha\text{gs}\text{-}s\text{-open}. \)

**Proof:** Let \( A \) be a \( \tau_j\text{-gs}\text{-closed} \) set such that \( F \subset \tau_j\text{-cl}(A) - A. \) It follows that \( F = \phi. \) Therefore \( F \subset (\tau_j\text{-clint}(\tau_j\text{-clint}(A) - A). \) Thus \( \tau_j\text{-cl}(A) - A \) is \( (\tau_i, \tau_j) \text{-} \alpha\text{gs}\text{-}s\text{-open}. \)

### § 5.3 Applications of \((\tau_i, \tau_j)\text{-} \alpha\text{gs}\text{-}s\text{-Closed Sets}

In this section we introduce a new space such as \((\tau_i, \tau_j)\text{-} \alpha\text{gs}\text{-}s\text{-Closed \(\text{-} T^*{\frac{1}{2}}\text{-} \) spaces as an application and investigate some of their properties.

**Definition 5.3.1:** A bitopological space \((X, \tau_1, \tau_2)\) is said to be a \( (\tau_i, \tau_j)\text{-} \alpha\text{gs}\text{-}s\text{-Closed \(\text{-} T^*{\frac{1}{2}}\text{-} \) space if every \((\tau_i, \tau_j)\text{-} \alpha\text{gs}\text{-}s\text{-closed set is } \tau_j\text{-closed.} \)

**Example 5.3.2:** Let \( X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}. \) Then the space \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\text{-} \alpha\text{gs}\text{-}s\text{-Closed \(\text{-} T^*{\frac{1}{2}}\text{-} \) space.
Remark 5.3.3: If \( \tau_1 = \tau_2 \) in the Definition 1.4.36, we obtain the definition of \( \text{ags} T^*_{1/2} \)-space.

Theorem 5.3.4: If a bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-space, then \(\{x\}\) is \(\tau_1\)-open or \(\tau_1\)-gs-closed for each \(x \in X\).

Proof: Suppose that \(\{x\}\) is not \(\tau_1\)-gs-closed set of \((X, \tau_1, \tau_2)\). Then \(\{x\}^c\) is not \(\tau_1\)-gs-open set in \((X, \tau_1, \tau_2)\). Therefore \(X\) is the only \(\tau_1\)-gs-open set containing \(\{x\}^c\). Then \(\{x\}^c\) is \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-closed by Theorem 5.2.26. Since \((X, \tau_1, \tau_2)\) is a \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-space, \(\{x\}^c\) is \(\tau_1\)-closed. Therefore \(\{x\}\) is \(\tau_1\)-open.

The converse of the above theorem need not be true as shown from the following example

Example 5.3.5: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then every singleton set of \((X, \tau_1, \tau_2)\) is \(\tau_2\)-open set or \(\tau_1\)-gs-closed but \((X, \tau_1, \tau_2)\) is not a \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-space.

Remark 5.3.6: \((X, \tau_1, \tau_2)\) is not generally \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-space even if both \((X, \tau_1)\) and \((X, \tau_2)\) are \(\text{ags} T^*_{1/2} \)-spaces as seen from the following example.

Example 5.3.7: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\). Then the topological spaces \((X, \tau_1)\) and \((X, \tau_2)\) are \(\text{ags} T^*_{1/2} \)-spaces, but the bitopological space \((X, \tau_1, \tau_2)\) is not a \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-space.

Remark 5.3.8: The following examples shows that the concept of \((\tau_1, \tau_2) - T_{1/2} \)-spaces and \((\tau_1, \tau_2) - \text{ags} T^*_{1/2} \)-spaces are independent of each other.
**Example 5.3.9:** In the Example 5.3.7, the space \((X, \tau_1, \tau_2)\) is a \((\tau_1, \tau_2) - T_{1/2}\)-space but not a \((\tau_1, \tau_2) - \alpha_{gs}T^*_{1/2}\)-space.

**Example 5.3.10:** Let \(X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}\). Then space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - \alpha_{gs}T^*_{1/2}\)-space but not a \((\tau_1, \tau_2) - T_{1/2}\)-space.

**Remark 5.3.11:** The concepts of \((\tau_1, \tau_j) - T^*_{1/2}\)-spaces and \((\tau_1, \tau_j) - \alpha_{gs}T^*_{1/2}\)-spaces are independent of each other as seen from the following examples.

**Example 5.3.12:** Let \(X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}\}\). Then the space \((X, \tau_1, \tau_2)\) is a \((\tau_1, \tau_2) - T^*_{1/2}\)-space but not a \((\tau_1, \tau_2) - \alpha_{gs}T^*_{1/2}\)-space.

**Example 5.3.13:** Let \(X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}\}\). Then the space \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2) - \alpha_{gs}T^*_{1/2}\)-space but not a \((\tau_1, \tau_2) - T^*_{1/2}\)-space.

**Remark 5.3.14:** From the above results, we have the following diagram.

\[
\begin{array}{c}
(\tau_1, \tau_2) - T_{1/2}\text{-space} \quad \rightarrow \quad (\tau_1, \tau_2) - \alpha_{gs}T^*_{1/2}\text{-space} \quad \rightarrow \quad (\tau_1, \tau_2) - T^*_{1/2}\text{-space}
\end{array}
\]
§ 5.4 αg*s-Continuous Functions in Bitopological Spaces

In this section, we introduce a new class of continuous functions, called αg*s-continuous functions in bitopological spaces. During this process, some of their properties are obtained.

**Definition 5.4.1:** A function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called \( \alpha^*(\tau_1, \tau_2) \)-αg*s-continuous (αg*s-continuous) if the inverse image of every \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \) is a \( (\tau_1, \tau_2) \)-αg*s-closed set in \( (X, \tau_1, \tau_2) \).

**Remark 5.4.2:** Suppose that \( \tau_1 = \tau_2 = \tau \) and \( \sigma_1 = \sigma_2 = \sigma \) in Definition 5.4.1, then αg*s-continuous functions of bitopological spaces coincides with αg*s-continuity of topological spaces.

**Theorem 5.4.3:** If \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha^*(\tau_1, \tau_2) \)-\( \sigma_k \)-continuous, then \( f \) is \( \alpha^*(\tau_1, \tau_2) \)-\( \sigma_k \)-continuous function.

**Proof:** Let \( V \) be a \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( f^{-1}(V) \) is \( \tau_j \)-closed set. By Theorem 5.2.3, \( f^{-1}(V) \) is \( (\tau_1, \tau_2) \)-αg*s-closed set in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( \alpha^*(\tau_1, \tau_2) \)-\( \sigma_k \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 5.4.4:** Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}\} \), \( \tau_2 = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}\} \), \( \sigma_2 = \{Y, \phi, \{a, b\}\} \). Define a function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then \( f \) is \( \alpha^*(\tau_1, \tau_2) \)-\( \sigma_2 \)-continuous but not a \( \tau_1 \)-\( \sigma_2 \)-continuous, since for the \( \sigma_2 \)-closed set \( \{c\} \) in \( (Y, \sigma_1, \sigma_2) \), \( f^{-1}(\{c\}) = \{b\} \) is not a \( \tau_1 \)-closed set in \( (X, \tau_1, \tau_2) \) but it is \( (\tau_1, \tau_2) \)-αg*s-closed set in \( (X, \tau_1, \tau_2) \).
**Theorem 5.4.5:** A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha^*(\tau_1, \tau_j)\)-\( \sigma_k \)-continuous function if and only if the inverse image of every \( \sigma_k \)-open set in \( (Y, \sigma_1, \sigma_2) \) is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-open set in \( (X, \tau_1, \tau_2) \).

**Proof:** Let \( G \) be a \( \sigma_k \)-open set in \( (Y, \sigma_1, \sigma_2) \). Then \( G^c \) is \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Since \( f \) is \( \alpha^*(\tau_1, \tau_j)\)-\( \sigma_k \)-continuous, \( f^1(G^c) \) is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-closed set in \( (X, \tau_1, \tau_2) \). But \( f^1(G^c) = (f^1(G))^c \) which is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-closed set in \( (X, \tau_1, \tau_2) \). Thus \( f^1(G) \) is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-open set in \( (X, \tau_1, \tau_2) \).

Conversely, assume that the inverse image of every \( \sigma_k \)-open set in \( (Y, \sigma_1, \sigma_2) \) is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-open set in \( (X, \tau_1, \tau_2) \). Let \( F \) be a \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( F^c \) is \( \sigma_k \)-open set in \( (Y, \sigma_1, \sigma_2) \). By hypothesis, \( f^1(F^c) = (f^1(F))^c \) which is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-closed set in \( (X, \tau_1, \tau_2) \). So \( f^1(F) \) is \( (\tau_1, \tau_j) \)-\( \alpha \text{g}^s \)-closed set in \( (X, \tau_1, \tau_2) \). Thus \( f \) is \( \alpha^*(\tau_1, \tau_j)\)-\( \sigma_k \)-continuous function.

**Theorem 5.4.6:** If a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha^*(\tau_1, \tau_j)\)-\( \sigma_k \)-continuous, then \( f \) is \( (\tau_1, \tau_j)\)-\( \text{gs} \)-\( \sigma_k \)-continuous but not conversely.

**Proof:** Let \( V \) be a \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( f^1(V) \) is \( (\tau_1, \tau_j)\)-\( \alpha \text{g}^s \)-closed set in \( (X, \tau_1, \tau_2) \) as \( f \) is \( \alpha^*(\tau_1, \tau_j)\)-\( \sigma_k \)-continuous. By Theorem 5.2.5, \( f^1(G) \) is \( (\tau_1, \tau_j)\)-\( \text{gs} \)-\( \sigma_k \)-closed set in \( (X, \tau_1, \tau_2) \). Hence \( f \) is \( (\tau_1, \tau_j)\)-\( \text{gs} \)-\( \sigma_k \)-continuous function.

**Example 5.4.7:** Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\} \), \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma_1 = \{Y, \emptyset, \{a\}, \{b\}\} \), \( \sigma_2 = \{Y, \emptyset, \{a, c\}\} \). Then the identity function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( (\tau_1, \tau_2)\)-\( \text{gs} \)-\( \sigma_2 \)-continuous but not \( \alpha^*(\tau_1, \tau_2)\)-\( \sigma_2 \)-continuous, since for the \( \sigma_2 \)-closed set \( \{b\} \) in \( (Y, \sigma_1, \sigma_2) \), \( f^1(\{b\}) = \{b\} \) is not \( (\tau_1, \tau_2)\)-\( \alpha \text{g}^s \)-closed set but it is \( (\tau_1, \tau_2)\)-\( \text{gs} \)-closed set in \( (X, \tau_1, \tau_2) \).
Theorem 5.4.8: If a function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha^*(\tau_1, \tau_j)-\sigma_k \)-continuous, then \( f \) is \( \alpha(\tau_i, \tau_j)-\sigma_k \)-continuous but not conversely.

Proof: Let \( K \) be a \( \sigma_k \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( f^{-1}(K) \) is \( (\tau_i, \tau_j) \)-\( \alpha_g^s \)-closed set in \( (X, \tau_1, \tau_2) \) as \( f \) is \( \alpha^*(\tau_i, \tau_j)-\sigma_k \)-continuous. By Theorem 5.2.7, \( f^{-1}(K) \) is \( (\tau_i, \tau_j)\)-\( \alpha_g^s \)-closed set in \( (X, \tau_1, \tau_2) \). Hence \( f \) is \( \alpha(\tau_i, \tau_j)-\sigma_k \)-continuous function.

Example 5.4.9: Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{a, c\}\} \), \( \tau_2 = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}\} \), \( \sigma_2 = \{Y, \phi, \{a\}, \{b, c\}\} \). Then the identity function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha(\tau_i, \tau_2)-\sigma_2 \)-continuous but not \( \alpha^*(\tau_1, \tau_2)-\sigma_2 \)-continuous, since for the \( \sigma_2 \)-closed set \( \{a\} \) in \( (Y, \sigma_1, \sigma_2) \), \( f^{-1}(\{a\}) = \{a\} \) is not a \( (\tau_1, \tau_2)\)-\( \alpha g^s \)-closed set in \( (X, \tau_1, \tau_2) \) but it is \( (\tau_1, \tau_2)\)-\( \alpha g^s \)-closed set in \( (X, \tau_1, \tau_2) \).

Remark 5.4.10: The following examples show that \( \alpha^*(\tau_i, \tau_j)-\sigma_k \)-continuous functions and \( D(\tau_i, \tau_j)-\sigma_k \)-continuous functions are independent of each other.

Example 5.4.11: Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{a, b\}\} \), \( \tau_2 = \{X, \phi, \{a\}\} \) and \( \sigma_1 = \{Y, \phi, \{a, b\}\} \), \( \sigma_2 = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\} \). Then the identity function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \alpha^*(\tau_1, \tau_2)-\sigma_2 \)-continuous but not a \( D(\tau_1, \tau_2)-\sigma_2 \)-continuous, since for the \( \sigma_2 \)-closed set \( \{b\} \) in \( (Y, \sigma_1, \sigma_2) \), \( f^{-1}(\{b\}) = \{b\} \) is not a \( (\tau_1, \tau_2)\)-\( g \)-closed set in \( (X, \tau_1, \tau_2) \) but it is \( (\tau_1, \tau_2)\)-\( \alpha g^s \)-closed set in \( (X, \tau_1, \tau_2) \).

Example 5.4.12: In the Example 5.4.9, the function \( f \) is \( D(\tau_1, \tau_2)-\sigma_2 \)-continuous but not a \( \alpha^*(\tau_1, \tau_2)-\sigma_2 \)-continuous, since for the \( \sigma_2 \)-closed set
Remark 5.4.13: The concept of $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous functions and $D^*(\tau_i, \tau_j)$-$\sigma_k$-continuous functions are independent of each other as seen from the following examples.

Example 5.4.14: In the Example 5.4.11, the function $f$ is $\alpha^*(\tau_1, \tau_2)$-$\sigma_2$-continuous but not a $D^*(\tau_1, \tau_2)$-$\sigma_2$-continuous function.

Example 5.4.15: In the Example 5.4.9, the function $f$ is $D^*(\tau_i, \tau_j)$-$\sigma_2$-continuous but not a $\alpha^*(\tau_i, \tau_j)$-$\sigma_2$-continuous function.

Theorem 5.4.16: If a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous, then $f$ is $\zeta(\tau_i, \tau_j)$-$\sigma_k$-continuous but not conversely.

Proof: Let $V$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$ Then $f^{-1}(V)$ is $(\tau_i, \tau_j)$-$\alpha g^* s$-closed set in $(X, \tau_1, \tau_2)$ as $f$ is $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous. By Theorem 5.2.11, $f^{-1}(V)$ is $(\tau_i, \tau_j)$-gpr-closed set in $(X, \tau_1, \tau_2)$ Hence $f$ is $\zeta(\tau_i, \tau_j)$-$\sigma_k$-continuous function.

Example 5.4.17: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{c\}, \{a, c\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\zeta(\tau_1, \tau_2)$-$\sigma_2$-continuous but not a $\alpha^*(\tau_1, \tau_2)$-$\sigma_2$-continuous function.

Theorem 5.4.18: If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\alpha^*(\tau_i, \tau_j)$-$\sigma_k$-continuous and $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-$\alpha g^* T^*_{1/2}$-space, then $f$ is $\tau_j$-$\sigma_k$-continuous.
**Proof:** Let $f$ be a $\alpha^*(\tau, \tau_j)$-$\sigma_k$-continuous function and $F$ be a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Then $f^1(F)$ is $(\tau, \tau_j)$-$\alpha g^*$-closed set in $(X, \tau_1, \tau_2)$. Since $(X, \tau_1, \tau_2)$ is $(\tau, \tau_j)$-$\alpha g T^*$-$1/2$-space, $f^1(F)$ is $\tau_j$-closed set in $(X, \tau_1, \tau_2)$. Hence $f$ is $\tau_j$-$\sigma_k$-continuous function.

**Remark 5.4.19:** From the above results, we have the following diagram.

- $(\tau, \tau_j)$-gs-$\sigma_k$-cont
- $\tau_j$-$\sigma_k$-cont
- $D(\tau, \tau_j)$-$\sigma_k$-cont

where $A \rightarrow B$ represents $A$ implies $B$ but not conversely and $A \leftrightarrow B$ represents $A$ and $B$ are independent of each other.
§ 5.5 Some Stronger forms of \( \alpha_g^s \)-Continuous Functions in Bitopological Spaces

In this section, we introduce \( \alpha_g^s \)-bi-continuous functions, \( \alpha_g^s \)-strongly bi continuous functions in bitopological spaces. We also study their relations with some existing functions in bitopological spaces. Further we introduce and study the pairwise \( \alpha_g^s \)-irresolute functions in bitopological spaces.

**Definition 5.5.1:** A function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called

i) \( \alpha_g^s \)-bi-continuous if \( f \) is \( \alpha^s(\tau_1, \tau_2)-\sigma_2 \)-continuous and \( \alpha^s(\tau_2, \tau_1)-\sigma_1 \)-continuous.

ii) \( \alpha_g^s \)-strongly-bi-continuous (briefly \( \alpha_g^s \)-s-bi-continuous) if \( f \) is \( \alpha_g^s \)-bi-continuous, \( \alpha^s(\tau_2, \tau_1)-\sigma_2 \)-continuous and \( \alpha^s(\tau_1, \tau_2)-\sigma_1 \)-continuous.

**Theorem 5.5.2:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function.

i) If \( f \) is bi-continuous, then \( f \) is \( \alpha_g^s \)-bi-continuous.

ii) If \( f \) is s-bi-continuous, then \( f \) is \( \alpha_g^s \)-s-bi-continuous.

**Proof:** i) Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a bi-continuous. Then \( f \) is \( \tau_1-\sigma_1 \)-continuous and \( \tau_2-\sigma_2 \)-continuous. By Theorem 5.4.3, \( f \) is \( \alpha^s(\tau_1, \tau_2)-\sigma_2 \)-continuous and \( \alpha^s(\tau_2, \tau_1)-\sigma_1 \)-continuous. Thus \( f \) is \( \alpha_g^s \)-bi-continuous.

(ii) Similar to (i), using Theorem 5.4.3.

The converse of the above theorem need not be true as seen from the following examples.
Example 5.5.3: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{b, c\}\}$ and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}\}$, $\sigma_2 = P(Y)$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = p$ and $f(b) = f(c) = q$. Then $f$ is $\alpha g^* s$-bi-continuous but not bi-continuous.

Example 5.5.4: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}\}$, $\sigma_2 = \{Y, \phi, \{a, b\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then the function $f$ is $\alpha g^* s$-s-bi-continuous but not $s$-bi-continuous.

Theorem 5.5.5: Every $\alpha g^* s$-s-bi-continuous function is $\alpha g^* s$-bi-continuous function but not conversely.

Proof: The proof follows from the definitions.

Example 5.5.6: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma_1 = \{Y, \phi, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{a\}, \{b, c\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then the function $f$ is $\alpha g^* s$-bi-continuous but not $\alpha g^* s$-s-bi-continuous function.

Theorem 5.5.7: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function.

i) If $f$ is $\alpha g^* s$-bi-continuous, then $f$ is $\alpha g s$-bi-continuous.

ii) If $f$ is $\alpha g^* s$-s-bi-continuous, then $f$ is $\alpha g s$-s-bi-continuous.

Proof: i) Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $\alpha g^* s$-bi-continuous. Then $f$ is $\alpha^*(\tau_1, \tau_2)$-$\sigma_2$-continuous and $\alpha^*(\tau_2, \tau_1)$-$\sigma_1$-continuous. By Theorem 5.4.8, $f$ is $\alpha(\tau_1, \tau_2)$-$\sigma_2$-continuous and $\alpha(\tau_2, \tau_1)$-$\sigma_1$-continuous. Thus $f$ is $\alpha g s$-bi-continuous.

(ii) Similar to (i), using Theorem 5.4.8

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The converse of the above theorem need not be true as seen from the following example.

**Example 5.5.8:** Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{c\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, \{a, b\}\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b, c\}\}$, $\sigma_2 = \{Y, \phi, \{c\}, \{b, c\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\alpha g s$-bi-continuous but not $\alpha g s$-s-bi-continuous function. Also this function $f$ is $\alpha g s$-s-bi-continuous but not $\alpha g s$-s-bi-continuous function.

**Remark 5.5.9:** The concepts of $g$-bi-continuous (resp. $g$-s-bi-continuous) functions and $\alpha g s$-bi-continuous (resp. $\alpha g s$-s-bi-continuous) functions are independent of each other as seen from the following examples.

**Example 5.5.10:** In the Example 5.5.6, the function $f$ is $g$-bi-continuous (resp. $g$-s-bi-continuous) function but not $\alpha g s$-bi-continuous (resp. $\alpha g s$-s-bi-continuous) function.

**Example 5.5.11:** Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}\}$ and $\sigma_1 = \{Y, \phi, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{b, c\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then the function $f$ is $\alpha g s$-s-bi-continuous but not $g$-bi-continuous function.

**Example 5.5.12:** Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{a, c\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Then the identity function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\alpha g s$-s-bi-continuous but not $g$-s-bi-continuous function.
Remark 5.5.13: The concept of $g^*$-bi-continuous (resp. $g^*$-s-bi-continuous) functions and $\alpha g^*$-s-bi-continuous (resp. $\alpha g^*$-s-s-bi-continuous) functions are independent of each other as seen from the following examples.

Example 5.5.14: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{c\}, \{a, c\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \emptyset, \{a\}, \{a, b\}\}$, $\sigma_2 = \{Y, \emptyset, \{c\}, \{b, c\}\}$. Define a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then the function $f$ is $g^*$-bi-continuous (resp. $g^*$-s-bi-continuous) but not $\alpha g^*$-s-bi-continuous function (resp. $\alpha g^*$-s-s-bi-continuous) function.

Example 5.5.15: In the Example 5.5.11, the function $f$ is $\alpha g^*$-s-bi-continuous but not $g^*$-bi-continuous function.

Example 5.5.16: In the Example 5.5.12, the function $f$ is $\alpha g^*$-s-s-bi-continuous but not $g^*$-s-bi-continuous function.

Theorem 5.5.17: Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function.

i) If $f$ is $\alpha g^*$-s-bi-continuous, then $f$ is gpr-bi-continuous.

ii) If $f$ is $\alpha g^*$-s-s-bi-continuous, then $f$ is gpr-s-bi-continuous.

Proof: i) Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $\alpha g^*$-s-bi-continuous. Then $f$ is $\alpha^*(\tau_1, \tau_2)$-continuous and $\alpha^*(\tau_2, \tau_1)$-continuous. By Theorem 5.4.18, $f$ is $\zeta(\tau_1, \tau_2)$-continuous and $\zeta(\tau_2, \tau_1)$-continuous. Thus $f$ is gpr-bi-continuous.

(ii) Similar to (i), using Theorem 5.4.18.

The converse of the above theorem need not be true as seen from the following example.
Example 5.5.18: Let \( X = Y = \{a, b, c\}, \tau_1 = \{X, \phi, \{c\}, \{a, b\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \) and \( \sigma_1 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}, \sigma_2 = \{Y, \phi, \{a, b\}\} \). Define a function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a) = a, f(b) = c \) and \( f(c) = b \). Then the function \( f \) is gpr-bi-continuous (resp. gpr-s-bi-continuous) but not \( \alpha g^s \) bi-continuous (resp. \( \alpha g^s-s \)-bi-continuous) function.

Definition 5.5.19: A function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called pairwise \( \alpha g^s \)-irresolute function if \( f^{-1}(A) \in \alpha^s(\tau_i, \tau_j) \) in \( (X, \tau_i, \tau_j) \) for every \( A \in \alpha^s(\sigma_i, \sigma_j) \) in \( (Y, \sigma_i, \sigma_j) \).

Remark 5.5.20: If \( \tau_1 = \tau_2 = \tau \) and \( \sigma_1 = \sigma_2 = \sigma \) simultaneously, then \( f \) becomes a \( \alpha g^s \)-irresolute function in topological spaces.

Theorem 5.5.21: If \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise \( \alpha g^s \)-irresolute then \( f \) is \( \alpha^s(\tau_i, \tau_j) - \sigma_\alpha \)-continuous.

Proof: Let \( F \) be any \( \sigma_\alpha \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( F \) is \( (\sigma_1, \sigma_2) - \alpha g^s \)-closed in \( (Y, \sigma_1, \sigma_2) \). Since \( f \) is pairwise \( \alpha g^s \)-irresolute, \( f^{-1}(F) \in \alpha^s(\tau_i, \tau_j) \). Therefore \( f \) is \( \alpha^s(\tau_i, \tau_j) - \sigma_\alpha \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 5.5.22: Let \( X = Y = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{a, b\}\} \) and \( \sigma_1 = \{Y, \phi, \{c\}, \{a, c\}\}, \sigma_2 = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\} \). Then the identity function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( \alpha^s(\tau_i, \tau_j) - \sigma_\alpha \)-continuous but not a pairwise \( \alpha g^s \)-irresolute.
**Theorem 5.5.23:** A function \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise \( \alpha g^s \)-irresolute if and only if for every \((\sigma_k, \sigma_e)\)-\( \alpha g^s \)-open set \( A \) in \((Y, \sigma_1, \sigma_2)\), the inverse image \( f^{-1}(A) \) is \((\tau_n, \tau_j)\)-\( \alpha g^s \)-open in \((X, \tau_1, \tau_2)\).

**Remark 5.5.24:** From the above results we have the following diagram

\[ \text{g}^* \text{p bi-continuity} \quad \text{g bi-continuity} \quad \text{g}^* \text{bi-continuity} \]

\[ \text{bi-continuity} \quad \alpha g^s \text{s-bi-continuity} \quad \alpha g^s \text{ -bi-continuity} \]

\[ \text{gpr- bi-continuity} \]

\[ \text{s-bi-continuity} \quad \alpha g^s \text{s-s-bi-continuity} \quad \alpha g^s \text{s-bi-continuity} \]

\[ \text{g}^* \text{s-bi-continuity} \quad \text{g-s-bi-continuity} \quad \text{g}^* \text{p-s-bi-continuity} \]

\[ \text{gpr-s-bi-continuity} \]

**Theorem 5.5.25:** If \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) are pairwise \( \alpha g^s \)-irresolute functions, then their composition \( g \circ f \) is also pairwise \( \alpha g^s \)-irresolute.

**Proof:** Let \( A \in (\eta_m, \eta_n) \) in \((Z, \eta_1, \eta_2)\). Since \( g \) is pairwise \( \alpha g^s \)-irresolute, \( g^{-1}(A) \in \alpha^*(\sigma_k, \sigma_e) \) in \((Y, \sigma_1, \sigma_2)\). Again since \( f \) is pairwise \( \alpha g^s \)-irresolute,
\[ f^{-1}(g^{-1}(A)) \in \alpha^*(\tau_1, \tau_2) \text{ in } (X, \tau_1, \tau_2). \text{ That is } f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \in \alpha^*(\tau_1, \tau_2) \text{ in } (X, \tau_1, \tau_2). \text{ Hence gof is pairwise } \alpha g^s \text{-irresolute function.} \]

**Theorem 5.5.26:** If \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be any two functions. Then

i) \( \text{gof: } (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is \( \alpha^*(\tau_1, \eta_2) \)-\( \eta_n \)-continuous if \( g \) is \( \sigma_c-\eta_n \)-continuous and \( f \) is \( \alpha^*(\tau_k, \eta_k) \)-\( \sigma_k \)-continuous.

ii) \( \text{gof: } (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is \( \alpha^*(\tau_1, \tau_2) \)-\( \eta_n \)-continuous if \( f \) is pairwise \( \alpha g^s \)-irresolute and \( g \) is \( \alpha^*(\sigma_k, \sigma_c) \)-\( \eta_n \)-continuous.

**Proof:** i) Let \( A \) be any \( \eta_n \)-closed set in \( (Z, \eta_1, \eta_2) \). Since \( g \) is \( \sigma_c-\eta_n \)-continuous, we have \( g^{-1}(A) \) is \( \sigma_c \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Again since \( f \) is \( \alpha^*(\tau_k, \eta_k) \)-\( \sigma_k \)-continuous, \( f^{-1}(g^{-1}(A)) \) is \( (\tau_k, \eta_k) \)-\( \alpha^* g^s \)-closed set in \( (X, \tau_1, \tau_2) \).

But \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \). Hence \( \text{gof is } \alpha^*(\tau_1, \eta_2) \)-\( \eta_n \)-continuous.

(ii) Let \( A \) be any \( \eta_n \)-open set in \( (Z, \eta_1, \eta_2) \). Since \( g \) is \( \alpha^*(\sigma_k, \sigma_c) \)-\( \eta_n \)-continuous, \( g^{-1}(A) \) is \( (\sigma_k, \sigma_c) \)-\( \alpha^* g^s \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Again since \( f \) is pairwise \( \alpha^* g^s \)-irresolute, \( f^{-1}(g^{-1}(A)) \) is \( (\tau_k, \eta_1) \)-\( \alpha^* g^s \)-open in \( (X, \tau_1, \tau_2) \). But \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \). Hence \( \text{gof is } \alpha^*(\tau_1, \eta_2) \)-\( \eta_n \)-continuous.