CHAPTER-V
rw-CLOSED SETS AND rw-CONTINUOUS MAPS IN
BITOPOLOGICAL SPACES

5.1 Introduction.

The triple \( (X, \tau_1, \tau_2) \) where \( X \) is a set and \( \tau_1 \) and \( \tau_2 \) are topologies on \( X \) is called a bitopological space. Kelly [45] initiated the systematic study of such spaces in 1963. He generalized the topological concepts to bitopological setting and published a large number of papers. Following the work of Kelly on the bitopological spaces, various authors, like Arya and Nour [7], Di Maio and Noiri [24], Fukutake [34], Nagaveni [65], Maki, Sundaram and Balachandran [55], Sheik John [85], Sampath Kumar[82], Patty [77], Arockiarani [3], Gnanambal [40], Reilly [8], Rajamani and Viswanthan [80], and Popa [78] have turned their attention to the various concepts of topology by considering bitopological spaces instead of topological spaces.

In chapter II, we have introduced and studied the concept of rw-closed sets, rw-open sets and rw-closure operator in topological spaces. In chapter III, we have introduced and investigated some properties of rw-continuous maps and rw-irresolute maps in topological spaces.

In section 2 of this chapter, \( (i, j) \)-rw-closed sets in bitopological space have been introduced and studied. Among many other results it is observed that every \( (i, j) \)-w-closed set is \( (i, j) \)-rw-closed set which implies \( (i, j) \)-rg-closed set but not conversely.

In section 3 of this chapter, we have introduced \( (i, j) \)-rw-open sets in bitopological space and study some of their properties. In section 4 of this chapter, we shall use the \( (i, j) \)-rw-closed subsets of bitopological space \( (X, \tau_1, \tau_2) \) to define a new closure operator “(i, j)-rw-cl”, and thus

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new topology \( \tau_{rw}(i, j) \) on the space and shall examine some of the properties of this new topology.

In section 5 of this chapter, a new class of maps called \( D_{rw}(i, j) - \sigma_k \)-continuous maps in bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every \( C(i, j) - \sigma_k \)-continuous map is \( D_{rw}(i, j) - \sigma_k \)-continuous which implies \( D(i, j) - \sigma_k \)-continuous but not conversely. Also, we have introduced the concept of \( rw \)-bi-continuity, \( rw \)-s-bi-continuity and pairwise \( rw \)-irresolute in bitopological spaces and study some of their properties.

Throughout this chapter \((X, \tau_i, \tau_2), (Y, \sigma_1, \sigma_2)\) and \((Z, \eta_1, \eta_2)\) denote nonempty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and fixed integers \( i, j, k, e, m, n \in \{1, 2\} \).

5.2 \((\tau_i, \tau_j)\)-rw-closed sets and their basic properties.

In this section, we introduce and investigate the concept of \((\tau_i, \tau_j)\)-rw-closed sets which are introduced in a bitopological space in analogy with \( rw \)-closed sets in topological spaces. From now on, \( \tau \)-cl(A) denotes the closure of A relative to a topology \( \tau \).

5.2.1 Definition: Let \( i, j \in \{1, 2\} \) be fixed integers. In a bitopological space \((X, \tau_1, \tau_2)\), a subset \( A \subset X \) is said to be \((\tau_i, \tau_j)\)-rw-closed (briefly, \((i, j)\)-rw-closed) set if \( \tau_j\)-cl(A) \( \subset \) G and \( G \in \text{RSO}(X, \tau_i) \).

We denote the family of all \((i, j)\)-rw-closed sets in a bitopological space \((X, \tau_1, \tau_2)\) by \( D_{rw}(\tau_i, \tau_j) \) or \( D_{rw}(i, j) \).

5.2.2 Remark: By setting \( \tau_1 = \tau_2 \) in Definition 5.2.1, an \((i, j)\)-rw-closed set reduces to a \( rw \)-closed set in \( X \).
First we prove that the class of \((i, j)\)-rw-closed sets properly lies between the class of \((i, j)\)-w-closed sets and the class of \((i, j)\)-rg-closed sets.

5.2.3 Theorem: If \(A\) is \((i, j)\)-w-closed subset of \((X, \tau_1, \tau_2)\), then \(A\) is \((i, j)\)-rw-closed.

Proof: Let \(A\) be a \((i, j)\)-w-closed subset of \((X, \tau_1, \tau_2)\). Let \(G \in \text{RSO}(X, \tau_i)\) be such that \(A \subseteq G\). Since \(\text{RSO}(X, \tau_i) \subseteq \text{SO}(X, \tau_i)\), we have \(G \in \text{SO}(X, \tau_i)\). Then by hypothesis, \(\tau_j-\text{cl}(A) \subseteq G\). Therefore \(A\) is \((i, j)\)-rw-closed.

The converse of this theorem need not be true as seen from the following example.

5.2.4 Example: Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \phi, \{a\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). Then the subsets \{a\}, \{b\} and \{a, b\} are \((1, 2)\)-rw-closed sets, but not \((1, 2)\)-w-closed sets in the bitopological space \((X, \tau_1, \tau_2)\).

5.2.5 Theorem: If \(A\) is a \((i, j)\)-rw-closed subset of \((X, \tau_1, \tau_2)\), then \(A\) is \((i, j)\)-rg-closed.

Proof: Let \(A\) be a \((i, j)\)-rw-closed subset of \((X, \tau_1, \tau_2)\). Let \(G \in \text{RO}(X, \tau_i)\) be such that \(A \subseteq G\). Since \(\text{RO}(X, \tau_i) \subset \text{RSO}(X, \tau_i)\), we have \(G \in \text{RSO}(X, \tau_i)\). Then by hypothesis, \(\tau_j-\text{cl}(A) \subseteq G\). Therefore \(A\) is \((i, j)\)-rg-closed.

The converse of this theorem need not be true as seen from the following example.

5.2.6 Example: Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\) and \(\tau_2 = \{X, \phi, \{a, b\}, \{c, d\}\}\). Then the subsets \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{a, c\}, \{a, c, d\}, and \{b, c, d\} are \((1, 2)\)-rg-closed sets, but not \((1, 2)\)-rw-closed sets in the bitopological space \((X, \tau_1, \tau_2)\).
5.2.7 Theorem: If A is $\tau_j$-closed subset of a bitopological space $(X, \tau_1, \tau_2)$, then the set A is $(i, j)$-rw-closed.

Proof: Let $G \in \mathcal{RSO}(X, \tau_i)$ be such that $A \subseteq G$. Then by hypothesis, $\tau_j\text{-cl}(A) = A$, which implies $\tau_j\text{-cl}(A) \subseteq G$. Therefore A is $(i, j)$-rw-closed.

The converse of this theorem need not be true as seen from the following example.

5.2.8 Example: Let $X = \{a, b, c\}$, $\tau_i=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b, c\}\}$. Then the subset $\{a, b\}$ is $(1, 2)$-rw-closed set, but not a $\tau_2$-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.9 Remark: $\tau_j$-w-closed sets and $(i, j)$-rw-closed sets are independent as seen from the following examples.

5.2.10 Example: Let $X = \{a, b, c\}$, $\tau_i=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a, b\}\}$. Then the subset $\{a\}$ is $(1, 2)$-rw-closed set, but not $\tau_2$-w-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.11 Example: Let $X = \{a, b, c\}$, $\tau_i=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a, b\}, \{b, c\}\}$. Then the subsets $\{b\}, \{c\}$ and $\{a, c\}$ are $\tau_2$-w-closed sets but not $(1, 2)$-rw-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.12 Remark: $\tau_j$-rg-closed sets and $(i, j)$-rw-closed sets are independent as seen from the following examples.

5.2.13 Example: Let $X = \{a, b, c\}$, $\tau_i=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}\}$. Then the subsets $\{a\}, \{b\}, \{c\}$ and $\{a, c\}$ are $\tau_2$-rg-closed sets but not $(1, 2)$-rw-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.
5.2.14 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the subsets $\{a\}, \{b\}$ are $(1, 2)$-rw-closed sets, but not $\tau_2$-rg-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.15 Remark: (i, j)-g-closed sets and (i, j)-rw-closed sets are independent as seen from the following examples.

5.2.16 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}\}$. Then the subsets $\{b\}$ and $\{a, b\}$ are $(1, 2)$-rw-closed sets, but not $(1, 2)$-g-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.17 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}\}$. Then the subset $\{a, c\}$ is a $(1, 2)$-g-closed set but not a $(1, 2)$-rw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.18 Theorem: If $A$ is a $(i, j)$-rw-closed subset of $(X, \tau_i, \tau_2)$, then $A$ is $(i, j)$-gpr-closed.

Proof: Let $A$ be a $(i, j)$-rw-closed subset of $(X, \tau_i, \tau_2)$. Let $G \in RO(X, \tau_i)$ be such that $A \subset G$. Since $RO(X, \tau_i) \subset RSO(X, \tau_i)$, we have $G \in RSO(X, \tau_i)$. Then by hypothesis, $\tau_i$-cl($A$) $\subset G$. Also $\tau_i$-pcl($A$) $\subset \tau_i$-cl($A$) which implies $\tau_i$-pcl($A$) $\subset G$. Therefore $A$ is $(i, j)$-gpr-closed.

The converse of this theorem need not be true as seen from the following example.

5.2.19 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then the subsets $\{b\}, \{c\}$ and $\{a, c\}$ are $(1, 2)$-gpr-closed sets but not $(1, 2)$-rw-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.20 Remark: (i, j)-wg-closed sets and (i, j)-rw-closed sets are independent as seen from the following examples.
5.2.21 Example: Let $X= \{a, b, c\}$, $\tau_1=\{X, \emptyset, \{a\}, \{a, b\}\}$ and $\tau_2=\{X, \emptyset, \{b\}, \{a\}\}$. Then the subset $\{a, b\}$ is a (1, 2)-rw-closed set but not a (1, 2)-wg-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.22 Example: Let $X= \{a, b, c\}$, $\tau_1=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \emptyset, \{c\}, \{b, c\}\}$. Then the subsets $\{c\}$, $\{b, c\}$, and $\{a, c\}$ are (1, 2)-wg-closed sets but not (1, 2)-rw-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.23 Remark: $(i, j)$-gp-closed sets and $(i, j)$-rw-closed sets are independent as seen from the following examples.

5.2.24 Example: Let $X= \{a, b, c\}$, $\tau_1=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \emptyset, \{a\}, \{b, c\}\}$. Then the subset $\{a, c\}$ is a (1, 2)-gp-closed set, but not a (1, 2)-rw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.25 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \emptyset, \{a\}\}$ and $\tau_2=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $\{a\}$ is a (1, 2)-rw-closed set, but not a (1, 2)-gp-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.26 Remark: $(i, j)$-$g^*$-closed sets and $(i, j)$-rw-closed sets are independent as seen from the following example.

5.2.27 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \emptyset, \{c\}, \{b, c\}\}$. Then the subset $\{a\}$ is a (1, 2)-rw-closed set, but not a (1, 2)-$g^*$-closed set. Also the subset $\{b, c\}$ is a (1, 2)-$g^*$-closed set but not a (1, 2)-rw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.28 Remark: From the above discussions and known results we have the following implications. Here

\[ \text{A} \rightarrow \text{B} \text{ means A implies B, but not conversely and } \]

\[ \text{A} \leftrightarrow \text{B} \text{, means A and B are independent of each other} \]
5.2.29 Theorem: If $A, B \in D_{rw}(i, j)$, then $A \cup B \in D_{rw}(i, j)$.

Proof: Let $G \in RSO(X, \tau_i)$ be such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since $A, B \in D_{rw}(i, j)$, we have $\tau_i \text{-cl}(A) \subseteq G$ and $\tau_i \text{-cl}(B) \subseteq G$. That is $\tau_i \text{-cl}(A) \cup \tau_i \text{-cl}(B) \subseteq G$. Also $\tau_i \text{-cl}(A) \cup \tau_i \text{-cl}(B) = \tau_i \text{-cl}(A \cup B)$ and so $\tau_i \text{-cl}(A \cup B) \subseteq G$. Therefore $A \cup B \in D_{rw}(i, j)$.

5.2.30 Remark: The intersection of two $(i, j)$-rw-closed sets is generally not a $(i, j)$-rw-closed set as seen from the following example.

5.2.31 Example: Let $X=\{a, b, c, d\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the subsets $\{a, b\}$ and $\{a, c, d\}$ are $(1, 2)$-rw-closed sets, but $\{a, b\} \cap \{a, c, d\} = \{a\}$ is not a $(1, 2)$-rw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.32 Remark: The family $D_{rw}(1, 2)$ is generally not equal to the family $D_{rw}(2, 1)$ as seen from the following example.

5.2.33 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{c\}, \{b, c\}\}$. Then $D_{rw}(1, 2)=\{X, \phi, \{a\}, \{a, b\}\}$ and $D_{rw}(2, 1)=\{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Therefore $D_{rw}(1, 2) \neq D_{rw}(2, 1)$. 

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5.2.34 Theorem: If \( \tau_1 \subset \tau_2 \) and \( \text{RSO}(X, \tau_1) \subset \text{RSO}(X, \tau_2) \) in \((X, \tau_1, \tau_2)\), then \( D_{rw}(\tau_1, \tau_2) \supset D_{rw}(\tau_2, \tau_1) \).

Proof: Let \( A \in D_{rw}(\tau_2, \tau_1) \). That is \( A \) is a \((2, 1)\)-rw-closed set. To prove that \( A \in D_{rw}(\tau_1, \tau_2) \). Let \( G \in \text{RSO}(X, \tau_1) \) be such that \( A \subset G \). Since \( \text{RSO}(X, \tau_1) \subset \text{RSO}(X, \tau_2) \), we have \( G \in \text{RSO}(X, \tau_2) \). As \( A \) is a \((2, 1)\)-rw-closed set, we have \( \tau_1-cl(A) \subset G \). Since \( \tau_1 \subset \tau_2 \), we have \( \tau_2-cl(A) \subset \tau_1-cl(A) \) and it follows that \( \tau_2-cl(A) \subset G \). Hence \( A \) is \((1, 2)\)-rw-closed. That is \( A \in D_{rw}(\tau_1, \tau_2) \). Therefore \( D_{rw}(\tau_1, \tau_2) \supset D_{rw}(\tau_2, \tau_1) \).

5.2.35 Theorem: Let \( i, j \) be fixed integers of \( \{1, 2\} \). For each \( x \) of \((X, \tau_1, \tau_2)\), \( \{x\} \) is a regular semiopen in \((X, \tau_i)\) or \( \{x\}^c \) is \((i, j)\)-rw-closed.

Proof: Suppose \( \{x\} \) is not regular semiopen in \((X, \tau_i)\). Then \( \{x\}^c \) is not regular semiopen in \((X, \tau_i)\). Now regular semiopen in \((X, \tau_i)\) containing \( \{x\}^c \) is \( X \) alone. Also \( \{x\}^c \) is \((i, j)\)-rw-closed.

5.2.36 Theorem: If \( A \) is \((i, j)\)-rw-closed, then \( \tau_i-cl(A)-A \) contains no non-empty \( \tau_i \)-regular semiopen set.

Proof: Let \( A \) be a \((i, j)\)-rw-closed set. Suppose \( F \) is a non-empty \( \tau_i \)-regular semiopen set contained in \( \tau_i-cl(A)-A \). Now \( F \subset X-A \) which implies \( A \subset F^c \). Also \( F^c \) is a \( \tau_i \)-regular semiopen. Since \( A \) is a \((i, j)\)-rw-closed set, we have \( \tau_i-cl(A) \subset F^c \). Consequently \( F \subset \tau_i-cl(A) \cap (\tau_i-cl(A))^c = \emptyset \), which is a contradiction. Hence \( \tau_i-cl(A)-A \) does not contains any non-empty \( \tau_i \)-regular semiopen set.

The converse of this theorem does not hold as seen from the following example.

5.2.37 Example: Let \( X=\{a, b, c\}, \quad \tau_1=\{X, \phi, \{b\}, \{a, c\}\} \) and \( \tau_2=\{X, \phi, \{a, \}\} \). If \( A=\{b\} \), then \( \tau_i-cl(A)-A = \{b, c\}-\{b\} = \{c\} \) does not contain any
non-empty $\tau_i$-regular semiopen set. But $A$ is not a $(1, 2)$-rw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

5.2.38 Corollary: If $A$ is $(i, j)$-rw-closed in $(X, \tau_i, \tau_j)$, then $A$ is $\tau_j$-closed if and only if $\tau_j$-cl$(A)$ - $A$ is a $\tau_i$-regular semiopen set.

Proof: Suppose $A$ is $\tau_j$-closed. Then $\tau_j$-cl$(A)$ = $A$ and so $\tau_j$-cl$(A)$ - $A$ = $\emptyset$, which is a $\tau_i$-regular semiopen set.

Conversely, suppose $\tau_j$-cl$(A)$ - $A$ is a $\tau_i$-regular semiopen. Since $A$ is $(i, j)$-rw-closed, by Theorem 5.2.36, $\tau_j$-cl$(A)$ - $A$ does not contain any non-empty $\tau_i$-regular semiopen set. Therefore $\tau_j$-cl$(A)$ - $A$ = $\emptyset$. That is $\tau_j$-cl$(A)$ = $A$ and hence $A$ is $\tau_j$-closed.

5.2.39 Theorem: In a bitopological space $(X, \tau_i, \tau_j)$, $\text{RSO}(X, \tau_i) \subseteq \{F \subseteq X : F \in \tau_j\}$ if and only if every subset of $(X, \tau_i, \tau_j)$ is a $(i, j)$-rw-closed set.

Proof: Suppose that $\text{RSO}(X, \tau_i) \subseteq \{F \subseteq X : F \in \tau_j\}$. Let $A$ be any subset of $X$. Let $G \in \text{RSO}(X, \tau_i)$ be such that $A \subseteq G$. Then $\tau_j$-cl$(G)$ = $G$. Also $\tau_j$-cl$(A)$ $\subseteq$ $\tau_j$-cl$(G)$ = $G$. That is $\tau_j$-cl$(A)$ $\subseteq$ $G$. Therefore $A$ is a $(i, j)$-rw-closed set.

Conversely, suppose that every subset of $(X, \tau_i, \tau_j)$ is a $(i, j)$-rw-closed set. Let $G \in \text{RSO}(X, \tau_i)$. Since $G \subseteq G$ and $G$ is $(i, j)$-rw-closed, we have $\tau_j$-cl$(G)$ $\subseteq$ $G$. Thus $\tau_j$-cl$(G)$ = $G$ and so $G$ is $\tau_j$-closed. That is $G \in \{F \subseteq X : F \in \tau_j\}$. Hence $\text{RSO}(X, \tau_i) \subseteq \{F \subseteq X : F \in \tau_j\}$.

5.2.40 Theorem: Let $A$ be a $(i, j)$-rw-closed subset of a bitopological space $(X, \tau_i, \tau_j)$. If $A$ is $\tau_i$-regular semiopen, then $A$ is $\tau_j$-closed.

Proof: Let $A$ be $\tau_i$-regular semiopen. Now $A \subseteq A$. Then by hypothesis $\tau_j$-cl$(A)$ $\subseteq$ $A$. Therefore $\tau_j$-cl$(A)$ = $A$. That is $A$ is $\tau_j$-closed.
5.2.41 **Theorem:** If A is a (i, j)-rw-closed set and $\tau_i \subset \text{RSO}(X, \tau_i)$, then $\tau_j\text{-cl}({x}) \cap A \neq \emptyset$ for each $x \in \tau_j\text{-cl}(A)$.

**Proof:** Let A be (i, j)-rw-closed and $\tau_i \subset \text{RSO}(X, \tau_i)$. Suppose $\tau_i\text{-cl}({x}) \cap A = \emptyset$ for some $x \in \tau_j\text{-cl}(A)$, then $A \subset (\tau_i\text{-cl}({x}))^c$. Now $(\tau_i\text{-cl}({x}))^c \in \tau_i \subset \text{RSO}(X, \tau_i)$, by hypothesis. That is $(\tau_i\text{-cl}({x}))^c$ is $\tau_i$-regular semiopen. Since A is (i, j)-rw-closed, we have $\tau_j\text{-cl}(A) \subset (\tau_i\text{-cl}({x}))^c$. This shows that $x \notin \tau_j\text{-cl}(A)$. This contradicts the assumption.

5.2.42 **Theorem:** If A is a (i, j)-rw-closed set and $A \subset B \subset \tau_j\text{-cl}(A)$, then B is (i, j)-rw-closed.

**Proof:** Let G be a $\tau_i$-regular semiopen set such that $B \subset G$. As A is a (i, j)-rw-closed set and $A \subset G$, we have $\tau_j\text{-cl}(A) \subset G$. Now $B \subset \tau_j\text{-cl}(A)$ which implies, $\tau_j\text{-cl}(B) \subset \tau_j\text{-cl}(\tau_j\text{-cl}(A)) = \tau_j\text{-cl}(A) \subset G$. Thus $\tau_j\text{-cl}(B) \subset G$. Therefore B is a (i, j)-rw-closed set.

5.2.43 **Theorem:** Let $A \subset Y \subset X$ and suppose that A is (i, j)-rw-closed in $(X, \tau_i, \tau_2)$. Then A is (i, j)-rw-closed relative to Y provided Y is a $\tau_i$-regular open set.

**Proof:** Let $\tau_{i,Y}$ be the restriction of $\tau_i$ to Y. Let G be a $\tau_{i,Y}$-regular semiopen set such that $A \subset G$. Since $A \subset Y \subset X$ and Y is $\tau_i$-regular open, by the Lemma 2.2.34, G is $\tau_i$-regular semiopen. Since A is (i, j)-rw-closed, $\tau_j\text{-cl}(A) \subset G$. That is $Y \cap \tau_j\text{-cl}(A) \subset Y \cap G = G$. Also $Y \cap \tau_j\text{-cl}(A) = \tau_{i,Y}\text{-cl}(A)$. Thus $\tau_{i,Y}\text{-cl}(A) \subset G$. Hence A is (i, j)-rw-closed relative to Y.

5.2.44 **Theorem:** In a bitopological space $(X, \tau_i, \tau_2)$, if $\text{RSO}(X, \tau_i) = \{X, \emptyset\}$, then every subset of $(X, \tau_i, \tau_2)$ is (i, j)-rw-closed.

**Proof:** Let $\text{RSO}(X, \tau_2) = \{X, \emptyset\}$ in a bitopological space $(X, \tau_i, \tau_2)$. Let A be any subset of X. To prove that A is an (i, j)-rw-closed. Suppose $A = \emptyset$,
then A is (i, j)-rw-closed. Suppose A ≠ \emptyset, then X is the only \tau_i-regular semiopen set and \tau_i-cl(A) \subseteq X. Hence A is a (i, j)-rw-closed set.

The converse of the above theorem need not be true in general as seen from the following example.

5.2.45 Example: Let X= \{a, b, c\}, \tau_1= \{X, \emptyset, \{a\}, \{b, c\}\} and \tau_2=\{ X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} Then every subset of X is a (1, 2)-rw-closed set but RSO(X, \tau_1) = \{X, \emptyset, \{a\}, \{b, c\}\}.

5.2.46 Theorem: If A is \tau_i-regular open and (i, j)-rg-closed, then A is (i, j)-rw-closed.

Proof: Let G be a \tau_i-regular semiopen set such that A \subseteq G. Now A \subseteq A, A is \tau_i-regular open and (i, j)-rg-closed, we have \tau_i-cl(A) \subseteq A. That is \tau_i-cl(A) \subseteq G. Therefore A is (i, j)-rw-closed.

5.2.47 Theorem: If A is \tau_i-open and (i, j)-g-closed, then A is (i, j)-rw-closed.

Proof: Let G be a \tau_i-regular semiopen set such that A \subseteq G. Now A \subseteq A, A is \tau_i-open and (i, j)-g-closed, we have \tau_i-cl(A) \subseteq A. That is \tau_i-cl(A) \subseteq G. Therefore A is (i, j)-rw-closed.

5.2.48 Theorem: Suppose that B \subseteq A \subseteq X, B is a (i, j)-rw-closed set relative to A and that A is both \tau_i-clopen and \tau_j-closed. Then B is (i, j)-rw-closed set in (X, \tau_1, \tau_2).

Proof: Let \tau_{i,A}-be the restriction of \tau_i to A. Let B \subseteq G and G be \tau_i-regular semiopen. But it is given that B \subseteq A \subseteq X. Therefore B \subseteq G and B \subseteq A, which implies B \subseteq A \cap G. Now we show that A \cap G is \tau_{i,A-} regular semiopen. Since A is \tau_i-open and G is \tau_i-semiopen, A \cap G is \tau_i-semiopen. Since A is \tau_i-closed and G is \tau_i-semi-closed, A \cap G is \tau_i-semi-closed.
Thus $A \cap G$ is both $\tau_j$-semiopen and $\tau_j$-semi-closed and hence $A \cap G$ is $\tau_j$-regular semiopen. Since $A \cap G \subseteq A \subseteq X$, by Lemma 2.2.34, $A \cap G$ is $\tau_{i,A}$-regular semiopen.

Since $B$ is a $(i, j)$-rw-closed set relative to $A$, $\tau_{j,A} \text{-cl}(B) \subseteq A \cap G$—(i). But $\tau_{j,A} \text{-cl}(B) = A \cap \tau_j \text{-cl}(B)$—(ii). From (i) and (ii), it follows that $A \cap \tau_j \text{-cl}(B) \subseteq A \cap G$. Consequently $A \cap \tau_j \text{-cl}(B) \subseteq G$. Since $A$ is $\tau_j$-closed, $\tau_j \text{-cl}(A) = A$ and $\tau_j \text{-cl}(B) \subseteq \tau_j \text{-cl}(A) = A$, we have $A \cap \tau_j \text{-cl}(B) = \tau_j \text{-cl}(B)$. Thus $\tau_j \text{-cl}(B) \subseteq G$ and hence $B$ is $(i, j)$-rw-closed set in $(X, \tau_1, \tau_2)$.

5.3 $(\tau_i, \tau_j)$-rw-open sets and their basic properties.

In this section, we introduce $(i, j)$-rw-open sets in bitopological spaces and study some of their properties.

5.3.1 Definition: Let $i, j \in \{1, 2\}$ be fixed integers. In a bitopological space $(X, \tau_1, \tau_2)$, a subset $A \subseteq X$ is said to be $(\tau_i, \tau_j)$-rw-open (briefly, $(i, j)$-rw-open) if $A^c$ is $(i, j)$-rw-closed.

5.3.2 Theorem: In a bitopological space $(X, \tau_1, \tau_2)$, we have the following

(i) Every $(i, j)$-w-open set is $(i, j)$-rw-open but not conversely.

(ii) Every $(i, j)$-rw-open set is $(i, j)$-rg-open but not conversely.

(iii) Every $\tau_j$-open set is $(i, j)$-rw-open but not conversely.

(iv) Every $(i, j)$-rw-open set is $(i, j)$-gpr-open but not conversely.

Proof: The proof follows from the Theorems 5.2.3, 5.2.5, 5.2.7, and 5.2.18.

5.3.3 Theorem: If $A$ and $B$ are $(i, j)$-rw-open sets, then $A \cap B$ is $(i, j)$-rw-open.
Proof: The proof follows from the Theorem 5.2.29.

5.3.4 Remark: The union of two (i, j)-rw-open sets is generally not an (i, j)-rw-open set as seen from the following example.

5.3.5 Example Let $X=\{a, b, c, d\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the subsets $\{c, d\}$ and $\{b\}$ are (1, 2)-rw-open sets, but $\{c, d\} \cup \{b\} = \{b, c, d\}$ is not (1, 2)-rw-open set in the bitopological space $(X, \tau_1, \tau_2)$.

5.3.6 Theorem: A subset $A$ of $(X, \tau_1, \tau_2)$ is (i, j)-rw-open if and only if $F \subseteq \tau_j$-int$(A)$, whenever $F$ is $\tau$-regular semiopen and $F \subseteq A$.

Proof: Suppose that $F \subseteq \tau_j$-int$(A)$ whenever $F \subseteq A$ and $F$ is $\tau$-regular semiopen. To prove that $A$ is (i, j)-rw-open. Let $G$ be $\tau$-regular semiopen and $A^c \subseteq G$. Then $G^c \subseteq A$ and $G^c$ is $\tau$-regular semiopen, by Lemma 1.2.5. By hypothesis, $G^c \subseteq \tau_j$-int$(A)$. That is $(\tau_j$-int$(A))^c \subseteq G$, since $\tau_j$-cl$(A^c) = (\tau_j$-int$(A))^c$. Thus $A^c$ is (i, j)-rw-closed. That is $A$ is (i, j)-rw-open.

Conversely, suppose that $A$ is (i, j)-rw-open, $F \subseteq A$ and $F$ is $\tau$-regular semiopen. To prove that $A$ is (i, j)-rw-open. Let $G$ be $\tau$-regular semiopen and $A^c \subseteq G$. Then $G^c \subseteq (\tau_j$-int$(A))^c = (\tau_j$-int$(A))^c$. Since $A^c$ is (i, j)-rw-closed, we have $\tau_j$-cl$(A^c) \subseteq F^c$ and so $F \subseteq \tau_j$-int$(A)$, since $\tau_j$-cl$(A^c) = (\tau_j$-int$(A))^c$.

5.3.7 Theorem: Let $A$ and $G$ be two subsets of a bitopological space $(X, \tau_1, \tau_2)$. If the set $A$ is (i, j)-rw-open, then $G=X$, whenever $G$ is $\tau$-regular semiopen and $\tau_j$-int$(A) \cup A^c \subseteq G$.

Proof: Let $A$ be (i, j)-rw-open, $G$ be the $\tau$-regular semiopen and $\tau_j$-int$(A) \cup A^c \subseteq G$. Then $G^c \subseteq (\tau_j$-int$(A) \cup A^c)^c = (\tau_j$-int$(A))^c \setminus A^c$. Since $A^c$ is (i, j)-rw-closed and $G^c$ is $\tau$-regular semiopen, by Theorem 5.2.36, it follows that $G^c=\phi$, Therefore $G=X$. 

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The converse of the above theorem need not be true as seen from the following example.

5.3.8 Example: Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}\} \). If \( A = \{a, c\} \), then the only \( \tau_1 \)-regular semiopen set containing \( \tau_2 \text{-int}(A) \cup A^c \) is \( X \). But \( A \) is not \((1, 2)\)-rw-open set in \((X, \tau_1, \tau_2)\).

5.3.9 Theorem: If a subset \( A \) of \((X, \tau_1, \tau_2)\) is \((i, j)\)-rw-closed, then \( \tau_j \text{-cl}(A) - A \) is \((i, j)\)-rw-open.

Proof: Let \( A \) be a \((i, j)\)-rw-closed subset in \((X, \tau_1, \tau_2)\). Let \( F \) be a \( \tau_j \)-regular semiopen set such that \( F \subset \tau_j \text{-cl}(A) - A \). By Theorem 5.2.36, \( F = \phi \). Therefore \( F \subset \tau_j \text{-int}(\tau_j \text{-cl}(A) - A) \) and by Theorem 5.3.6, \( \tau_j \text{-cl}(A) - A \) is \((i, j)\)-rw-open.

The converse of the above theorem need not be true as seen from the following example.

5.3.10 Example: For the subset \( A = \{b\} \) in \( X = \{a, b, c\} \) of Example 5.3.8, \( \tau_2 \text{-cl}(A) - A = \{b, c\} - \{b\} = \{c\} \) is \((1, 2)\)-rw-open but \( A = \{b\} \), is not \((1, 2)\)-rw-closed.

5.3.11 Theorem: If \( \tau_j \text{-int}(A) \subset B \subset A \) and \( A \) is \((i, j)\)-rw-open in \((X, \tau_1, \tau_2)\), then \( B \) is \((i, j)\)-rw-open.

Proof: Let \( F \) be \( \tau_j \)-regular semiopen such that \( F \subset B \). Now \( F \subset B \subset A \). That is \( F \subset A \). Since \( F \) is \((i, j)\)-rw-open, by Theorem 5.3.6, \( F \subset \tau_j \text{-int}(A) \). By hypothesis \( \tau_j \text{-int}(A) \subset B \). Therefore \( \tau_j \text{-int} (\tau_j \text{-int}(A)) \subset \tau_j \text{-int}(B) \). That is \( \tau_j \text{-int}(A) \subset \tau_j \text{-int}(B) \) and hence \( F \subset \tau_j \text{-int}(B) \). Again by Theorem 5.3.6, \( B \) is a \((i, j)\)-rw-open set in \((X, \tau_1, \tau_2)\).

5.3.12 Corollary: Let \( A \) and \( B \) be subsets of a space \((X, \tau_1, \tau_2)\). If \( B \) is \((i, j)\)-rw-open and \( A \supset \tau_j \text{-int}(B) \), then \( A \cap B \) is \((i, j)\)-rw-open.
Proof: Let B be (i, j)-rw-open and A ⇒ xj-int(B). That is \( \tau_j\)-int(B) ⊆ A. Then \( \tau_j\)-int(B) ⊆ A ∩ B. Also \( \tau_j\)-int(B) ⊆ A ∩ B ⊆ B and B is (i, j)-rw-open. By Theorem 5.3.11, A ∩ B is also (i, j)-rw-open.

5.3.13 Theorem: Every singleton point set in a space \((X, \tau_1, \tau_2)\) is either (i, j)-rw-open or \(\tau_i\)-regular semiopen.

Proof: Let \((X, \tau_1, \tau_2)\) be a bitopological space. Let \(x \in X\). To prove \(\{x\}\) is either (i, j)-rw-open or \(\tau_i\)-regular semiopen. That is to prove \(X - \{x\}\) is either (i, j)-rw-closed or \(\tau_i\)-regular semiopen, which follows from Theorem 5.2.35.

5.4 \((\tau_i, \tau_j)\)-rw-closure in bitopological spaces.

W. Dunham [33] introduced the concept of generalized closure operator \(C^*\) and Fukutake [34] introduced and studied the concept of pairwise generalized closure operator \((\tau_i, \tau_j)\)-cl* in a bitopological spaces. Analogous to that we introduce the pairwise rw-closure operator \((i, j)\)-rw-cl in bitopological spaces and study some of their properties.

5.4.1 Definition: Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(i, j \in \{1, 2\}\) be fixed integers. For each subset \(E\) of \(X\), define \((\tau_i, \tau_j)\)-rw-cl \((E) = \bigcap\{A: E \subseteq A \in D_{rw}(i, j)\}\) (briefly: \((i, j)\)-rw-cl \((E)\)).

5.4.2 Theorem: If \(A\) and \(B\) be subsets of \((X, \tau_1, \tau_2)\). Then

(i) \((i, j)\)-rw-cl \((X) = X\) and \((i, j)\)-rw-cl \((\emptyset) = \emptyset\).

(ii) \(A \subseteq (i, j)\)-rw-cl \((A)\).

(iii) If \(B\) is any \((i, j)\)-rw-closed set containing \(A\), then \((i, j)\)-rw-cl \((A) \subseteq B\).

Proof: Follows form the Definition 5.4.1.
5.4.3 Theorem: Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers. If $A \subseteq B$, then $(i, j)$-rw-cl($A$) $\subseteq (i, j)$-rw-cl($B$).

Proof: Let $A \subseteq B$. By Definition 5.4.1, $(i, j)$-rw-cl($B$) $= \cap \{F: B \subseteq F \in D_{rw}(i, j)\}$
If $B \subseteq F \in D_{rw}(i, j)$, since $A \subseteq B$, $A \subseteq B \subseteq F \in D_{rw}(i, j)$, we have $(i, j)$-rw-cl($A$) $\subseteq F$. Therefore $(i, j)$-rw-cl($A$) $\subseteq \cap \{F: B \subseteq F \in D_{rw}(i, j)\} = (i, j)$-rw-cl($B$).
That is $(i, j)$-rw-cl($A$) $\subseteq (i, j)$-rw-cl($B$).

5.4.4 Theorem: Let $A$ be a subset of $(X, \tau_1, \tau_2)$. If $\tau_1 \subseteq \tau_2$ and $RSO(X, \tau_1) \subseteq RSO(X, \tau_2)$, then $(1, 2)$-rw-cl($A$) $\subseteq (2, 1)$-rw-cl($A$).

Proof: By definition 5.4.1, $(1, 2)$-rw-cl($A$) $= \cap \{F: A \subseteq F \in D_{rw}(1, 2)\}$. Since $\tau_1 \subseteq \tau_2$, by Theorem 5.2.34, $D_{rw}(2, 1) \subseteq D_{rw}(1, 2)$. Therefore $\cap \{F: A \subseteq F \in D_{rw}(1, 2)\} \subseteq \cap \{F: A \subseteq F \in D_{rw}(2, 1)\}$. That is $(1, 2)$-rw-cl($A$) $\subseteq \cap \{F: A \subseteq F \in D_{rw}(2, 1)\} = (2, 1)$-rw-cl($A$). Hence $(1, 2)$-rw-cl($A$) $\subseteq (2, 1)$-rw-cl($A$).

5.4.5 Theorem: Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers, then $A \subseteq \tau_j$-cl($A$) $\subseteq \tau_j$-cl($A$).

Proof: By Definition 5.4.1, it follows that $A \subseteq (i, j)$-rw-cl($A$). Now to prove that $(i, j)$-rw-cl($A$) $\subseteq \tau_j$-cl($A$). By definition of closure, $\tau_j$-cl($A$) $= \cap \{F \subseteq X: A \subseteq F$ and $F$ is $\tau_j$-closed}. If $A \subseteq F$ and $F$ is $\tau_j$-closed, then $F$ is $(i, j)$-rw-closed, as every $\tau_j$-closed set is is $(i, j)$-rw-closed. Therefore $(i, j)$-rw-cl($A$) $\subseteq \cap \{F \subseteq X: A \subseteq F$ and $F$ is $\tau_j$-closed} = $\tau_j$-cl($A$). Hence $(i, j)$-rw-cl($A$) $\subseteq \tau_j$-cl($A$).

5.4.6 Remark: Containment relation in the above theorem may be proper as seen from the following example.

5.4.7 Example: Let $X$={$a, b, c, d$}, $\tau_1$={$X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}$} and $\tau_2$={$X, \phi, \{a, b\}, \{c, d\}$}. Then $\tau_2$-closed sets are $X, \phi, \{a, b\}, \{c, d\}$ and $(1, 2)$-rw-closed sets are $X, \phi, \{a, b\}, \{c, d\}$ and $\{a, b, c\}$ and $\{a, b, d\}$.
Take $A = \{b, c\}$. Then $\tau_2\text{-cl}(A) = X$ and $(1, 2)\text{-rw-cl}(A) = \{a, b, c\}$. Now $A \subseteq (1, 2)\text{-rw-cl}(A)$, but $A \neq (1, 2)\text{-rw-cl}(A)$. Also $(1, 2)\text{-rw-cl}(A) \subseteq \tau_2\text{-cl}(A)$, but $(i, j)\text{-rw-cl}(A) \neq \tau_j\text{-cl}(A)$.

**5.4.8 Theorem:** Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers. If $A$ is $(i, j)$-rw-closed, then $(i, j)$-rw-cl$(A) = A$.

**Proof:** Let $A$ be a $(i, j)$-rw-closed subset of $(X, \tau_1, \tau_2)$. We know that $A \subseteq (i, j)$-rw-cl$(A)$. Also $A \subseteq A$ and $A$ is $(i, j)$-rw-closed. By the Theorem 5.4.2 (iii), $(i, j)$-rw-cl$(A) \subseteq A$. Hence $(i, j)$-rw-cl$(A) = A$.

**5.4.9 Remark:** The converse of the above Theorem 5.4.8 need not be true as seen from the following example.

**5.4.10 Example:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then $(1, 2)$-rw-closed sets are $X, \phi, \{a\}, \{a, b\}, \{b, c\}$. Take $A = \{b\}$. Now $(1, 2)$-rw-cl$(A) = X \cap \{a, b\} \cap \{b, c\} = \{b\}$, but $\{b\}$ is not a $(1, 2)$-rw-closed set.

**5.4.11 Theorem:** The operator $(i, j)$-rw-closure in Definition 5.4.1, is the Kuratowski closure operator on $X$.

**Proof:** (i) $(i, j)$-rw-cl$(\phi) = \phi$, by Theorem 5.4.2 (i).

(ii) $E \subset (i, j)$-rw-cl$(E)$ for any subset $E$ of $X$ by Theorem 5.4.2 (ii).

(iii) Suppose $E$ and $F$ are two subsets of $(X, \tau_1, \tau_2)$. It follows from Theorem 5.4.3, that $(i, j)$-rw-cl$(E) \subset (i, j)$-rw-cl$(E \cup F)$ and that $(i, j)$-rw-cl$(F) \subset (i, j)$-rw-cl$(E \cup F)$. Hence we have $(i, j)$-rw-cl$(E) \cup (i, j)$-rw-cl$(F) \subset (i, j)$-rw-cl$(E \cup F)$.

Now if $x$ does not belong to $(i, j)$-rw-cl$(E) \cup (i, j)$-rw-cl$(F)$, then $x \notin (i, j)$-rw-cl$(E)$ and $x \notin (i, j)$-rw-cl$(F)$, it follows that there exist $A, B \in D_{rw}(i, j)$ such that $E \subseteq A$, $x \notin A$ and $F \subseteq B$, $x \notin B$. Hence $E \cup F \subseteq A \cup B$, 

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x \not\in A \cup B. Since A \cup B is (i, j)-rw-closed, by Theorem 5.2.29, x does not belong to (i, j)-rw-cl(E \cup F). Then we have (i, j)-rw-cl(E \cup F) \subset (i, j)-rw-cl(E) \cup (i, j)-rw-cl(F). From the above discussion we have (i, j)-rw-cl(E \cup F) = (i, j)-rw-cl(E) \cup (i, j)-rw-cl(F).

(iv) Let E be any subset of (X, \tau_1, \tau_2). By the definition of (i, j)-rw-closure, (i, j)-rw-cl (E) = \cap \{A \subset X : E \subset A \in D_{rw}(i, j)\}. If E \subset A \in D_{rw}(i, j), then (i, j)-rw-cl (E) \subset A. Since A is a (i, j)-rw-closed set containing (i, j)-rw-cl (E), by Theorem 5.4.2 (iii), (i, j)-rw-cl\{(i, j)-rw-cl(E)\} \subset A, Hence (i, j)-rw-cl\{(i, j)-rw-cl(E)\} \subset \cap \{A \subset X : E \subset A \in D_{rw}(i, j)\} = (i, j)-rw-cl(E).

Conversely (i, j)-rw-cl (E) \subset (i, j)-rw-cl((i, j)-rw-cl(E)) is true by Theorem 5.4.2 (iii). Then we have (i, j)-rw-cl(E) = (i, j)-rw-cl((i, j)-rw-cl(E)). Hence (i, j)-rw-closure is a Kuratowski closure operator on X.

From the above Theorem 5.4.11, (i, j)-rw-closure defines the new topology on X.

5.4.12 Definition: Let i, j \in \{1, 2\} be two fixed integers. Let \tau_{rw}(i, j) be the topology on X generated by (i, j)-rw-closure in the usual manner. That is \tau_{rw}(i, j) = \{E \subset X : (i, j)-rw-cl(E^c) = E^c\}.

5.4.13 Theorem: Let (X, \tau_1, \tau_2) be a bitopological space and i, j \in \{1, 2\} be two fixed integers, then \tau_j \subset \tau_{rw}(i, j).

Proof: Let G \in \tau_j. It follows that G^c is \tau_j-closed. By Theorem 5.2.7, G^c is (i, j)-rw-closed. Therefore (i, j)-rw-cl (G^c) = G^c, by Theorem 5.4.8. That is G \in \tau_{rw}(i, j) and hence \tau_j \subset \tau_{rw}(i, j).

5.4.14 Remark: Containment relation in the above Theorem 5.4.13 may be proper as seen from the following example.

5.4.15 Example: Let X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} and \tau_2 = \{X, \phi, \{a\}, \{b, c\}\}. Then (1, 2)-rw-closed sets are X, \phi, \{a\}, \{a, b\}, \{b, c\} and
\[ \tau_{rw}(1, 2) = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}. \] Clearly \( \tau_2 \subseteq \tau_{rw}(1, 2) \), but \( \tau_2 \neq \tau_{rw}(1, 2) \).

5.4.16 Theorem: Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(i, j \in \{1, 2\}\) be two fixed integers. If a subset \(E\) of \(X\) is \((i, j)\)-rw-closed, then \(E\) is \(\tau_{rw}(i, j)\)-closed.

Proof: Let a subset \(E\) of \(X\) be \((i, j)\)-rw-closed. By Theorem 5.4.8, \((i, j)\)-rw-cl \((E) = E\). That is \((i, j)\)-rw-cl \((E^c)^c\) = \((E^c)^c\). It follows that \(E^c \in \tau_{rw}(i, j)\). Therefore \(E\) is \(\tau_{rw}(i, j)\)-closed.

5.4.17 Remark: The converse of the above theorem need not be true as seen from the following example.

5.4.18 Example: For \((X, \tau_1, \tau_2)\) of Example 5.4.15, the subset \(A = \{b\}\) is \(\tau_{rw}(1, 2)\)-closed, but not \((1, 2)\)-rw-closed.

5.4.19 Corollary: For any point \(x\) of \((X, \tau_1, \tau_2)\), \(\{x\}\) is \(\tau_i\)-regular semiopen or \(\tau_{rw}(i, j)\)-open.

Proof: Let \(x\) be any point of \((X, \tau_1, \tau_2)\). By Theorem 5.2.35, \(\{x\}\) is \(\tau_i\)-regular semiopen or \(\{x\}^c\) is \((i, j)\)-rw-closed. That is \(\{x\}^c\) is \(\tau_{rw}(i, j)\)-closed, by above Theorem 5.4.16. Therefore \(\{x\}\) is \(\tau_i\)-regular semiopen or \(\tau_{rw}(i, j)\)-open.

5.4.20 Theorem: If \(\tau_1 \subset \tau_2\) and \(\text{RSO}(X, \tau_1) \subset \text{RSO}(X, \tau_2)\) in \((X, \tau_1, \tau_2)\), then \(\tau_{rw}(2, 1) \subset \tau_{rw}(1, 2)\).

Proof: Let \(G \in \tau_{rw}(2, 1)\). Then \((2, 1)\)-rw-cl \((G^c) = G^c\). To prove that \(G \in \tau_{rw}(1, 2)\). That is to prove \((1, 2)\)-rw-cl \((G^c) = G^c\). Now \((1, 2)\)-rw-cl \((G^c) = \cap \{F \subset X : \text{G}^c \subset F \in D_{rw}(1, 2)\}\). Since \(\tau_1 \subset \tau_2\) and \(\text{RSO}(X, \tau_1) \subset \text{RSO}(X, \tau_2)\), by Theorem 5.2.34, \(D_{rw}(2, 1) \subset D_{rw}(1, 2)\). Thus \(\cap \{F \subset X : \text{G}^c \subset F \in D_{rw}(1, 2)\}\)
\( \{ F \subseteq X : G^c \subseteq \text{Fe } D_{rw}(2, 1) \} \). That is \((1, 2)\)-rw-\(\text{cl}(G^c) \subseteq (2, 1)\)-rw-\(\text{cl}(G^c) = G^c \), and so \((1, 2)\)-rw-\(\text{cl}(G^c) \subseteq G^c \).

Conversely \( G^c \subseteq (1, 2)\)-rw-\(\text{cl}(G^c) \) is true by the Theorem 5.4.2 (ii).

Then we have \((1, 2)\)-rw-\(\text{cl}(G^c) = G^c \). That is \( G \in \tau_{rw}(1, 2) \) and hence \( \tau_{rw}(2, 1) \subseteq \tau_{rw}(1, 2) \).

### 5.5 \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous maps and \( rw\)-bi-continuous maps.

In this section a new class of maps called \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous maps in bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every \( C(i, j)\)-\(\sigma_k\)-continuous map is \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous which implies \( D_r(i, j)\)-\(\sigma_k\)-continuous. Also, we introduce the concept of \( rw\)-bi-continuity and \( rw\)-s-bi-continuity in bitopological spaces and study some of their properties.

#### 5.5.1 Definition: A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous if the inverse image of every \( \sigma_k\)-closed set is an \((i, j)\)-rw-closed set in \((X, \tau_1, \tau_2)\).

#### 5.5.2 Remark: If \( \tau_1 = \tau_2 = \tau \) and \( \sigma_1 = \sigma_2 = \sigma \) in Definition 5.5.1, then the \( D_{rw}(i, j)\)-\(\sigma_k\)-continuity of maps coincides with \( rw\)-continuity of maps in topological spaces.

#### 5.5.3 Theorem: If a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( \tau_j\)-\(\sigma_k\)-continuous, then it is a \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous.

**Proof:** Let \( V \) be a \( \sigma_k\)-closed set. Since \( f \) is \( \tau_j\)-\(\sigma_k\)-continuous, \( f^{-1}(V) \) is \( \tau_j\)-closed. By Theorem 5.2.7, \( f^{-1}(V) \) is \((i, j)\)-rw-closed in \((X, \tau_1, \tau_2)\). Therefore \( f \) is \( D_{rw}(i, j)\)-\(\sigma_k\)-continuous.
The converse of this Theorem need not be true in general as seen from the following example.

**5.5.4 Example:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{b\}\}$ and $\tau_2 = \{X, \phi, \{b, c\}\}$, $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}\}$ and $\sigma_2 = \{Y, \phi, \{q\}\}$. Define a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = p$, $f(b) = f(c) = q$. Then $f$ is $D_{rw}(2, 1) - \sigma_2$-continuous but it is not $\tau_1 - \sigma_2$-continuous, since for the $\sigma_2$-closed set $\{p\}$, $f^{-1}(\{p\}) = \{a\}$ which is not $\tau_1$-closed.

**5.5.5 Theorem:** If a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $C(i, j) - \sigma_k$-continuous, then it is $D_{rw}(i, j) - \sigma_k$-continuous.

**Proof:** Let $V$ be a $\sigma_k$-closed set. Since $f$ is $C(i, j) - \sigma_k$-continuous, $f^{-1}(V)$ is $(i, j)$-w-closed. By Theorem 5.2.3, $f^{-1}(V)$ is $(i, j)$-rw-closed in $(X, \tau_1, \tau_2)$. Therefore $f$ is $D_{rw}(i, j) - \sigma_k$-continuous.

The converse of this Theorem need not be true in general as seen from the following example.

**5.5.6 Example:** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a, b\}\}$ and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}\}$ and $\sigma_2 = \{Y, \phi, \{p, \{q\}\}\}$. Define a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = f(b) = q$ and $f(c) = p$. Then $f$ is $D_{rw}(1, 2) - \sigma_1$-continuous but it is not $C(1, 2) - \sigma_1$-continuous, since for the $\sigma_1$-closed set $\{q\}$, $f^{-1}(\{q\}) = \{a, b\}$ which is not $(1, 2)$-w-closed set.

**5.5.7 Theorem:** If a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D_{rw}(i, j) - \sigma_k$-continuous, then it is $D_l(i, j) - \sigma_k$-continuous.

**Proof:** Let $V$ be a $\sigma_k$-closed set. Since $f$ is $D_{rw}(i, j) - \sigma_k$-continuous, $f^{-1}(V)$ is $(i, j)$-rw-closed. By Theorem 5.2.5, $f^{-1}(V)$ is $(i, j)$-rg-closed in $(X, \tau_1, \tau_2)$. Therefore $f$ is $D_l(i, j) - \sigma_k$-continuous.
The converse of this Theorem need not be true in general as seen from the following example.

5.5.8 Example: Let $X=\{a, b, c, d\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\tau_2=\{X, \phi, \{a, b\}, \{c, d\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi, \{p\}\}$ and $\sigma_2=\{Y, \phi\}$. Define a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a)=f(c)=p$ and $f(b)=f(d)=q$. Then $f$ is $D_r(1, 2)-\sigma_1$-continuous but it is not $D_{rw}(1, 2)-\sigma_1$-continuous, since for the $\sigma_1$-closed set $\{p\}$, $f^{-1}(\{p\})=\{a, c\}$ which is not a $(1, 2)$-rw-closed set.

5.5.9 Theorem: If a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $D_r(i, j)-\sigma_k$-continuous, then it is $\zeta(i, j)-\sigma_k$-continuous.

Proof: Let $V$ be a $\sigma_k$-closed set. Since $f$ is $D_r(i, j)-\sigma_k$-continuous, $f^{-1}(V)$ is $(i, j)$-rw-closed. By Theorem 5.2.18, $f^{-1}(V)$ is $(i, j)$-gpr-closed in $(X, \tau_1, \tau_2)$. Therefore $f$ is $\zeta(i, j)-\sigma_k$-continuous.

The converse of this Theorem need not be true in general as seen from the following example.

5.5.10 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a, b\}, \{c\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi\}$ and $\sigma_2=\{Y, \phi, \{p\}\}$. Define a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a)=f(b)=p$ and $f(c)=q$. Then this function $f$ is $\zeta(1, 2)-\sigma_2$-continuous but it is not $D_{rw}(1, 2)-\sigma_2$-continuous, since for the $\sigma_2$-closed set $\{q\}$, $f^{-1}(\{q\})=\{c\}$ which is not $(1, 2)$-rw-closed in $(X, \tau_1, \tau_2)$.

5.5.11 Remark: $D_{rw}(i, j)-\sigma_k$-continuous maps and $D(i, j)-\sigma_k$-continuous maps are independent.

5.5.12 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi\}$ and $\sigma_2=\{Y, \phi, \{q\}\}$. Define a map $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a)=f(b)=p$ and $f(c)=q$. Then this function
f is $D_{rw}(1, 2)$-$\sigma_2$-continuous but it is not $D(1, 2)$-$\sigma_2$-continuous, since for the $\sigma_2$-closed set \{p\}, $f^{-1}(\{p\})=\{a, b\}$ which is not $(1, 2)$-g-closed in $(X, \tau_1, \tau_2)$.

5.5.13 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b, c\}\}$ and $Y=\{p, q\}$, $\sigma_1=P(Y)$ and $\sigma_2=\{Y, \phi, \{q\}\}$. Define a map $f:(X, \tau_1, \tau_2)\to(Y, \sigma_1, \sigma_2)$ by $f(a)=f(c)=p$ and $f(b)=q$. Then this function $f$ is $D(1, 2)$-$\sigma_2$-continuous but it is not $D_{rw}(1, 2)$-$\sigma_2$-continuous, since for the $\sigma_2$-closed set \{p\}, $f^{-1}(\{p\})=\{a, c\}$ which is not $(1, 2)$-rw-closed in $(X, \tau_1, \tau_2)$.

5.5.14 Remark: $D_{rw}(i, j)$-$\sigma_k$-continuous maps and $W(i, j)$-$\sigma_k$-continuous maps are independent.

5.5.15 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}\}$ and $Y=\{p, q\}$, $\sigma_1=P(Y)$ and $\sigma_2=\{Y, \phi, \{q\}\}$. Define a map $f:(X, \tau_1, \tau_2)\to(Y, \sigma_1, \sigma_2)$ by $f(a)=f(c)=p$ and $f(b)=q$. Then this function $f$ is $D_{rw}(1, 2)$-$\sigma_2$-continuous but it is not $W(1, 2)$-$\sigma_2$-continuous, since for the $\sigma_2$-closed set \{p\}, $f^{-1}(\{p\})=\{a, c\}$ which is not $(1, 2)$-wg-closed in $(X, \tau_1, \tau_2)$.

5.5.16 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{c\}, \{b, c\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi\}$ and $\sigma_2=\{Y, \phi, \{p\}\}$. Define a map $f:(X, \tau_1, \tau_2)\to(Y, \sigma_1, \sigma_2)$ by $f(a)=p$ and $f(b)=f(c)=q$. Then this function $f$ is $W(1, 2)$-$\sigma_2$-continuous but it is not $D_{rw}(1, 2)$-$\sigma_2$-continuous, since for the $\sigma_2$-closed set \{q\}, $f^{-1}(\{q\})=\{b, c\}$ which is not $(1, 2)$-rw-closed in $(X, \tau_1, \tau_2)$. 

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5.5.17 **Remark:** From the above discussions and known results we have the following implications. From

\[ A \rightarrow B \] means A implies B, but not conversely and from

\[ A \leftrightarrow B \] means A and B are independent of each other

\[
\begin{align*}
C(i,j)-\sigma_k\text{-continuity} \quad & \quad \zeta(i,j)-\sigma_k\text{-continuity} \\
\tau_j-\sigma_k\text{-continuity} \quad & \quad D_{rw}(i,j)-\sigma_k\text{-continuity} \\
W(i,j)-\sigma_k\text{-continuity} \quad & \quad D(i,j)-\sigma_k\text{-continuity}
\end{align*}
\]

Figure-5.2

5.5.18 **Theorem:** The following statements are equivalent

(i) A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( D_{rw}(i,j)-\sigma_k\text{-continuous} \).

(ii) The inverse image of every \( \sigma_k\text{-open} \) set in \( Y \) is \( (i,j)-rw\text{-open} \) in \( X \).

**Proof:** (i) \( \Rightarrow \) (ii) Let \( G \) be a \( \sigma_k\text{-open} \) set in \( Y \). Then \( G^C \) is \( \sigma_k\text{-closed} \) set in \( Y \). Since \( f \) is \( D_{rw}(i,j)-\sigma_k\text{-continuous} \), \( f^{-1}(G^C) \) is \( (i,j)-rw\text{-closed} \) in \( X \). That is \( f^{-1}(G^C) = (f^{-1}(G))^C \) and so \( f^{-1}(G) \) is \( (i,j)-rw\text{-open} \) in \( (X, \tau_1, \tau_2) \).

(ii) \( \Rightarrow \) (i) Let \( F \) be a \( \sigma_k\text{-closed} \) set in \( Y \). Then \( F^C \) is \( \sigma_k\text{-open} \) set in \( Y \). By hypothesis, \( f^{-1}(F^C) \) is \( (i,j)-rw\text{-open} \) in \( X \). That is \( f^{-1}(F^C) = (f^{-1}(F))^C \) and so \( f^{-1}(F) \) is \( (i,j)-rw\text{-closed} \) in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D_{rw}(i,j)-\sigma_k\text{-continuous} \).

5.5.19 **Theorem:** If a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( D_{rw}(i,j)-\sigma_k\text{-continuous} \), then \( f((i,j)-rw\text{-cl}(A)) \subset \sigma_k\text{-cl}(f(A)) \) holds for every subset \( A \) of \( X \).

**Proof:** Let \( A \) be any subset of \( X \). Then \( f(A) \subset \sigma_k\text{-cl}(f(A)) \) and \( \sigma_k\text{-cl}(f(A)) \) is \( \sigma_k\text{-closed} \) set in \( Y \). Also \( f^{-1}(f(A)) \subset f^{-1}(\sigma_k\text{-cl}(f(A))) \). That is
Since $f$ is $D_{rw}(i, j)-\sigma_k$-continuous, $f^{-1}(\sigma_k-cl(f(A)))$ is a $(i, j)$-rw-closed set in $(X, \tau_1, \tau_2)$. By Theorem 5.4.2 (iii), $(i, j)$-rw-cl$(A) \subset f^{-1}(\sigma_k-cl(f(A)))$. Therefore $f((i, j)$-rw-cl$(A)) \subset f_{\sigma_k}(\sigma_k-cl(f(A))) \subset \sigma_k-cl(f(A))$. Hence $f((i, j)$-rw-cl$(A)) \subset \sigma_k-cl(f(A))$ for every subset $A$ of $(X, \tau_1, \tau_2)$.

5.5.20 Remark: Converse of the Theorem 5.5.19 is not true in general as seen from the following example.

5.5.21 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b, c\}\}$ and $Y=\{p, q\}$, $\sigma_1=\mathcal{P}(Y)$ and $\sigma_2=\{Y, \phi, \{p\}\}$. Then $D_{rw}(1, 2)=\{X, \phi, \{a\}, \{a, b\}, \{b, c\}\}$. Define a map $f:(X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)$ by $f(a)=f(c)=p$ and $f(b)=q$. Then $f((1, 2)$-rw-cl$(A)) \subset \sigma_2-cl(f(A))$ for every subset $A$ of $X$. But $f$ is not $D_{rw}(1, 2)$-$\sigma_2$-continuous, since for the $\sigma_2$-closed set $\{q\}$, $f^{-1}([q])=[b]$ which is not a $(1, 2)$-rw-closed set in $(X, \tau_1, \tau_2)$.

5.5.22 Theorem: If a map $f:(X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)$ is $D_{rw}(i, j)-\sigma_k$-continuous and $g: (Y, \sigma_1, \sigma_2)\rightarrow(Z, \eta_1, \eta_2)$ is $\sigma_k-$\eta_1-continuous, then $gof$ is $D_{rw}(i, j)-\eta_1$-continuous.

Proof: Let $F$ be $\eta_1$-closed set in $(Z, \eta_1, \eta_2)$. Since $g$ is $\sigma_k-$\eta_1-continuous, $g^{-1}(F)$ is a $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Since $f$ is $D_{rw}(i, j)-\sigma_k$-continuous, $f^{-1}(g^{-1}(F))=(gof)^{-1}(F)$ is a $(i, j)$-rw-closed set in $(X, \tau_1, \tau_2)$ and hence $gof$ is $D_{rw}(i, j)-\eta_1$-continuous.

5.5.23 Definition: (i) A map $f:(X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)$ is called rw-bi-continuous if $f$ is $D_{rw}(1, 2)-\sigma_2$-continuous and is $D_{rw}(2, 1)-\sigma_1$-continuous.

(ii) A map $f:(X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)$ is called rw-strongly-bi-continuous (briefly rw-s-bi-continuous) if $f$ is rw-continuous, $D_{rw}(2, 1)-\sigma_2$-continuous and $D_{rw}(1, 2)-\sigma_1$-continuous.
5.5.24 Theorem: Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a map.

(i) If \( f \) is bi-continuous then \( f \) is rw-bi-continuous.

(ii) If \( f \) is s-bi-continuous then \( f \) is rw-s-bi-continuous.

Proof: (i) Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a bi-continuous map. Then \( f \) is \( \tau_1-\sigma_1 \)-continuous and \( \tau_2-\sigma_2 \)-continuous and so by Theorem 5.5.3, \( f \) is \( D_{rw}(1, 2)-\sigma_2 \)-continuous and \( D_{rw}(2, 1)-\sigma_1 \)-continuous. Thus \( f \) is rw-bi-continuous.

(ii) Similar to (i), using Theorem 5.5.3.

The converse of this Theorem need not be true in general as seen from the following example.

5.5.25 Example: Let \( X=\{a, b, c\}, \tau_1=\{X, \phi, \{a\}\} \) and \( \tau_2=\{X, \phi, \{a, b\}\} \) and \( Y=\{p, q\}, \sigma_1=\{Y, \phi, \{p\}\} \) and \( \sigma_2=\{Y, \phi, \{p, q\}\} \). Define a map \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a)=p \) and \( f(b)=f(c)=q \). Then \( f \) is rw-s-bi-continuous but not s-bi-continuous. This map is also rw-bi-continuous but not bi-continuous.

5.5.26 Theorem: Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a map.

(i) If \( f \) is w-bi-continuous then \( f \) is rw-bi-continuous.

(ii) If \( f \) is w-s-bi-continuous then \( f \) is rw-s-bi-continuous.

Proof: (i) Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be w-bi-continuous map. Then \( f \) is \( C(2, 1)-\sigma_1 \)-continuous and \( C(1, 2)-\sigma_2 \)-continuous and so by Theorem 5.5.5, \( f \) is \( D_{rw}(1, 2)-\sigma_2 \)-continuous and \( D_{rw}(2, 1)-\sigma_1 \)-continuous. Thus \( f \) is rw-bi-continuous.

(ii) Similar to (i), using Theorem 5.5.5.

The converse of this Theorem need not be true in general as seen from the following example.
5.5.27 Example: Let $X=\{a, b, c\}$, $\tau_1=\{X, \phi, \{a\}\}$ and $\tau_2=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi\}$ and $\sigma_2=\{Y, \phi, \{p\}\}$. Define a map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a)=f(b)=q$ and $f(c)=p$. Then this function $f$ is rw-bi-continuous but it is not w-bi-continuous, since $f$ is not $C(2, 1)$-$\sigma_1$-continuous. This map is also rw-s-bi-continuous but not w-s-bi-continuous.

5.5.28 Theorem: Let $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.

(i) If $f$ is rw-bi-continuous then $f$ is rg-bi-continuous.

(ii) If $f$ is rw-s-bi-continuous then $f$ is rg-s-bi-continuous.

Proof: (i) Let $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be rw-bi-continuous map. Then $f$ is $D_{rw}(2, 1)$-$\sigma_1$-continuous and $D_{rw}(1, 2)$-$\sigma_2$-continuous and so by Theorem 5.5.7, $f$ is $D(1, 2)$-$\sigma_2$-continuous and $D(2, 1)$-$\sigma_1$-continuous. Therefore $f$ is rg-bi-continuous.

(ii) Similar to (i), using Theorem 5.5.7.

The converse of this Theorem need not be true in general as seen from the following example.

5.5.29 Example: Let $X=\{a, b, c, d\}$, $\tau_1=\{X, \phi, \{a\}, \{b\}, \{a, b, c\}\}$ and $\tau_2=\{X, \phi, \{a, b\}, \{c, d\}\}$ and $Y=\{p, q\}$, $\sigma_1=\{Y, \phi\}$ and $\sigma_2=\{Y, \phi, \{p\}\}$. Define a map $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a)=f(b)=f(d)=p$ and $f(c)=q$. Then $f$ is rg-bi-continuous but it is not rw-bi-continuous, since $f$ is not $D_{rw}(1, 2)$-$\sigma_2$-continuous. This map is also rg-s-bi-continuous but not rw-s-bi-continuous.
5.5.30 **Remark:** The following diagram summarizes the above discussions.

![Diagram](image)

**Figure-5.3**

5.5.31 **Definition:** A map \( \phi : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is called pairwise rw-irresolute if \( f^{-1}(A) \in D_{rw}(i, j) \) in \((X, \tau_1, \tau_2)\) for every \( A \in D_{rw}(k, e) \) in \((Y, \sigma_1, \sigma_2)\).

5.5.32 **Remark:** If \( \tau_1 = \tau_2 \) and \( \sigma_1 = \sigma_2 \) simultaneously, then \( f \) becomes a rw-irresolute map.

5.5.33 **Theorem:** If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise rw-irresolute, then \( f \) is \( D_{rw}(i, j) \)-\( \sigma_c \)-continuous.

**Proof:** Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be pairwise rw-irresolute and \( F \) be a \( \sigma_c \)-closed set in \((Y, \sigma_1, \sigma_2)\). Then \( F \) is \((k, e)\)-rw-closed set in \((Y, \sigma_1, \sigma_2)\) by Theorem 5.2.7. By hypothesis, \( f^{-1}(F) \) is a \((i, j)\)-rw-closed set in \((X, \tau_1, \tau_2)\). Therefore \( f \) is \( D_{rw}(i, j) \)-\( \sigma_c \)-continuous.

Converse of this Theorem is not true in general as seen from the following example.

5.5.34 **Example:** Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}\} \) and \( Y = \{p, q\} \), \( \sigma_1 = \{Y, \phi, \} \) and \( \sigma_2 = \{Y, \phi, \{q\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(b) = p \) and \( f(c) = q \). Then \( f \) is \((1, 2)\)-\( \sigma_2 \)-continuous but it is not pairwise rw-irresolute, since for the \((1, 2)\)-rw-closed set \( \{q\} \) in \((Y, \sigma_1, \sigma_2)\), \( f^{-1}(\{q\}) = \{c\} \) which is not a \((1, 2)\)-rw-closed set in \((X, \tau_1, \tau_2)\).
5.5.35 Theorem: The following statements are equivalent
(i) A map \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise rw-irresolute
(ii) The inverse image of every \((k, e)\)-rw-open set in \((Y, \sigma_1, \sigma_2)\) is a \((i, j)\)-rw-open set in \((X, \tau_1, \tau_2)\).

Proof: Proof is similar to that of Theorem 5.5.18.

5.5.36 Theorem: If \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) are two pairwise rw-irresolute maps, then their composition \( gof \) is also pairwise rw-irresolute.

Proof: Let \( A \in Drw(m, n) \) in \((Z, \eta_1, \eta_2)\). Since \( g \) is pairwise rw-irresolute, \( g^{-1}(A) \in Drw(k, e) \) in \((Y, \sigma_1, \sigma_2)\). Since \( f \) is pairwise rw-irresolute, \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \in Drw(i, j) \). Hence \( gof \) is pairwise rw-irresolute.

5.5.37 Theorem: If a map \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise rw-irresolute and \( g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) is \( Drw(k, e)\)-\( \eta_n \)-continuous, then \( gof: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is \( Drw(i, j)\)-\( \eta_n \)-continuous.

Proof: Let \( F \) be a \( \eta_n \)-closed set in \((Z, \eta_1, \eta_2)\). Since \( g \) is \( Drw(k, e)\)-\( \eta_n \)-continuous, \( g^{-1}(F) \in Drw(k, e) \) in \((Y, \sigma_1, \sigma_2)\). Since \( f \) is pairwise rw-irresolute, \( f^{-1}(g^{-1}(A)) = (gof)^{-1}(A) \in Drw(i, j) \) in \((X, \tau_1, \tau_2)\) and hence \( gof \) is \( Drw(i, j)\)-\( \eta_n \)-continuous.