CHAPTER-VI
FUZZY TOPOLOGICAL SPACES

6.1 Introduction and preliminaries.

The concept of a fuzzy subset was introduced and studied by L.A.Zadeh [98] in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of science and engineering. C.L.Chang [19] introduced and studied fuzzy topological spaces in 1968 as a generalization of topological spaces. Many researchers like R.H.Warren ([95], [96]), K.K.Azad ([9],[10]), G.Balasubramanian and P.Sundaram [14], S.R.Malghan and S.S.Benchalli ([58], [59]), M.N.Mukherjee and B.Ghosh [63], Anjan.Mukherjee [62], A.N.Zahren [99], J.A.Goguen [42] and many others have contributed to the development of fuzzy topological spaces.

In section 1 of this chapter, the concept of fuzzy subset is illustrated, various operations on fuzzy sets such as union, intersection and complementation of fuzzy sets are included and a list of related properties is included. The concept of image and the inverse image of a fuzzy set under a function are included and the properties proved by C.L.Chang [19] and R.H.Warren [95] are given. Further the basic concepts and results on fuzzy topological spaces from the work of C.L.Chang [19], R.H.Warran [95], K.K.Azad [10] and A.N.Zahern [99] are presented, which are required in the subsequent chapters.

In section 2 of this chapter, fuzzy rw-closed sets and fuzzy rw-open sets have been introduced and studied. Among many other results it is observed that every fuzzy closed set is fuzzy rw-closed but not conversely.
In section 3 of this chapter, fuzzy rw-continuous maps and fuzzy rw-irresolute maps have been introduced and studied. Among many other results it is observed that every fuzzy continuous map is fuzzy rw-continuous but not conversely. We prove that the composition of fuzzy rw-irresolute map is fuzzy rw-irresolute.

In section 4 of this chapter, we introduce fuzzy rw-open maps and fuzzy rw-closed maps in fuzzy topological spaces and obtain certain characterizations of these maps.

In section 5 of this chapter, we introduce and study two new fuzzy homeomorphisms, namely fuzzy rw-homeomorphism and fuzzy rwc-homeomorphism. It is observed that every fuzzy homeomorphism is fuzzy rw-homeomorphism but not conversely. We prove that the composition of two fuzzy rwc-homeomorphism is a fuzzy rwc-homeomorphism.

Let $X$ be set and $\mu_A: X \rightarrow [0, 1]$ be a function from $X$ into the closed interval $[0, 1]$, which may take any value between 0 and 1, for an element of $X$. Such a function is called a membership function or membership characteristic function. A fuzzy subset $A$ in $X$ is characterized by a membership function $\mu_A: X \rightarrow [0, 1]$ which associates with each point $x$ in $X$, a real number $\mu_A(x)$ between 0 and 1 which represents the degree or grade of membership or belongingness of $x$ to $A$. If $A$ is an ordinary subset of $X$, then $\mu_A$ can take either 1 or 0 according as $x$ does or does not belong to $A$. Then, in this case, $\mu_A$ reduces to the usual characteristic function $\chi_A$ of $A$.

6.1.1 Definition: [19] A fuzzy subset $A$ in set $X$ is defined to be a function $A: X \rightarrow [0, 1]$. 
A fuzzy subset \( A \) in set \( X \) is empty iff its membership function is identically zero on \( X \) and is denoted by 0 or \( \mu_\emptyset \). The set \( X \) can be considered as a fuzzy subset of \( X \) whose membership function is 1 on \( X \) and is denoted by 1 or \( 1_X \) or \( \mu_X \).

In fact, every subset of \( X \) is fuzzy subset of \( X \) but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

6.1.2 Definition: [98] If \( A \) and \( B \) are any two fuzzy subsets of a set \( X \), then \( \text{"}A\text{" is said to be included in \( B \) \text{"} or \( \text{"}A\text{" is contained in \( B \) \text{"} or \( \text{"}A\text{" is less then or equal to \( B \) \text{"} iff } A(x) \leq B(x) \text{ for all } x \in X \text{ and is denoted by } A \leq B. \text{ Equivalently, } A \leq B \text{ iff } \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X. \) \)

Note that every fuzzy subset is included itself and empty fuzzy subset is included in every fuzzy subset.

6.1.3 Definition: [98] Two fuzzy subsets \( A \) and \( B \) of a set \( X \) are said to be equal, written \( A=B \), if \( A(x)=B(x) \) for every \( x \) in \( X \).

6.1.4 Definition: [98] The complement of a fuzzy subset \( A \) in a set \( X \), denoted by \( 1-A \), is the fuzzy subset of \( X \) defined by \( 1-A(x) \) for all \( x \) in \( X \). Note that \( 1-(1-A)=A \).

6.1.5 Definition: [98] The union of two fuzzy subsets \( A \) and \( B \) in a set \( X \), denoted by \( A \cup B \), is a fuzzy subset in \( X \) defined by

\[
(A \cup B)(x) = \text{Max} \{ A(x), B(x) \}, \text{ for all } x \text{ in } X.
\]

In general, the union of a family of fuzzy subsets \( \{A_\lambda: \lambda \in \Lambda\} \) is a fuzzy subset denoted by \( \bigvee_{\lambda \in \Lambda} A_\lambda \) and defined by \( (\bigvee_{\lambda \in \Lambda} A_\lambda)(x) = \text{Sup} \{ A_\lambda(x): \lambda \in \Lambda \} \), for all \( x \) in \( X \).
6.1.6 Definition: [98] The intersection of two fuzzy subsets $A$ and $B$ in a set $X$, denoted by $A \cap B$, is fuzzy subset in $X$ defined by

$$(A \cap B)(x) = \min\{A(x), B(x)\}, \text{ for all } x \in X.$$ 

In general, the intersection of a family of fuzzy subsets $\{A_\lambda: \lambda \in \Lambda\}$ is a fuzzy subset denoted by $\bigwedge_{\lambda \in \Lambda} A_\lambda$ and defined by

$$\left( \bigwedge_{\lambda \in \Lambda} A_\lambda \right)(x) = \inf\{A_\lambda(x): \lambda \in \Lambda\}, \text{ for all } x \in X.$$ 

6.1.7 Theorem: ([44], [47] and [98]) Let $X$ be any set and $A$, $B$, $C$ be fuzzy subsets of $X$. The following results hold good.

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. $A \cap X = A$
6. $A \cup X = X$
7. $1 - (A \cup B) = (1 - A) \cap (1 - B)$
8. $1 - (A \cap B) = (1 - A) \cup (1 - B)$
9. $A - B = A \cap (1 - B)$
10. $A \cap 0 = 0$, where $0$ is the empty fuzzy set
11. $A \cup 0 = A$, where $0$ is the empty fuzzy set.

6.1.8 Definition: [98] Let $f: X \rightarrow Y$ be a function from a set $X$ into a set $Y$. Let $A$ be a fuzzy set in $X$ and $B$ be a fuzzy set in $Y$.

1. The inverse image of $B$ under $f$, written $f^{-1}(B)$ is a fuzzy set in $X$, defined by $(f^{-1}(B))(x) = B(f(x)) = (f^{-1}(B))(x)$, for each $x \in X$.
2. The image of $A$ under $f$, written $f(A)$ is a fuzzy set in $Y$, defined by $(f(A))(y) = \sup\{A(z): z \in f^{-1}(y)\}$, for each $y \in Y$ where $f^{-1}(y) = \{x \in X: f(x) = y\}$.

6.1.9 Theorem: [19] Let $f$ be a function from a set $X$ into a set $Y$. The following results hold good.
(1) \( f^{-1}(1-B) = 1 - f^{-1}(B) \), for any fuzzy set \( B \) in \( Y \).

(2) \( f(1-A) \geq 1 - f(A) \), for any fuzzy set \( A \) in \( X \).

(3) \( A \leq B \) implies \( f(A) \leq f(B) \), for any two fuzzy sets \( A, B \) in \( X \).

(4) \( C \leq D \) implies \( f^{-1}(C) \leq f^{-1}(D) \), for any two fuzzy sets \( C, D \) in \( Y \).

(5) \( A \leq f^{-1}(f(A)) \), for any fuzzy set \( A \) in \( X \).

(6) \( B \geq f(f^{-1}(B)) \), for any fuzzy set \( B \) in \( Y \).

(7) Let \( g \) be a function from \( Y \) to \( Z \). Then \( (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \), for any fuzzy set \( C \) in \( Z \).

6.1.10 Theorem: [95] Let \( f \) be a function from a set \( X \) into a set \( Y \). If \( A, A_i, i \in I \) are fuzzy sets in \( X \) and \( B, B_j, j \in J \) are fuzzy sets in \( Y \), where \( I \) and \( J \) denotes the indexed sets, then the following results are true.

(1) \( f(f^{-1}(B)) = B \), if \( f \) is onto.

(2) \( f(\bigwedge_{i \in I} A_i) \leq \bigwedge_{i \in I} f(A_i) \)

(3) \( f^{-1}(\bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} f^{-1}(B_j) \)

(4) \( f(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} f(A_i) \)

(5) \( f^{-1}(\bigvee_{j \in J} B_j) = \bigvee_{j \in J} f^{-1}(B_j) \)

(6) \( f(f^{-1}(B) \land A) = B \land f(A) \)

C.L. Chang [19] in the year 1968, introduced the notion of fuzzy topological spaces as an application of fuzzy sets to general topological spaces. Since then several researchers have contributed to the development of fuzzy topological spaces. Some of basic concepts on fuzzy topological spaces, which may be used in the sequel, are included.

6.1.11 Definition: [19] Let \( X \) be a set and \( T \) be a family of fuzzy subsets of \( X \). The family \( T \) is called a fuzzy topology on \( X \) iff \( T \) satisfies the following axioms
(i) $0, 1 \in T$ (ii) If $\{A_\lambda : \lambda \in \Lambda\} \subseteq T$ then $\bigvee_{\lambda \in \Lambda} A_\lambda \in T$ and

(iii) If $G, H \in T$ then $G \wedge H \in T$.

The pair $(X, T)$ is called a fuzzy topological space (abbreviated as fts). The members of $T$ are called fuzzy open sets in $X$. A fuzzy set $A$ in $X$ is said to be fuzzy closed set in $X$ iff $1-A$ is a fuzzy open set in $X$.

6.1.12 Remark: [19] Every topological space is a fuzzy topological space but not conversely.

6.1.13 Definition: [55] Let $X$ be a fts and $A$ be a fuzzy set in $X$. Then $\{B: B \text{ is a fuzzy closed set in } X \text{ and } B \geq A\}$ is called closure of $A$ and is denoted by $\overline{A}$ or $\text{cl}(A)$.

6.1.14 Theorem: [55] Let $A$ and $B$ be two fuzzy sets in fts $(X, T)$. Then the following results are true.

(1) $\text{cl}(A)$ is a fuzzy closed set in $X$

(2) $\text{cl}(A)$ is the least fuzzy closed set containing $A$

(3) $A$ is a fuzzy closed iff $A=\text{cl}(A)$

(4) $\text{cl}(0)=0$, $0$ is the empty fuzzy set.

(5) $\text{cl}(\text{cl}(A))=\text{cl}(A)$

(6) $\text{cl}(A) \vee \text{cl}(B)=\text{cl}(A \vee B)$

(7) $\text{cl}(A) \wedge \text{cl}(B) \geq \text{cl}(A \wedge B)$

6.1.15 Definition: [19] Let $A$ and $B$ be two fuzzy sets in fuzzy topological space $(X, T)$ and let $A \geq B$. Then $B$ is called an interior fuzzy set of $A$ if there exists $H \in T$ such that $A \geq H \geq B$. The least upper bound of all interior fuzzy sets of $A$ is called the interior of $A$ and is denoted by $A^\circ$ or $\text{int}(A)$.
6.1.16 Theorem: ([19], [45]) Let X be fts, A and B be two fuzzy sets in X. The following results hold good.

(1) $A^\circ$ is a fuzzy open set in X

(2) $A^\circ$ is the largest fuzzy open set in X which is less than or equal to A.

(3) A is a fuzzy open set iff $A = A^\circ$

(4) $A \leq B$ implies $A^\circ \leq B^\circ$

(5) $(A^\circ)^\circ = A$

(6) $A^\circ \wedge B^\circ = (A \wedge B)^\circ$

(7) $A^\circ \vee B^\circ \leq (A \vee B)^\circ$

(8) $(1-A)^\circ = 1-\text{cl}(A)$

(9) $\text{cl}(1-A) = 1-A^\circ$

6.1.17 Definition: [10] A fuzzy set A in a fts X is said to be fuzzy semiopen if and only if there exists a fuzzy open set V in X such that $V \leq A \leq \text{cl}(V)$.

6.1.18 Definition: [10] A fuzzy set A in a fts X is said to be fuzzy semi-closed if and only if there exists a fuzzy closed set V in X such that $\text{int}(V) \leq A \leq V$.

It is seen that a fuzzy set A is fuzzy semiopen if and only if $1-A$ is a fuzzy semi-closed.

6.1.19 Theorem: [10] The following are equivalent:

(a) $\lambda$ is a fuzzy semiclosed set,  
    (c) $\text{int}(\text{cl}(\lambda)) \leq \lambda$

(b) $\lambda^C$ is a fuzzy semiopen set,  
    (d) $\text{cl}(\text{int}(\lambda^C)) \geq \lambda^C$
6.1.20 **Theorem:** [10] (a) Any union of fuzzy semiopen sets is a fuzzy semiopen set and (b) any intersection of fuzzy semiclosed sets is a fuzzy semiclosed.

6.1.21 **Remark:** [10] (i) Every fuzzy open set is a fuzzy semiopen but not conversely.

(ii) Every fuzzy closed set is a fuzzy semi-closed set but not conversely.

(iii) The closure of a fuzzy open set is fuzzy semiopen set

(iv) The interior of a fuzzy closed set is fuzzy semi-closed set

6.1.22 **Definition:** [10] A fuzzy set $\lambda$ of a fts $X$ is called a fuzzy regular open set of $X$ if $\text{int}(\text{cl} (\lambda)) = \lambda$.

6.1.23 **Definition:** [10] A fuzzy set $\lambda$ of fts $X$ is called a fuzzy regular closed set of $X$ if $\text{cl}(\text{int} (\lambda)) = X$.

6.1.24 **Theorem:** [10] A fuzzy set $\lambda$ of a fts $X$ is a fuzzy regular open if and only if $\lambda^C$ fuzzy regular closed set.

6.1.25 **Remark:** [10] (i) Every fuzzy regular open set is a fuzzy open set but not conversely.

(ii) Every fuzzy regular closed set is a fuzzy closed set but not conversely.

6.1.26 **Theorem:** [10] (i) The closure of a fuzzy open set is a fuzzy regular closed.

(ii) The interior of a fuzzy closed set is a fuzzy regular open set

6.1.27 **Definition:** [99] A fuzzy set $\alpha$ of a fuzzy topological space $X$ is said to be a fuzzy regular semiopen set in fts $X$ if there exists a fuzzy regular open set $\sigma$ in $X$ such that $\sigma \leq \alpha \leq \text{cl}(\sigma)$. We denote the class of fuzzy regular semiopen sets in fts $X$ by $\text{FRSO}(X)$. 

- 169 -
6.1.28 **Theorem:** [99] (i) Every fuzzy regular semiopen set is a fuzzy semiopen set but not conversely.

(ii) Every fuzzy regular closed set is a fuzzy regular semiopen set but not conversely.

(iii) Every fuzzy regular open set is a fuzzy regular semiopen set but not conversely.

6.1.29 **Theorem:** [99] A fuzzy set $\alpha$ of fts $X$ is fuzzy regular semiopen if and only if $\alpha$ is both fuzzy semiopen and fuzzy semi-closed.

6.1.30 **Theorem:** [99] If $\alpha$ is fuzzy regular semiopen in fts $X$, then $1-\alpha$ is fuzzy regular semiopen in $X$.

6.1.31 **Definition:** [14] A fuzzy set $\alpha$ in fts $X$ is called fuzzy generalized closed (gf-closed) if $\text{cl}(\alpha) < p$ whenever $\alpha < p$ and $p$ fuzzy open and $\alpha$ is fuzzy generalized open if $1-\alpha$ is fuzzy generalized closed.

6.1.32 **Definition:** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is called

(i) fuzzy continuous [19] if $f^{-1}(\alpha)$ is fuzzy open in $X$ for each fuzzy open set $\alpha$ in $Y$.

(ii) fuzzy generalized continuous (gf-continuous) [14] if $f^{-1}(\alpha)$ is fuzzy generalized closed in $X$ for each fuzzy closed set $\alpha$ in $Y$.

(iii) fuzzy semi continuous [10] if $f^{-1}(\alpha)$ is fuzzy semiopen in $X$ for each fuzzy open set $\alpha$ in $Y$.

(iv) fuzzy almost continuous [10] if $f^{-1}(\alpha)$ is fuzzy open in $X$ for each fuzzy regular open set $\alpha$ in $Y$.

(v) fuzzy irresolute [64] if $f^{-1}(\alpha)$ is fuzzy semiopen in $X$ for each fuzzy semiopen set $\alpha$ in $Y$.  

(vi) fuzzy ge-irresolute [14] if $f^{-1}(\alpha)$ is fuzzy generalized closed in $X$ for each fuzzy generalized closed set $\alpha$ in $Y$.

**6.1.33 Definition:** [62] A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy completely semi continuous if and only if $f^{-1}(\alpha)$ is a fuzzy regular semiopen subset of $X$ for every fuzzy open subset of $\alpha$ in $Y$.

**6.1.34 Remark:** [62] A mapping $f:X \rightarrow Y$ is fuzzy completely semi continuous if and only if $f^{-1}(\beta)$ is a fuzzy regular semiopen subset of $X$ for every fuzzy closed subset of $\beta$ in $Y$.

**6.1.35 Definition:** A mapping $f:X \rightarrow Y$ from a fts $X$ to a fts $Y$ is called

(i) fuzzy open mapping [19] if $f(\lambda)$ is fuzzy open in $Y$ for every fuzzy open set in $\lambda$ in $X$.

(ii) fuzzy semiopen mapping [10] if $f(\lambda)$ is fuzzy semiopen in $Y$ for every fuzzy open set in $\lambda$ in $X$.

**6.1.36 Definition:** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is called

(i) fuzzy closed mapping [19] if $f(\mu)$ is fuzzy closed in $Y$ for every fuzzy closed set in $\mu$ in $X$.

(ii) fuzzy semi-closed mapping [10] if $f(\mu)$ is fuzzy semi-closed in $Y$ for every fuzzy closed set in $\mu$ in $X$.

**6.1.37 Definition:** [19] A bijective mapping $f: X \rightarrow Y$ from a fuzzy topological space $X$ to another fuzzy topological space $Y$ is called fuzzy homeomorphism if $f$ and $f^{-1}$ are fuzzy continuous.
6.2 Fuzzy rω-closed sets and fuzzy rω-open sets in fts.

In this section, we introduce new class of fuzzy sets called fuzzy rω-closed sets in fuzzy topological spaces and investigate certain basic properties of these fuzzy sets. Also we introduce fuzzy rω-open sets in fuzzy topological spaces and study some of their properties.

6.2.1 Definition: Let \((X, T)\) be a fuzzy topological space. A fuzzy set \(\alpha\) of \(X\) is called fuzzy regular \(r\omega\)-closed (briefly, fuzzy rω-closed) if \(\text{cl}(\alpha) \leq \sigma\) whenever \(\alpha \leq \sigma\) and \(\sigma\) is fuzzy regular semiopen in fts \(X\).

We denote the class of all fuzzy regular \(r\omega\)-closed sets in fts \(X\) by \(\text{FRWC}(X)\).

6.2.2 Theorem: Every fuzzy closed set is a fuzzy rω-closed set in a fts \(X\).

Proof: Let \(\alpha\) be a fuzzy closed set in a fts \(X\). Let \(\beta\) be a fuzzy regular semiopen set in \(X\) such that \(\alpha \leq \beta\). Since \(\alpha\) is fuzzy closed, \(\text{cl}(\alpha) = \alpha\). Therefore \(\text{cl}(\alpha) \leq \beta\). Hence \(\alpha\) is fuzzy rω-closed in fts \(X\).

The converse of the above Theorem need not be true in general as seen from the following example.

6.2.3 Example: Let \(X = \{a, b, c\}\). Define a fuzzy set \(\alpha\) in \(X\) by

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases}
\]

Let \(T = \{1, 0, \alpha\}\). Then \((X, T)\) is a fuzzy topological space. Define a fuzzy set \(\beta\) in \(X\) by

\[
\beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases}
\]

Then \(\beta\) is a fuzzy rω-closed set but it is not a fuzzy closed set in fts \(X\).
6.2.4 Corollary: By Remark 6.1.25. (ii), it has been proved that every fuzzy regular closed set is a fuzzy closed set but not conversely. By Theorem 6.2.2 every fuzzy closed set is a fuzzy rw-closed set but not conversely and hence every fuzzy regular closed set is a fuzzy rw-closed set but not conversely.

6.2.5 Remark: Fuzzy generalized closed sets and fuzzy rw-closed sets are independent.

6.2.6 Example: Let \( X = \{a, b, c, d\} \) and the functions \( \alpha, \beta, \gamma : X \rightarrow [0, 1] \) be defined as

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}
\]

Consider \( T = \{1, 0, \alpha, \beta, \gamma\} \). Then \( (X, T) \) is a fuzzy topological space. In this fts \( X \), the fuzzy set \( \lambda : X \rightarrow [0, 1] \) define by

\[
\lambda(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \lambda \) is a fuzzy generalized closed set in fts \( X \). In this fts, the fuzzy set \( \delta : X \rightarrow [0, 1] \) define by

\[
\delta(x) = \begin{cases} 
1 & \text{if } x = a, c, \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \delta \) is a fuzzy regular semiopen set containing \( \lambda \), but \( \delta \) does not contain \( cl(\lambda) \) which is \( \gamma^c \). Therefore \( \delta \) is not a fuzzy rw-closed set in fts \( X \).
6.2.7 Example: Let \( X = I = [0, 1] \). Define a fuzzy set \( \lambda \) in \( X \) by

\[
\lambda(x) = \begin{cases} 
\frac{1}{2} & \text{if } x = \frac{2}{3} \\
0 & \text{otherwise}
\end{cases}
\]

Let \( T = \{1, 0, \lambda\} \). Then \((X, T)\) is a fuzzy topological space.

Let \( \alpha(x) = \begin{cases} 
\frac{1}{3} & \text{if } x = \frac{2}{3} \\
0 & \text{otherwise}
\end{cases} \)

Then \( \alpha \) is a fuzzy rw-closed set in fts \( X \). Now \( \text{cl}(\alpha) = \lambda^c \) and \( \lambda \) is a fuzzy open set containing \( \alpha \) but \( \lambda \) does not contain \( \text{cl}(\alpha) \) which is \( \lambda^c \). Therefore \( \alpha \) is not a fuzzy generalized closed.

6.2.8 Remark: Fuzzy rw-closed sets and fuzzy semi-closed sets are independent.

6.2.9 Example: Consider the fuzzy topological space \((X, T)\) defined in Example 6.2.3. Then the fuzzy set \( \alpha = \{(a, 1), (b, 0), (c, 0)\} \) is a fuzzy rw-closed but it is not a fuzzy semi-closed set in fts \( X \).

6.2.10 Example: Consider the fuzzy topological space \((X, T)\) defined in Example 6.2.6. In this fts \( X \), the fuzzy set \( \mu : X \rightarrow [0, 1] \) is define by

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = a, c \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \mu \) is a fuzzy semi-closed in fts \( X \). \( \mu \) is also fuzzy regular semiopen set containing \( \mu \) which is does not contain \( \text{cl}(\mu) = \beta^c = \{(a, 1), (b, 0), (c, 1), (d, 1)\} \). Therefore \( \mu \) is not a fuzzy rw-closed set in fts \( X \).

6.2.11 Remark: From the above discussions and known results we have the following implications

In the following diagram, by \( \rightarrow \) we mean \( A \) implies \( B \) but not conversely and
6.2.12 Theorem: If α and β are fuzzy rw-closed sets in fts $X$, then $\alpha \lor \beta$ is fuzzy rw-closed set in fts $X$.

Proof: Let $\sigma$ be a fuzzy regular semiopen set in fts $X$ such that $\alpha \lor \beta \leq \sigma$. Now $\alpha \leq \sigma$ and $\beta \leq \sigma$. Since $\alpha$ and $\beta$ are fuzzy rw-closed sets in fts $X$, $\text{cl}(\alpha) \leq \sigma$ and $\text{cl}(\beta) \leq \sigma$. Therefore $\text{cl}(\alpha) \lor \text{cl}(\beta) \leq \sigma$. But $\text{cl}(\alpha) \lor \text{cl}(\beta) = \text{cl}(\alpha \lor \beta)$. Thus $\text{cl}(\alpha \lor \beta) \leq \sigma$. Hence $\alpha \lor \beta$ is a fuzzy rw-closed set in fts $X$.

6.2.13 Remark: If $\alpha$ and $\beta$ are fuzzy rw-closed sets in fts $X$, then $\alpha \land \beta$ need not be a fuzzy rw-closed set in general as seen from the following example.

6.2.14 Example: Consider the fuzzy topological space $(X, T)$ defined in Example 6.2.6. In this fts $X$, the fuzzy sets $\delta_1, \delta_2 : X \rightarrow [0, 1]$ are defined by

$$
\delta_1(x) = \begin{cases} 
1 & \text{if } x = c, d \\
0 & \text{otherwise}
\end{cases} \quad \delta_2(x) = \begin{cases} 
1 & \text{if } x = a, b, c \\
0 & \text{otherwise}
\end{cases}
$$

Then $\delta_1$ and $\delta_2$ are the fuzzy rw-closed sets in fts in $X$.

Let $\lambda = \delta_1 \land \delta_2$. Then $\lambda(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases}$

Then $\lambda = \delta_1 \land \delta_2$ is not a fuzzy rw-closed set in fts $X$. 

- 175 -
6.2.15 Theorem: If a fuzzy set $\alpha$ of fts $X$ is both fuzzy regular open and fuzzy rw-closed, then $\alpha$ is a fuzzy regular closed set in fts $X$.

Proof: Suppose a fuzzy set $\alpha$ of fts $X$ is both fuzzy regular open and fuzzy rw-closed. As every fuzzy regular open set is a fuzzy regular semiopen set and $\alpha \leq \alpha$, we have $\text{cl}(\alpha) \leq \alpha$. Also $\alpha \leq \text{cl}(\alpha)$. Therefore $\text{cl}(\alpha) = \alpha$. That is $\alpha$ is fuzzy closed. Since $\alpha$ is fuzzy regular open, $\text{int}(\alpha) = \alpha$. Now $\text{cl}(\text{int}(\alpha)) = \text{cl}(\alpha) = \alpha$. Therefore $\alpha$ is a fuzzy regular closed set in fts $X$.

6.2.16 Theorem: If a fuzzy set $\alpha$ of a fts $X$ is both fuzzy regular semiopen and fuzzy rw-closed, then $\alpha$ is a fuzzy closed set in fts $X$.

Proof: Suppose a fuzzy set $\alpha$ of a fts $X$ is both fuzzy regular semiopen and fuzzy rw-closed. Now $\alpha \leq \alpha$, we have $\text{cl}(\alpha) \leq \alpha$. Also $\alpha \leq \text{cl}(\alpha)$. Therefore $\text{cl}(\alpha) = \alpha$ and hence $\alpha$ is a fuzzy closed set in fts $X$.

6.2.17 Theorem: If a fuzzy set $\alpha$ of a fts $X$ is both fuzzy open and fuzzy generalized closed, then $\alpha$ is a fuzzy rw-closed set in fts $X$.

Proof: Suppose a fuzzy set $\alpha$ of a fts $X$ is both fuzzy open and fuzzy generalized closed. Now $\alpha \leq \alpha$, by hypothesis we have $\text{cl}(\alpha) \leq \alpha$. Also $\alpha \leq \text{cl}(\alpha)$. Therefore $\text{cl}(\alpha) = \alpha$. That is $\alpha$ is a fuzzy closed set and hence $\alpha$ is a fuzzy rw-closed set in fts $X$, as every fuzzy closed set is a fuzzy rw-closed set.

6.2.18 Remark: If a fuzzy set $\gamma$ is both fuzzy regular open and fuzzy rw-closed set in fts $X$, then $\gamma$ need not be a fuzzy generalized closed set in general as seen from the following example.

6.2.19 Example: Let $X = \{a, b, c\}$ and the functions $\alpha, \beta, \gamma: X \rightarrow [0, 1]$ be defined as
\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases}
\]
\[
\gamma(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}
\]

Consider \( T=\{1, 0, \alpha, \beta, \gamma\} \). Then \( (X, T) \) is a fuzzy topological space. In this fts \( X \), \( \gamma \) is both fuzzy open and fuzzy rw-closed set in fts \( X \) but it is not fuzzy generalized closed.

6.2.20 Theorem: Let \( \alpha \) be a fuzzy rw-closed set of a fts \( X \) and suppose \( \alpha \leq \beta \leq \text{cl}(\alpha) \). Then \( \beta \) is also a fuzzy rw-closed set in fts \( X \).

Proof: Let \( \alpha \leq \beta \leq \text{cl}(\alpha) \) and \( \alpha \) be a fuzzy rw-closed set of fts \( X \). Let \( \sigma \) be any fuzzy regular semiopen set such that \( \beta \leq \sigma \). Then \( \alpha \leq \sigma \) and \( \alpha \) is fuzzy rw-closed, we have \( \text{cl}(\alpha) \leq \sigma \). But \( \text{cl}(\beta) \leq \text{cl}(\alpha) \) and thus \( \text{cl}(\beta) \leq \sigma \). Hence \( \beta \) is a fuzzy rw-closed set in fts \( X \).

6.2.21 Theorem: In a fuzzy topological space \( X \) if \( \text{FRSO}(X)=\{1, 0\} \), where \( \text{FRSO}(X) \) is the family of all fuzzy regular semiopen sets then every fuzzy subset of \( X \) is fuzzy rw-closed.

Proof: Let \( X \) be a fuzzy topological space and \( \text{FRSO}(X)=\{1, 0\} \). Let \( \alpha \) be any fuzzy subset of \( X \). Suppose \( \alpha = 0 \). Then 0 is a fuzzy rw-closed set in fts \( X \). Suppose \( \alpha \neq 0 \). Then 1 is the only fuzzy regular semiopen set containing \( \alpha \) and so \( \text{cl}(\alpha) \leq 1 \). Hence \( \alpha \) is a fuzzy rw-closed set in fts \( X \).

6.2.22 Remark: The converse of the above Theorem 6.2.21 need not be true in general as seen from the following example.

6.2.23 Example: Let \( X=\{a, b, c\} \) and the functions \( \alpha, \beta : X \to [0, 1] \) be defined as...
\[
\alpha(x) = \begin{cases} 
1 \text{ if } x = a \\ 
0 \text{ otherwise} 
\end{cases}, \quad \beta(x) = \begin{cases} 
1 \text{ if } x = b, c \\ 
0 \text{ otherwise} 
\end{cases}
\]

Consider T={ 1, 0, \alpha, \beta}. Then (X, T) is a fuzzy topological space. In this fts X, every fuzzy subset of X is a fuzzy rw-closed set in fts X, but FRSO= {1, 0, \alpha, \beta}.

**6.2.24 Theorem:** If \( \alpha \) is a fuzzy rw-closed set of fts X and \( \text{cl}(\alpha) \land (1-\text{cl}(\alpha))=0 \), then \( \text{cl}(\alpha)-\alpha \) does not contain any non-zero fuzzy regular semiopen set in fts X.

**Proof:** Suppose \( \alpha \) is a fuzzy rw-closed set of fts X and \( \text{cl}(\alpha) \land (1-\text{cl}(\alpha))=0 \). We prove the result by contradiction. Let \( \beta \) be a fuzzy regular semiopen set such that \( \text{cl}(\alpha)-\alpha \geq \beta \) and \( \beta \neq 0 \). Now \( \beta \leq \text{cl}(\alpha) - \alpha \), i.e. \( \beta \leq 1 - \alpha \) which implies \( \alpha \leq 1 - \beta \). Since \( \beta \) is a fuzzy regular semiopen set, by Theorem 6.1.30, \( 1-\beta \) is also fuzzy regular semiopen set in fts X. Since \( \alpha \) is a fuzzy rw-closed set in fts X, by definition \( \text{cl}(\alpha) \leq 1 - \beta \). So \( \beta \leq 1 - \text{cl}(\alpha) \). Therefore \( \beta \leq (\text{cl}(\alpha) \land (1-\text{cl}(\alpha)))=0 \), by hypothesis. This shows that \( \beta=0 \) which is a contradiction. Hence \( \text{cl}(\alpha)-\alpha \) does not contain any non-zero fuzzy regular semiopen set in fts X.

**6.2.25 Corollary:** If \( \alpha \) is a fuzzy rw-closed set of fts X and \( \text{cl}(\alpha) \land (1-\text{cl}(\alpha))=0 \), then \( \text{cl}(\alpha)-\alpha \) does not contain any non-zero fuzzy regular open set in fts X.

**Proof:** Follows from the Theorem 6.2.24 and the fact that every fuzzy regular open set is a fuzzy regular semiopen set in fts X.

**6.2.26 Corollary:** If \( \alpha \) is a fuzzy rw-closed set of a fts X and \( \text{cl}(\alpha) \land (1-\text{cl}(\alpha))=0 \), then \( \text{cl}(\alpha)-\alpha \) does not contain any non-zero fuzzy regular closed set in fts X.
Proof: Follows form the Theorem 6.2.24 and the fact that every fuzzy regular closed set is a fuzzy regular semiopen set in fts X.

6.2.27 Theorem: Let $\alpha$ be a fuzzy rw-closed set of fts X and $\text{cl}(\alpha) \wedge (1 - \text{cl}(\alpha)) = 0$ Then $\alpha$ is a fuzzy closed set if and only if $\text{cl}(\alpha) - \alpha$ is a fuzzy regular semiopen set in fts X.

Proof: Suppose $\alpha$ is a fuzzy closed set in fts X. Then $\text{cl}(\alpha) = \alpha$ and so $\text{cl}(\alpha) - \alpha = 0$, which is a fuzzy regular semiopen set in fts X.

Conversely suppose $\text{cl}(\alpha) - \alpha$ is a fuzzy regular semiopen set in fts X. Since $\alpha$ is fuzzy rw-closed, by Theorem 6.2.24 $\text{cl}(\alpha) - \alpha$ does not contain any non-zero fuzzy regular open set in fts X. Then $\text{cl}(\alpha) - \alpha = 0$. That is $\text{cl}(\alpha) = \alpha$ and hence $\alpha$ is a fuzzy closed set in fts X.

We introduce a fuzzy rw-open set in fuzzy topological space X as follows.

6.2.28 Definition: A fuzzy set $\alpha$ of a fuzzy topological space X is called a fuzzy regular w-open (briefly, fuzzy rw-open) set if its complement $\alpha^C$ is a fuzzy rw-closed set in fts X.

We denote the family of all fuzzy rw-open sets in fts X by $\text{FRWO}(X)$.

6.2.29 Theorem: If a fuzzy set $\alpha$ of a fuzzy topological space X is fuzzy open, then it is fuzzy rw-open but not conversely.

Proof: Let $\alpha$ be a fuzzy open set of fts X. Then $\alpha^C$ is fuzzy closed. Now by Theorem 6.2.2, $\alpha^C$ is fuzzy rw-closed. Therefore $\alpha$ is a fuzzy rw-open set in fts X.

The converse of the above Theorem need not be true in general as seen from the following example.
6.2.30 **Example:** Let $X = \{a, b, c\}$. Define a fuzzy set $\alpha$ in $X$ by

$$\alpha(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}$$

Let $T = \{1, 0, \alpha\}$. Then $(X, T)$ is a fuzzy topological space. Define a fuzzy set $\beta$ in $X$ by

$$\beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases}$$

Then $\beta$ is a fuzzy rw-open set but it is not fuzzy open set in fts $X$.

6.2.31 **Corollary:** By Remark 6.1.25. (i), it has been proved that every fuzzy regular open set is a fuzzy open set but not conversely. By Theorem 6.2.29, every fuzzy open set is a fuzzy rw-open set but not conversely and hence every fuzzy regular open set is a fuzzy rw-open set but not conversely.

6.2.32 **Theorem:** A fuzzy set $\alpha$ of a fuzzy topological space $X$ is fuzzy rw-open if and only if $\delta \leq \text{int}(\alpha)$ whenever $\delta \leq \alpha$ and $\delta$ is a fuzzy regular semiopen set in fts $X$.

**Proof:** Suppose that $\delta \leq \text{int}(\alpha)$ whenever $\delta \leq \alpha$ and $\delta$ is a fuzzy regular semiopen set in fts $X$. To prove that $\alpha$ is fuzzy rw-open in fts $X$. Let $\alpha^C \leq \beta$ and $\beta$ is any fuzzy regular semiopen set in fts $X$. Then $\beta^C \leq \alpha$. By Theorem 6.1.30, $\beta^C$ is also fuzzy regular semiopen set in fts $X$. By hypothesis, $\beta^C \leq \text{int}(\alpha)$ which implies $(\text{int}(\alpha))^C \leq \beta$. That is $\text{cl}(\alpha^C) \leq \beta$, since $\text{cl}(\alpha^C) = (\text{int}(\alpha))^C$. Thus $\alpha^C$ is a fuzzy rw-closed and hence $\alpha$ is fuzzy rw-open in fts $X$.

Conversely, suppose that $\alpha$ is fuzzy rw-open. Let $\beta \leq \alpha$ and $\beta$ is any fuzzy regular semiopen in fts $X$. Then $\alpha^C \leq \beta^C$. By Theorem 6.1.30,
\( \beta^c \) is also fuzzy regular semiopen. Since \( \alpha^c \) is fuzzy rw-closed, we have \( \text{cl}(\alpha^c) \leq \beta^c \) and so \( \beta \leq \text{int}(\alpha) \), since \( \text{cl}(\alpha^c) = (\text{int}(\alpha))^c \).

6.2.33 Theorem: If \( \alpha \) and \( \beta \) are fuzzy rw-open sets in a fts \( X \), then \( \alpha \land \beta \) is also a fuzzy rw-open set in fts \( X \).

Proof: Let \( \alpha \) and \( \beta \) be two fuzzy rw-open sets in a fts \( X \). Then \( \alpha^c \) and \( \beta^c \) are fuzzy rw-closed sets in fts \( X \). By Theorem 6.2.12, \( \alpha^c \lor \beta^c \) is also a fuzzy rw-closed set in fts \( X \). That is \( (\alpha^c \lor \beta^c) = (\alpha \land \beta)^c \) is a fuzzy rw-closed set in \( X \). Therefore \( \alpha \land \beta \) is also a fuzzy rw-open set in fts \( X \).

6.2.34 Remark: The union of two fuzzy rw-open sets in a fts \( X \) is generally not a fuzzy rw-open set in fts \( X \).

6.2.35 Example: Consider the fuzzy topological space \( (X, T) \) defined as in Example 6.2.19. In this fts \( X \), the fuzzy sets \( \delta_1, \delta_2 : X \to [0, 1] \) are defined by

\[
\delta_1(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases}
\]

\[
\delta_2(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \delta_1 \) and \( \delta_2 \) are the rw-open fuzzy sets in fts \( X \). Let \( \lambda = \delta_1 \lor \delta_2 \) Then

\[
\lambda(x) = \begin{cases} 
1 & \text{if } x = a, c \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \lambda = \delta_1 \lor \delta_2 \) is not a fuzzy rw-open set in fts \( X \).

6.2.36 Theorem: If \( \text{int}(\alpha) \leq \beta \leq \alpha \) and \( \alpha \) is a fuzzy rw-open set in a fts \( X \), then \( \beta \) is also a fuzzy rw-open set in fts \( X \).

Proof: Suppose \( \text{int}(\alpha) \leq \beta \leq \alpha \) and \( \alpha \) is a fuzzy rw-open set in a fts \( X \). To prove that \( \beta \) is a fuzzy rw-open set in fts \( X \). Let \( \sigma \) be any fuzzy regular semiopen set in fts \( X \) such that \( \sigma \leq \beta \). Now \( \sigma \leq \beta \leq \alpha \). That is \( \sigma \leq \alpha \). Since \( \alpha \) is fuzzy rw-open set of fts \( X \), \( \sigma \leq \text{int}(\alpha) \) by Theorem 6.2.32. By
hypothesis \( \text{int}(\alpha) \leq \beta \). Then \( \text{int}(\text{int}(\alpha)) \leq \text{int}(\beta) \). That is \( \text{int}(\alpha) \leq \text{int}(\beta) \). Then \( \sigma \leq \text{int}(\beta) \). Again by Theorem 6.2.32 \( \beta \) is a fuzzy rw-open set in fts \( X \).

6.2.37 **Theorem:** If a fuzzy subset \( \alpha \) of a fts \( X \) is fuzzy rw-closed and \( \text{cl}(\alpha) \wedge (1 - \text{cl}(\alpha)) = 0 \), then \( \text{cl}(\alpha) - \alpha \) is a fuzzy rw-open set in fts \( X \).

**Proof:** Let \( \alpha \) be a fuzzy rw-closed set in a fts \( X \) and \( \text{cl}(\alpha) \wedge (1 - \text{cl}(\alpha)) = 0 \). Let \( \beta \) be any fuzzy regular semiopen set of fts \( X \) such that \( \beta \leq (\text{cl}(\alpha) - \alpha) \). Then by Theorem 6.2.24, \( \text{cl}(\alpha) - \alpha \) does not contain any non-zero fuzzy regular semiopen set and so \( \beta = 0 \). Therefore \( \beta \leq \text{int}(\text{cl}(\alpha) - \alpha) \). By Theorem 6.2.32, \( \text{cl}(\alpha) - \alpha \) is fuzzy rw-open.

6.2.38 **Theorem:** Let \( \alpha \) and \( \beta \) be two fuzzy subsets of a fts \( X \). If \( \beta \) is a fuzzy rw-open set and \( \alpha \geq \text{int}(\beta) \), then \( \alpha \wedge \beta \) is a fuzzy rw-open set in fts \( X \).

**Proof:** Let \( \beta \) be a fuzzy rw-open set of a fts \( X \) and \( \alpha \geq \text{int}(\beta) \). That is \( \text{int}(\beta) \leq \alpha \wedge \beta \). Also \( \text{int}(\beta) \leq \alpha \wedge \beta \leq \beta \) and \( \beta \) is a fuzzy rw-open set. By Theorem 6.2.36, \( \alpha \wedge \beta \) is also a fuzzy rw-open set in fts \( X \).

### 6.3 Fuzzy rw-continuous maps and fuzzy rw-irresolute maps in fts.

In this section, we introduce the concepts of fuzzy rw-continuous maps and fuzzy rw-irresolute maps in fuzzy topological spaces. We prove that the composition two fuzzy rw-continuous maps need not be fuzzy rw-continuous and study some of their properties.

**6.3.1 Definition:** Let \( X \) and \( Y \) be fuzzy topological spaces. A map \( f: X \rightarrow Y \) is said to be fuzzy rw-continuous if the inverse image of every fuzzy open set in \( Y \) is fuzzy rw-open in \( X \).

**6.3.2 Theorem:** If a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy continuous, then \( f \) is fuzzy rw-continuous.
Proof: Let $\mu$ be a fuzzy open set in a fts $Y$. Since $f$ is fuzzy continuous, $f^{-1}(\mu)$ is a fuzzy open set in fts $X$. As every fuzzy open set is fuzzy rw-open, we have $f^{-1}(\mu)$ is fuzzy a rw-open set in fts $X$. Therefore $f$ is fuzzy rw-continuous.

The converse of the above Theorem need not be true in general as seen from the following example.

6.3.3 Example: Let $X=Y=\{a, b, c\}$ and the functions $\alpha, \beta, \gamma: X\to[0, 1]$ be defined as

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases},
\beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases},
\gamma(x) = \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}
\]

Consider $\tau = \{0, 1, \alpha\}$ $\sigma = \{0, 1, \beta, \gamma\}$. Now $(X, \tau)$ and $(Y, \sigma)$ are the fuzzy topological spaces. Define a map $f: (X, \tau)\to(Y, \sigma)$ by $f(a)=b$, $f(b)=c$ and $f(c)=a$. Then $f$ is fuzzy rw-continuous but not fuzzy continuous as the inverse image of the fuzzy set $\gamma$ in $(Y, \sigma)$ is $\delta: X\to[0, 1]$ define as

\[
\delta(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}
\]

This is not a fuzzy open set in $(X, \tau)$.

6.3.4 Theorem: A map $f: (X, \tau)\to(Y, \sigma)$ is fuzzy rw-continuous if and only if the inverse image of every fuzzy closed set in a fts $Y$ is a fuzzy rw-closed set in fts $X$.

Proof: Let $\delta$ be a fuzzy closed set in a fuzzy topological space $Y$. Then $\delta^C$ is fuzzy open in fts $Y$. Since $f$ is fuzzy rw-continuous, $f^{-1}(\delta^C)$ is fuzzy rw-open in fts $X$. But $f^{-1}(\delta^C) = 1-f^{-1}(\delta)$ and so $f^{-1}(\delta)$ is a fuzzy rw-closed set in fts $X$. 

- 183 -
Conversely, assume that the inverse image of every fuzzy closed set in Y is fuzzy rw-closed in fts X. Let μ be a fuzzy open set in fts Y. Then μ^C is fuzzy closed in Y. By hypothesis f^(-1)(μ^C) = 1−f^(-1)(μ) is fuzzy rw-closed in X and so f^(-1)(μ) is a fuzzy rw-open set in fts X. Thus f is fuzzy rw-continuous.

6.3.5 Theorem: If a function f:(X, τ)→(Y, σ) is fuzzy almost continuous, then it is fuzzy rw-continuous.

Proof: Let a function f: (X, τ)→(Y, σ) be a fuzzy almost continuous and μ be a fuzzy open set in fts Y. Then f^(-1)(μ) is a fuzzy regular open set in fts X. Now f^(-1)(μ) is fuzzy rw-open in X, as every fuzzy regular open set is fuzzy rw-open. Therefore f is fuzzy rw-continuous.

The converse of the above Theorem need not be true in general as seen from the following example.

6.3.6 Example: Consider the fuzzy topological spaces (X, τ) and (Y, σ) as defined in Example 6.3.3. Define a map f:(X, τ)→(Y, σ) by f(a)=b, f(b)=c and f(c)=a. Then f is fuzzy rw-continuous but it is not almost continuous.

6.3.7 Remark: Fuzzy semi continuous maps and fuzzy rw-continuous maps are independent as seen from the following examples.

6.3.8 Example: Let X= Y={a, b, c} and the functions α, β: X→[0, 1] be defined as

α(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases}, \quad β(x) = \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}

Consider τ = {0, 1, α} and σ = {0, 1, β}. Now (X, τ) and (Y, σ) are the fuzzy topological spaces. Define a map f: (X, τ)→(Y, σ) by f(a)=a,
\( f(b) = b \) and \( f(c) = c \). Then \( f \) is fuzzy rw-continuous but it is not fuzzy semi continuous, as the inverse image of fuzzy set \( \beta \) in \((Y, \sigma)\) is \( \delta : X \to [0,1] \) defined as

\[
\delta(x) = \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}
\]

This is not a fuzzy semiopen set in fts \( X \).

**6.3.9 Example:** Let \( X = Y = \{a, b, c\} \) and the functions \( \alpha, \beta, \gamma, \delta : X \to [0,1] \) be defined as

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases}, \quad \beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases}, \quad \gamma(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}, \quad \delta(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases}
\]

and \( \delta(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases} \)

Consider \( \tau = \{0,1, \alpha, \beta, \gamma\} \) and \( \sigma = \{0,1, \delta\} \). Now \((X, \tau)\) and \((Y, \sigma)\) are the fuzzy topological spaces. Define a map \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(c) = c \) and \( f(b) = b \). Then \( f \) is fuzzy semi continuous but it is not fuzzy rw-continuous, as the inverse image of fuzzy set \( \delta \) in \((Y, \sigma)\) is \( \mu : X \to [0,1] \) defined as

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = a, c \\
0 & \text{otherwise}
\end{cases}
\]

This is not a fuzzy rw-open set in fts \( X \).

**6.3.10 Remark:** Fuzzy generalized continuous maps and fuzzy rw-continuous maps are independent as seen from the following examples.

**6.3.11 Example:** Consider the fuzzy topological spaces \((X, \tau)\) and \((Y, \sigma)\) as defined in Example 6.3.9. Define a map \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = a, \)
\( f(b)=b \) and \( f(c)=c \). Then \( f \) is fuzzy \( rw \)-continuous but it is not fuzzy \( g \)-continuous as the inverse image of fuzzy set \( \delta \) in \((Y, \sigma)\) is \( \mu : X \rightarrow [0, 1] \) defined as

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = c \\
0 & \text{otherwise}
\end{cases}
\]

This is not a fuzzy \( g \)-open set in \( fts \) \( X \).

**6.3.12 Example:** Let \( X = \{a, b, c, d\} \) and the functions \( \alpha, \beta, \gamma : X \rightarrow [0, 1] \) be defined as

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases}, \quad \beta(x) = \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases}, \quad \gamma(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases}
\]

Let \( Y = \{a, b, c\} \) and the function \( \delta : Y \rightarrow [0, 1] \) be defined as

\[
\delta(x) = \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}
\]

Consider \( \tau = \{0, 1, \alpha, \beta, \gamma\} \) and \( \sigma = \{0, 1, \delta\} \). Now \((X, \tau)\) and \((Y, \sigma)\) are the fuzzy topological spaces. Define a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a)=f(d)=c, f(b)=b \) and \( f(c)=c \). Then \( f \) is fuzzy \( g \)-continuous but it is not fuzzy \( rw \)-continuous as the inverse image of fuzzy set \( \delta \) in \((Y, \sigma)\) is \( \mu : X \rightarrow [0, 1] \) defined as

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}
\]

This is not a fuzzy \( rw \)-open set in \( fts \) \( X \).

**6.3.13 Remark:** From the above discussions and known results we have the following implications

In the following diagram,
6.3.14 **Theorem:** If a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy rw-continuous and fuzzy completely semi continuous then it is fuzzy continuous.

**Proof:** Let a function \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a fuzzy rw-continuous and fuzzy completely semi continuous. Let \( \mu \) be a fuzzy closed set in fts \( Y \). Then \( f^{-1}(\mu) \) is both fuzzy regular semiopen and fuzzy rw-closed set in fts \( X \). By Theorem 6.2.16, \( f^{-1}(\mu) \) is a fuzzy closed set in fts \( X \). Therefore \( f \) is fuzzy continuous.

6.3.15 **Theorem:** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy rw-continuous and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) is fuzzy continuous, then their composition \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is fuzzy rw-continuous.

**Proof:** Let \( \mu \) be a fuzzy open set in fts \( Z \). Since \( g \) is fuzzy continuous, \( g^{-1}(\mu) \) is a fuzzy open set in fts \( Y \). Since \( f \) is fuzzy rw-continuous, \( f^{-1}(g^{-1}(\mu)) \) is a fuzzy rw-open set in fts \( X \). But \( (g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)) \). Thus \( g \circ f \) is fuzzy rw-continuous.

We introduce fuzzy rw-irresolute map as follows.

6.3.16 **Definition:** Let \( X \) and \( Y \) be fuzzy topological spaces. A map \( f: X \rightarrow Y \) is said to be a fuzzy rw-irresolute map if the inverse image of every fuzzy rw-open set in \( Y \) is a fuzzy rw-open set in \( X \).
6.3.17 Theorem: If a map \( f: X \rightarrow Y \) is fuzzy rw-irresolute, then it is fuzzy rw-continuous.

Proof: Let \( \beta \) be a fuzzy open set in \( Y \). Since every fuzzy open set is fuzzy rw-open, \( \beta \) is a fuzzy rw-open set in \( Y \). Since \( f \) is fuzzy rw-irresolute, \( f^{-1}(\beta) \) is fuzzy rw-open in \( X \). Thus \( f \) is fuzzy rw-continuous.

The converse of the above Theorem need not be true in general as seen from the following example.

6.3.18 Example: Let \( X=Y=\{a, b, c\} \) and the functions \( \alpha, \beta, \gamma : X \rightarrow [0, 1] \) be defined as:

\[
\alpha(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \quad \beta(x) = \begin{cases} 1 & \text{if } x = b \\ 0 & \text{otherwise} \end{cases} \quad \gamma(x) = \begin{cases} 1 & \text{if } x = a, b \\ 0 & \text{otherwise} \end{cases}
\]

Consider \( T_1 = \{0, 1, \alpha, \beta, \gamma\} \) and \( T_2 = \{0, 1, \alpha\} \). Now \( (X, T_1) \) and \( (Y, T_2) \) are fuzzy topological spaces. Let \( f:(X, T_1) \rightarrow (Y, T_2) \) be the identity map. Then \( f \) is fuzzy rw-continuous but it is not fuzzy rw-irresolute. Since for the fuzzy rw-open set \( \mu: Y \rightarrow [0, 1] \) defined by

\[
\mu(x) = \begin{cases} 1 & \text{if } x = b, c \\ 0 & \text{otherwise} \end{cases}
\]

in \( Y \), \( f^{-1}(\mu) \neq \mu \) is not fuzzy rw-open in \( (X, T_1) \).

6.3.19 Theorem: Let \( X, Y \) and \( Z \) be fuzzy topological spaces. If \( f: X \rightarrow Y \) is fuzzy rw-irresolute and \( g: Y \rightarrow Z \) is fuzzy rw-continuous then their composition \( g \circ f: X \rightarrow Z \) is fuzzy rw-continuous.

Proof: Let \( \alpha \) be any fuzzy open set in \( fts X \). Since \( g \) is fuzzy rw-continuous, \( g^{-1}(\alpha) \) is a fuzzy rw-open set in \( fts Y \). Since \( f \) is fuzzy rw-irresolute, \( f^{-1}(g^{-1}(\alpha)) \) is a fuzzy rw-open set in \( fts X \). But \( (g \circ f)^{-1}(\alpha) = f^{-1}(g^{-1}(\alpha)) \). Thus \( g \circ f \) is fuzzy rw-continuous.
6.3.20 Theorem: Let X, Y and Z be fuzzy topological spaces and \( f: X \to Y \) and \( g: Y \to Z \) be fuzzy rw-irresolute maps, then their composition \( g \circ f: X \to Z \) is fuzzy rw-irresolute map.

Proof: Let \( \alpha \) be a fuzzy rw-open set in fts Z. Since \( g \) is fuzzy rw-irresolute, \( g^{-1}(\alpha) \) is a fuzzy rw-open set in fts Y. Since \( f \) is fuzzy rw-irresolute, \( f^{-1}(g^{-1}(\alpha)) \) is a fuzzy rw-open set in fts X. But \( (g \circ f)^{-1}(\alpha) = f^{-1}(g^{-1}(\alpha)) \). Thus \( g \circ f \) is fuzzy rw-irresolute.

6.4 Fuzzy rw-open maps and fuzzy rw-closed maps in fts.

In this section we introduce fuzzy rw-open maps and fuzzy rw-closed maps in fuzzy topological spaces and obtain certain characterizations of these maps.

6.4.1 Definition: Let X and Y be two fuzzy topological spaces. A map \( f:(X, T_1) \to (Y, T_2) \) is called fuzzy rw-open if the image of every fuzzy open set in X is fuzzy rw-open in Y.

6.4.2 Theorem: Every fuzzy open map is a fuzzy rw-open map.
Proof: Let \( f:(X, T_1) \to (Y, T_2) \) be a fuzzy open map and \( \mu \) be a fuzzy open set in fts X. Then \( f(\mu) \) is a fuzzy open set in fts Y. Since every fuzzy open set is fuzzy rw-open, \( f(\mu) \) is a fuzzy rw-open set in fts Y. Hence \( f \) is a fuzzy rw-open map.

The converse of the above Theorem need not be true in general as seen from the following example.

6.4.3 Example: Let \( X=Y=\{a, b, c\} \) and the functions \( \alpha, \beta, \gamma : X \to [0, 1] \) be defined as
\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \\
\beta(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases} \\
\gamma(x) = \begin{cases} 
1 & \text{if } x = a, c \\
0 & \text{otherwise}
\end{cases}
\]
Consider $T_1=\{0, 1, \alpha, \beta, \gamma\}$ and $T_2=\{0, 1, \alpha\}$. Then $(X, T_1)$ and $(Y, T_2)$ are fuzzy topological spaces. Let $f:(X, T_1)\to(Y, T_2)$ be defined as $f(a)=f(b)=a$ and $f(c)=c$. Then this function is fuzzy rw-open but it is not fuzzy open, since the image of the fuzzy open set $\gamma$ in $X$ is the fuzzy set $\gamma$ in $Y$ which is not fuzzy open.

6.4.4 Remark: Let $f:(X, T_1)\to(Y, T_2)$ and $g:(Y, T_2)\to(Z, T_3)$ be two fuzzy rw-open maps. Then $g\circ f$ need not be fuzzy rw-open as seen from the following example.

6.4.5 Example: Let $X=Y=\{a, b, c\}$ and the functions $\alpha, \beta, \gamma, \delta : X \to [0, 1]$ be defined as

\[
\begin{align*}
\alpha(x) &= \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \\
\beta(x) &= \begin{cases} 
1 & \text{if } x = b \\
0 & \text{otherwise}
\end{cases} \\
\gamma(x) &= \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases} \\
\delta(x) &= \begin{cases} 
1 & \text{if } x = b, c \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Consider $T_1=\{0, 1, \alpha, \delta\}$ and $T_2=\{0, 1, \alpha\}$ and $T_2=\{0, 1, \alpha, \beta, \gamma\}$. Then $(X, T_1)$, $(Y, T_2)$ and $(Z, T_3)$ are fuzzy topological spaces. Let $f:(X, T_1)\to(Y, T_2)$ and $g:(Y, T_2)\to(Z, T_3)$ be the identity maps. Then $f$ and $g$ are fuzzy rw-open maps but their composition $g\circ f:(X, T_1)\to(Z, T_3)$ is not fuzzy rw-open as $\delta : X \to [0, 1]$ is fuzzy open in $X$ but $(g\circ f)(\delta) = \delta$ is not fuzzy rw-open in $Z$.

6.4.6 Theorem: If $f:(X, T_1)\to(Y, T_2)$ is fuzzy open map and $g:(Y, T_2)\to(Z, T_3)$ is fuzzy rw-open map, then their composition $g\circ f:(X, T_1)\to(Z, T_3)$ is fuzzy rw-open map.

Proof: Let $\alpha$ be fuzzy open set in $(X, T_1)$. Since $f$ is fuzzy open map, $f(\alpha)$ is a fuzzy open set in $(Y, T_2)$. Since $g$ is a fuzzy rw-open map,
6.4.7 Definition: Let \( X \) and \( Y \) be two fuzzy topological spaces. A map \( f : (X, T_1) \rightarrow (Y, T_2) \) is called fuzzy rw-closed if the image of every fuzzy closed set in \( X \) is a fuzzy rw-closed set in \( Y \).

6.4.8 Theorem: Let \( f : (X, T_1) \rightarrow (Y, T_2) \) be fuzzy closed map. Then \( f \) is a fuzzy rw-closed map.

Proof: Let \( \alpha \) be a fuzzy closed set in \( (X, T_1) \). Since \( f \) is a fuzzy closed map, \( f(\alpha) \) is a fuzzy closed set in \( (Y, T_2) \). Since every fuzzy closed set is fuzzy rw-closed, \( f(\alpha) \) is a fuzzy rw-closed set in \( (Y, T_2) \). Hence \( f \) is a fuzzy rw-closed map.

The converse of the above Theorem need not be true in general as seen from the following example.

6.4.9 Example: Let \( X = [0, 1] \) and \( Y = [0, 1] \). The fuzzy sets \( \alpha : X \rightarrow [0, 1] \) \( \beta : Y \rightarrow [0, 1] \) are defined as:

\[
\alpha(x) = \begin{cases} 
0.5 \text{ if } x = 2/3 \\
1 \text{ otherwise}
\end{cases} 
\quad \beta(x) = \begin{cases} 
0.7 \text{ if } x = 2/3 \\
1 \text{ otherwise}
\end{cases}
\]

Consider \( T_1 = \{0, 1, \alpha\} \) and \( T_2 = \{0, 1, \beta\} \). Then \( (X, T_1) \) and \( (Y, T_2) \) are fuzzy topological spaces. Let \( f : (X, T_1) \rightarrow (Y, T_2) \) be the identity map. Then \( f \) is a fuzzy rw-closed map but it is not a fuzzy closed map, since the image of the fuzzy closed set \( \alpha^C \) in \( X \) is not a fuzzy closed set in \( Y \).

6.4.10 Remark: The composition of two fuzzy rw-closed maps need not be a fuzzy rw-closed map.

6.4.11 Example: Consider the fuzzy topological spaces \( (X, T_1) \), \( (Y, T_2) \) \( (Z, T_3) \) and mappings defined in Example 6.4.5. The maps \( f \) and \( g \) are correct and complete.
6.4.12 Theorem: If \( f : (X, T_1) \rightarrow (Y, T_2) \) and \( g : (Y, T_2) \rightarrow (Z, T_3) \) be two maps. Then \( g \circ f : (X, T_1) \rightarrow (Z, T_3) \) is fuzzy rw-closed map if \( f \) is fuzzy closed and \( g \) is fuzzy rw-closed.

**Proof:** Let \( \alpha \) be a fuzzy closed set in \( (X, T_1) \). Since \( f \) is a fuzzy closed map, \( f(\alpha) \) is a fuzzy closed set in \( (Y, T_2) \). Since \( g \) is a fuzzy rw-closed map, \( g(f(\alpha)) \) is a fuzzy rw-closed set in \( (Z, T_3) \). But \( g(f(\alpha)) = (g \circ f)(\alpha) \). Thus \( g \circ f \) is fuzzy rw-closed map.

6.4.13 Theorem: A map \( f : X \rightarrow Y \) is fuzzy rw-closed if for each fuzzy set \( \delta \) of \( Y \) and for each fuzzy open set \( \mu \) of \( X \) such that \( \mu \geq f^{-1}(\delta) \), there is a fuzzy rw-open set \( \alpha \) of \( Y \) such that \( \delta \leq \alpha \) and \( f^{-1}(\alpha) \leq \mu \).

**Proof:** Suppose that \( f \) is fuzzy rw-closed. Let \( \delta \) be a fuzzy subset of \( Y \) and \( \mu \) is a fuzzy open set of \( X \) such that \( f^{-1}(\delta) \leq \mu \). Let \( \alpha = 1-f(1-\mu) \) is fuzzy rw-open set in fts \( Y \). Note that \( f^{-1}(\delta) \leq \mu \) which implies \( \delta \leq \alpha \) and \( f^{-1}(\alpha) \leq \mu \).

For the converse, suppose that \( \mu \) is a fuzzy closed set in \( X \). Then \( f^{-1}(1-f(\mu)) \leq 1-\mu \) and \( 1-\mu \) is fuzzy open. By hypothesis, there is a fuzzy rw-open set \( \alpha \) of \( Y \) such that \( 1-f(\mu) \leq \alpha \) and \( f^{-1}(\alpha) \leq 1-\mu \). Therefore \( \mu \leq 1-f^{-1}(\alpha) \). Hence \( 1-\alpha \leq f(\mu) \), \( f(1-f^{-1}(\alpha)) \leq 1-\alpha \) which implies \( f(\mu) = 1-\alpha \). Since \( 1-\alpha \) is fuzzy rw-closed, \( f(\mu) \) is fuzzy rw-closed and thus \( f \) is fuzzy rw-closed.

6.4.14 Lemma: Let \( f : (X, T_1) \rightarrow (Y, T_2) \) be fuzzy irresolute and \( \alpha \) be fuzzy regular semiopen in \( Y \). Then \( f^{-1}(\alpha) \) is fuzzy regular semiopen in \( X \).
Proof: Let $\alpha$ be fuzzy regular semiopen in $Y$. To prove $f^{-1}(\alpha)$ is fuzzy regular semiopen in $X$. That is to prove $f^{-1}(\alpha)$ is both fuzzy semiopen and fuzzy semi-closed in $X$. Now $\alpha$ is fuzzy semiopen in $Y$. Since $f$ is fuzzy irresolute, $f^{-1}(\alpha)$ is fuzzy semiopen in $X$.

Now $\alpha$ is fuzzy semi-closed in $Y$, as fuzzy regular semiopen set is fuzzy semi-closed. Then $1-\alpha$ is fuzzy semiopen in $Y$. Since $f$ is fuzzy irresolute, $f^{-1}(1-\alpha)$ is fuzzy semiopen in $X$. But $f^{-1}(1-\alpha) = 1-f^{-1}(\alpha)$ is fuzzy semiopen in $X$ and so $f^{-1}(\alpha)$ is semi-closed in $X$. Thus $f^{-1}(\alpha)$ is both fuzzy semiopen and fuzzy semi-closed in $X$ and hence $f^{-1}(\alpha)$ is fuzzy regular semiopen in $X$.

6.4.15 Theorem: If a map $f:(X, T_1)\rightarrow(Y, T_2)$ is fuzzy irresolute and fuzzy rw-closed and $\alpha$ is fuzzy rw-closed set of $X$, then $f(\alpha)$ is a fuzzy rw-closed set in $Y$.

Proof: Let $\alpha$ be a fuzzy closed set of $X$. Let $f(\alpha) \leq \mu$, where $\mu$ is fuzzy regular semiopen in $Y$. Since $f$ is fuzzy irresolute, $f^{-1}(\mu)$ is a fuzzy regular semiopen in $X$, by Lemma 6.4.14 and $\alpha \leq f^{-1}(\mu)$. Since $\alpha$ is a fuzzy rw-closed set in $X$, $cl(\alpha) \leq f^{-1}(\mu)$. Since $f$ is fuzzy rw-closed, $f(cl(\alpha))$ is a fuzzy rw-closed set contained in the fuzzy regular semiopen set $\mu$, which implies $cl(f(cl(\alpha))) \leq \mu$ and hence $cl(f(\alpha)) \leq \mu$. Therefore $f(\alpha)$ is a fuzzy rw-closed set in $Y$.

6.4.16 Corollary: If a map $f:(X, T_1)\rightarrow(Y, T_2)$ is fuzzy irresolute and fuzzy closed and $\alpha$ is a fuzzy rw-closed set in fts $X$, then $f(\alpha)$ is a fuzzy rw-closed set in fts $Y$.

Proof: The proof follows from the Theorem 6.4.15 and the fact that every fuzzy closed map is a fuzzy rw-closed map.
6.4.17 Theorem: Let \(f:X \to Y\) and \(g:Y \to Z\) be two mappings such that \(g \circ f:X \to Z\) is a fuzzy rw-closed map, then

(i) if \(f\) is fuzzy continuous and surjective, then \(g\) is fuzzy rw-closed and

(ii) if \(g\) is fuzzy rw-irresolute and injective, then \(f\) is fuzzy rw-closed.

Proof: (i) Let \(\mu\) be a fuzzy closed set in \(Y\). Since \(f\) is fuzzy continuous, \(f^{-1}(\mu)\) is a fuzzy closed set in \(X\). Since \(g \circ f\) is a fuzzy rw-closed map, \((g \circ f)(f^{-1}(\mu))\) is a fuzzy rw-closed set in \(Z\). But \((g \circ f)(f^{-1}(\mu)) = g(\mu)\), as \(f\) is surjective. Thus \(g\) is fuzzy rw-closed.

(ii) Let \(\beta\) be a fuzzy closed set of \(X\). Then \((g \circ f)(\beta)\) is a fuzzy rw-closed set in \(Z\), since \(g \circ f\) is a fuzzy rw-closed map. Since \(g\) is fuzzy rw-irresolute, \(g^{-1}((g \circ f)(\beta))\) is fuzzy rw-closed in \(Y\). But \(g^{-1}((g \circ f)(\beta)) = f(\beta)\), as \(g\) is injective. Thus \(f\) is fuzzy rw-closed map.

6.5 Fuzzy rw-homeomorphisms in fts.

In this section, we introduce and study two new fuzzy homeomorphisms, namely fuzzy rw-homeomorphism and fuzzy rwc-homeomorphism. We prove that fuzzy homeomorphism is fuzzy rw-homeomorphism and we prove that the composition of two fuzzy rwc-homeomorphism is a rw-homeomorphism.

6.5.1 Definition: Let \(X\) and \(Y\) be fuzzy topological spaces. A bijection map \(f: (X, T_1) \to (Y, T_2)\) is called fuzzy rw-homeomorphism if \(f\) and \(f^{-1}\) are fuzzy rw-continuous.

The family of all fuzzy rw-homeomorphism from \((X, T)\) onto itself is denoted by \(\text{frw-h}(X, T)\).
6.5.2 **Theorem:** Every fuzzy homeomorphism is fuzzy rw-homeomorphism.

**Proof:** Let a map $f: (X, T_1) \to (Y, T_2)$ be a fuzzy homeomorphism. Then $f$ and $f^{-1}$ are fuzzy continuous. Since every fuzzy continuous map is fuzzy rw-continuous, $f$ and $f^{-1}$ are fuzzy rw-continuous. Therefore $f$ is fuzzy a rw-homeomorphism.

The converse of the above Theorem need not be true as seen from the following example.

6.5.3 **Example:** Let $X = Y = \{a, b, c\}$ and the functions $\alpha, \beta, \gamma: X \to [0, 1]$ be defined as

\[
\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{otherwise}
\end{cases} \quad \beta(x) = \begin{cases} 
1 & \text{if } x = a, b \\
0 & \text{otherwise}
\end{cases} \quad \gamma(x) = \begin{cases} 
1 & \text{if } x = a, c \\
0 & \text{otherwise}
\end{cases}
\]

Consider $T_1 = \{0, 1, \alpha, \gamma\}$ and $T_2 = \{0, 1, \beta\}$. Then $(X, T_1)$ and $(Y, T_2)$ are fuzzy topological spaces. Define a map $f: (X, T_1) \to (Y, T_2)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Here the function $f$ is a fuzzy rw-homeomorphism but it is not a fuzzy homeomorphism, as the image of a fuzzy open set $\alpha$ in $(X, T_1)$ is $\alpha$ which is not a fuzzy open set in $(Y, T_2)$.

6.5.4 **Theorem:** Let $X$ and $Y$ be fuzzy topological spaces and $f: (X, T_1) \to (Y, T_2)$ be a bijective map. Then the following statements are equivalent.

(a) $f^{-1}$ is fuzzy rw-continuous

(b) $f$ is a fuzzy rw-open map

(c) $f$ is a fuzzy rw-closed map.
Proof: (a) ⇒ (b) Let α be any fuzzy open set in X. Since $f^{-1}$ is fuzzy rw-continuous, $(f^{-1})^{-1}(\alpha) = f(\alpha)$ is fuzzy rw-open in Y. Hence f is a fuzzy rw-open map.

(b)⇒ (c) Let α be any fuzzy closed set in X. Then $1-\alpha$ is fuzzy rw-open in X. Since f is a fuzzy rw-open map, $f(1-\alpha)$ is fuzzy rw-open in Y. But $f(1-\alpha) = 1-f(\alpha)$, as f is a bijection map. Hence $f(\alpha)$ is fuzzy rw-closed in Y. Therefore f is fuzzy rw-closed.

(c)⇒(a) Let α be any fuzzy closed set in X. Then $f(\alpha)$ is a fuzzy rw-closed set in Y. But $(f^{-1})^{-1}(\alpha) = f(\alpha)$. Therefore $f^{-1}$ is fuzzy rw-continuous.

6.5.5 Theorem: Let $f:(X, T_1)\rightarrow(Y, T_2)$ be a bijection and fuzzy rw-continuous map. Then the following statements are equivalent.

(a) f is a fuzzy rw-open map
(b) f is a fuzzy rw-homeomorphism
(c) f is a fuzzy rw-closed map.

Proof: (a) ⇒ (b) By hypothesis and assumption f is a fuzzy rw-homeomorphism.

(b)⇒ (c) Since f is a fuzzy rw-homeomorphism; it is fuzzy rw-open. So by the above Theorem 6.5.4, it is a fuzzy rw-closed map.

(c) ⇒ (a) Let σ be a fuzzy open set in X, so that $1-\sigma$ is a closed set and f being rw-closed, f($1-\sigma$) is fuzzy rw-closed in Y. But $f(1-\sigma) = 1-f(\sigma)$ Thus $f(\sigma)$ is fuzzy rw-open in Y. Therefore f is a fuzzy rw-open map.

6.5.6 Definition: A bijection map $f: (X, T_1)\rightarrow(Y, T_2)$ is called a fuzzy rwc-homeomorphism if f and $f^{-1}$ are fuzzy rw-irresolute. We say that
spaces \((X, T_1)\) and \((Y, T_2)\) are fuzzy rwc-homeomorphism if there exist a fuzzy rwc-homeomorphism from \((X, T_1)\) onto \((Y, T_2)\).

The family of all fuzzy rwc-homeomorphism from \((X, T)\) onto itself is denoted by \(\text{frwc-h}(X, T)\).

6.5.7 **Theorem:** Every fuzzy rwc-homeomorphism is fuzzy rw-homeomorphism but not conversely.

**Proof:** The proof follows from the fact that every fuzzy rw-irresolute map is fuzzy rw-continuous but not conversely.

6.5.8 **Theorem:** Let \((X, T_1), (Y, T_2)\) and \((Z, T_3)\) be fuzzy topological spaces and \(f: (X, T_1) \rightarrow (Y, T_2)\), \(g: (Y, T_2) \rightarrow (Z, T_3)\) be fuzzy rwc-homeomorphisms. Then their composition \(gof: (X, T_1) \rightarrow (Z, T_3)\) is a fuzzy rwc-homeomorphism.

**Proof:** Let \(\mu\) be a fuzzy rw-open set in \((Z, T_3)\). Since \(g\) is a fuzzy rw-irresolute, \(g^{-1}(\mu)\) is a fuzzy rw-open set in \((Y, T_2)\). Since \(f\) is a fuzzy rw-irresolute, \(f^{-1}(g^{-1}(\mu))\) is a fuzzy rw-open set in \((X, T_1)\). But \(f^{-1}(g^{-1}(\mu)) = (gof)^{-1}(\mu)\). Therefore \(gof\) is fuzzy rw-irresolute.

To prove that \((gof)^{-1}\) is fuzzy rw-irresolute. Let \(\alpha\) be a fuzzy rw-open set in \((X, T_1)\). Since \(f^{-1}\) is fuzzy rw-irresolute, \((f^{-1})^{-1}(\alpha)\) is a fuzzy rw-open set in \((Y, T_2)\). Also \((f^{-1})^{-1}(\alpha) = f(\alpha)\). Since \(g^{-1}\) is fuzzy rw-irresolute, \(((g^{-1})^{-1})(f(\alpha))\) is a fuzzy rw-open set in \((Z, T_3)\). That is \(((g^{-1})^{-1})(f(\alpha)) = g(f(\alpha)) = (gof)(\alpha) = ((gof)^{-1})^{-1}(\alpha)\). Therefore \((gof)^{-1}\) is fuzzy rw-irresolute. Thus \(gof\) and \((gof)^{-1}\) are fuzzy rw-irresolute. Hence \(gof\) is fuzzy rwc-homeomorphism.

6.5.9 **Theorem:** The set \(\text{frwc-h}(X, T)\) is a group under the composition of maps.
Proof: Define a binary operation*: \(frwc-h(X, T) \times frwc-h(X, T) \rightarrow frwc-h(X, T)\) by \(f * g = gof\) for all \(f, g \in frwc-h(X, T)\) and \(o\) is the usual operation of composition of maps. Then by Theorem 6.5.8, \(g of \in frwc-h(X, T)\). We know that the composition of maps is associative and the identity map \(I: (X, T) \rightarrow (X, T)\) belonging to \(frwc-h(X, T)\) serves as the identity element. If \(f \in frwc-h(X, T)\), then \(f^{-1} \in frwc-h(X, T)\) such that \(fof^{-1} = f^{-1}of = I\) and so inverse exists for each element of \(frwc-h(X, T)\). Therefore \((frwc-h(X, T), o)\) is a group under the operation of composition of maps.

6.5.10 Theorem: Let \(f: (X, T_1) \rightarrow (Y, T_2)\) be a fuzzy rwc-homeomorphism. Then \(f\) induces an isomorphism from the group \(frwc-h(X, T_1)\) on to the group \(frwc-h(Y, T_2)\).

Proof: Using the map \(f\), we define a map \(\psi_f: frwc-h(X, T_1) \rightarrow frwc-h(Y, T_2)\) by \(\psi_f(h) = fo ho f^{-1}\) for every \(h \in frwc-h(X, T_1)\). Then \(\psi_f\) is a bijection. Further, for all \(h_1, h_2 \in frwc-h(X, T_1)\), \(\psi_f(h_1 oh_2) = fo(h_1 oh_2) o f^{-1} = (foh_1 o f^{-1}) o (foh_2 o f^{-1}) = \psi_f(h_1) o \psi_f(h_2)\). Therefore \(\psi_f\) is a homeomorphism and so it is an isomorphism induced by \(f\).