3.1 Introduction.

N. Levine [49] introduced semi-continuous functions using semiopen sets. Balachandran, Sundaram and Maki [13] introduced the concepts of generalized continuous maps and gc-irresolute maps on topological spaces. Several authors ([3], [22], [23], [31], [49], [60], and [72]) working in the field of general topology have shown interest in studying the concepts of generalizations of continuous maps.


In section 2 of this chapter, a new class of maps called regular w-continuous (briefly, rw-continuous) maps are introduced and studied their relations with various generalized continuous maps and investigate some of their properties.

In section 3 of this chapter, we introduce the concepts of rw-irresolute maps and strongly rw-continuous maps in topological spaces and investigate some of their properties.

In section 4 of this chapter, a new class of maps called rw-closed maps is introduced and their relations with various generalized closed maps are studied. We prove that the composition of two rw-closed maps
need not be rw-closed map. We also introduce rw*-closed maps, rw-open maps and rw*-open maps in topological spaces and obtain certain characterizations of these maps.

In section 5 of this chapter, we introduce the concept of rw-homeomorphism and study the relationship with homeomorphisms, w-homeomorphisms, g-homeomorphisms \( w^* \)-homeomorphisms, and \( rwg \)-homeomorphisms. Also we introduce a new class of maps \( rwc \)-homeomorphisms which form a subclass of \( rw \)-homeomorphism. This class of maps is closed under composition of maps. We prove that the set of all \( rwc \)-homeomorphisms form a group under the operation composition of maps.

### 3.2 \( rw \)-continuous maps and some of their properties.

In this section, a new class of maps called regular \( w \)-continuous (briefly, \( rw \)-continuous) maps are introduced and studied their relations with various generalized continuous maps. We prove that the composition of two \( rw \)-continuous maps need not be \( rw \)-continuous. We also discuss some properties of \( rw \)-continuous maps.

#### 3.2.1 Definition: A map \( f:(X, \tau) \rightarrow (Y, \sigma) \) is said to be \( rw \)-continuous (regular \( w \)-continuous) if the inverse image of every closed set in \( Y \) is a \( rw \)-closed set in \( X \).

#### 3.2.2 Theorem: If a map \( f:(X, \tau) \rightarrow (Y, \sigma) \) is continuous, then it is \( rw \)-continuous.

**Proof:** Let \( F \) be a closed subset in \( Y \). Since \( f \) is continuous, \( f^{-1}(F) \) is a closed set in \( X \). By Corollary 2.2.6, \( f^{-1}(F) \) is \( rw \)-closed. Therefore \( f \) is \( rw \)-continuous.
The converse of the above theorem need not be true as seen from the following example.

3.2.3 Example: Let $X=Y=\{a, b, c\}$ be with the topologies $\tau=\{X, \emptyset, \{a\}\}$ and $\sigma=\{Y, \emptyset, \{b\}, \{b, c\}\}$. Let $f:(X, \tau)\rightarrow(Y, \sigma)$ be defined by $f(a)=b$, $f(b)=c$ and $f(c)=a$. Then $f$ is rw-continuous, but not continuous, as the inverse image of closed set $\{a\}$ in $Y$ is $\{c\}$, which is not a closed set in $X$.

Thus the class of all rw-continuous maps properly contains the class of all continuous maps.

3.2.4 Theorem: A map $f:(X, \tau)\rightarrow(Y, \sigma)$ is rw-continuous if and only if the inverse image of every open set in $Y$ is a rw-open set in $X$.

Proof: Suppose a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is rw-continuous. Let $U$ be an open set in $Y$. Then $U^c$ is a closed set in $Y$. Since $f$ is rw-continuous, $f^{-1}(U^c)$ is rw-closed in $X$. But $f^{-1}(U^c)=X-f^{-1}(U)$ and so $f^{-1}(U)$ is rw-open in $X$.

Conversely assume that the inverse image of every open set in $Y$ is rw-open in $X$. Let $F$ be a closed set in $Y$. Then $F^c$ is open in $Y$. By hypothesis $f^{-1}(F^c)=X-f^{-1}(F)$ is rw-open in $X$. That is $f^{-1}(F)$ is closed in $X$. Thus $f$ is rw-continuous.

3.2.5 Theorem: If a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is w-continuous, then $f$ is rw-continuous.

Proof: Suppose a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is w-continuous. Let $F$ be a closed set in $Y$. Since $f$ is w-continuous, $f^{-1}(F)$ is w-closed in $X$. As every w-closed set is rw-closed in $X$, $f^{-1}(F)$ is rw-closed in $X$. Therefore $f$ is rw-continuous.

The converse of the above theorem need not be true as seen from the following example.
3.2.6 Example: Let $X=Y=\{a, b, c\}$ be with the topologies $\tau=\{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma=\{Y, \phi, \{a, c\}\}$. Let $f:(X, \tau)\to(Y, \sigma)$ be defined by $f(a)=a$, $f(b)=b$ and $f(c)=c$. Then $f$ is rw-continuous but not w-continuous, as the inverse image of the closed $\{b\}$ in $Y$ is $\{b\}$ which is not w-closed in $X$.

Thus the class of all rw-continuous maps properly contains the class of all w-continuous maps.

3.2.7 Theorem: If a map $f:(X, \tau)\to(Y, \sigma)$ is rw-continuous, then $f$ is rwg-continuous but not conversely.

Proof: Suppose a map $f:(X, \tau)\to(Y, \sigma)$ is rw-continuous. Let $F$ be a closed set in $Y$. Since $f$ is rw-continuous, $f^{-1}(F)$ is a rw-closed set in $X$. As every rw-closed set is rwg-closed, $f^{-1}(F)$ is rwg-closed set in $X$. Thus $f$ is rwg-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.2.8 Example: Let $X=\{a, b, c, d\}$, $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ $Y=\{p, q\}$ and $\sigma=\{Y, \phi, \{q\}\}$. Define $f:(X, \tau)\to(Y, \sigma)$ by $f(a)=p$, $f(b)=q$, $f(c)=q$, and $f(d)=p$. Then $f$ is rwg-continuous but not rw-continuous as the inverse image of the closed set $\{p\}$ in $Y$ is $\{a, d\}$ in $X$, which is not rw-closed in $X$.

3.2.9 Theorem: If a map $f:(X, \tau)\to(Y, \sigma)$ is rw-continuous, then it is rg-continuous.

Proof: Suppose a map $f:(X, \tau)\to(Y, \sigma)$ is rw-continuous. Let $F$ be a closed set in $Y$. Since $f$ is rw-continuous, $f^{-1}(F)$ is rw-closed in $X$. As every rw-closed set is rg-closed in $X$, $f^{-1}(F)$ is rg-closed in $X$, $f^{-1}(F)$ is a rg-closed set in $X$. Therefore $f$ is rg-continuous.
The converse of the above theorem need not be true as seen from the following example.

3.2.10 Example: Let $X=\{a, b, c, d\}$, $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, $Y=\{p, q\}$ and $\sigma=\{Y, \phi, \{p\}, \{q\}\}$. Define a map $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=f(b)=f(d)=q$ and $f(c)=p$. Then $f$ is rg-continuous, but not rw-continuous, as the inverse image of the closed set $\{p\}$ in $Y$ is $\{c\}$ in $X$, which is not a rw-closed set in $X$.

3.2.11 Theorem: If a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is rw-continuous, then it is gpr-continuous.

Proof: Suppose a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is rw-continuous. Let $F$ be a closed set in $Y$. Since $f$ is rw-continuous, $f^{-1}(F)$ is rw-closed in $X$. As every rw-closed set is gpr-closed in $X$, $f^{-1}(F)$ is a gpr-closed set in $X$. Therefore $f$ is gpr-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.2.12 Example: Let $X=\{a, b, c, d\}$, $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, $Y=\{p, q, r\}$ and $\sigma=\{Y, \phi, \{p\}\}$. Define $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=q$, $f(b)=f(c)=p$ and $f(d)=r$. Then $f$ is gpr-continuous, but not rw-continuous, as the inverse image of the closed set $\{q, r\}$ in $Y$ is $\{a, d\}$ in $X$, which is not a rw-closed set in $X$.

3.2.13 Theorem: If a map $f:(X, \tau)\rightarrow(Y, \sigma)$ is completely continuous, then $f$ is rw-continuous.

Proof: Let a map $f:(X, \tau)\rightarrow(Y, \sigma)$ be completely continuous. Let $F$ be a closed set in $Y$. Then $f^{-1}(F)$ is regular closed in $X$ and hence $f^{-1}(F)$ is rw-closed in $X$. Therefore $f$ is rw-continuous.
3.2.14 **Theorem:** If a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $w$-irresolute, then $f$ is $rw$-continuous.

**Proof:** Suppose that a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $w$-irresolute and let $V$ be an open set in $Y$. Then $V$ is $w$-open in $Y$. Since $f$ is $w$-irresolute $f^{-1}(V)$ is $w$-open and hence $rw$-open in $X$. Thus $f$ is $rw$-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.2.15 **Example:** Let $X=\{a, b, c,\}$, $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $Y=\{p, q\}$ and $\sigma=\{Y, \phi, \{p\}, \{q\}\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=f(b)=p$ and $f(c)=q$. Then $f$ is $rw$-continuous, but not $w$-irresolute, as the inverse image $w$-closed set $\{p\}$ in $Y$ is $\{a, b\}$ in $X$, which is not a $w$-closed set in $X$.

3.2.16 **Remark:** The following examples (3.2.17, 3.2.18 and 3.2.19) show that $rw$-continuous function and $g$-continuous, $g^*$-continuous, $wg$-continuous, semi-continuous, $\alpha$-continuous, $g\alpha$-continuous, $\alpha g$-continuous, $sg$-continuous, $gs$-continuous, $gsp$-continuous, $\beta$-continuous, pre-continuous, $gp$-continuous functions are independent.

3.2.17 **Example** Let $X=\{a, b, c, d\}$ be with topology, $\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $Y=\{a, b, c\}$ with topology $\sigma=\{Y, \phi, \{a\}\}$. A map $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a)=c$, $f(b)=b$, and $f(c)=f(d)=a$. Then the inverse image of every closed set in $Y$ is a $rw$-closed set in $X$ and hence $f$ is $rw$-continuous. Let $\{b, c\}$ be closed set in $Y$, $f^{-1}(\{b, c\})=\{a, b\}$ is not $g$-closed, $g^*$-closed, $wg$-closed, semi-closed, $\alpha$-closed, $g\alpha$-closed, $\alpha g$-closed, $sg$-closed, $gs$-closed, $gsp$-closed, $\beta$-closed, pre-closed, $gp$-closed in $X$. Thus $f$ is not $g$-continuous, $g^*$-continuous, $wg$-continuous semi-continuous, $\alpha$-continuous, $g\alpha$-continuous, $\alpha g$-
continuous, sg-continuous, gs-continuous, gsp-continuous, \(\beta\)-continuous, pre-continuous, gp- continuous.

3.2.18 Example: Let \(X=\{a, b, c, d\}\) be with topology, \(\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\), and \(Y=\{a, b, c\}\) with topology \(\sigma=\{Y, \phi, \{a, b\}\}\). A map \(f:(X, \tau)\rightarrow(Y, \sigma)\) be defined by \(f(a)=f(d)=c, f(b)=b\) and \(f(c)=a\). Then the inverse image of every closed set in \(Y\) is \(g\)-closed, \(g^*\)-closed, \(wg\)-closed, semi-closed, \(\alpha g\)-closed, sg-closed, gs-closed, gsp-closed, \(\beta\)-closed, gp-closed in \(X\) and hence \(f\) is \(g\)-continuous, \(g^*\)-continuous, \(wg\)-continuous, semi-continuous, \(\alpha g\)-continuous, sg-continuous, gs-continuous, gsp-continuous, \(\beta\)-continuous, gp-continuous. But \(f\) is not \(rw\)-continuous as the inverse image of the closed set \(\{c\}\) in \(Y\) is \(\{a, d\}\) in \(X\) which is not \(rw\)-closed.

3.2.19 Example: Let \(X=\{a, b, c, d\}\), with topology, \(\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\) and \(Y=\{a, b, c\}\) with topology \(\sigma=\{Y, \phi, \{a, b\}\}\). A map \(f:(X, \tau)\rightarrow(Y, \sigma)\) be defined by \(f(a)=a, f(d)=f(b)=b\) and \(f(c)=c\). Then the inverse image of every closed set in \(Y\) is \(\alpha\)-closed, \(g\alpha\)-closed, pre-closed in \(X\) and hence \(f\) is \(\alpha\)-continuous, \(g\alpha\)-continuous, pre-continuous. But \(f\) is not \(rw\)-continuous, as the inverse image of the closed set \(\{c\}\) in \(Y\) is \(\{c\}\) in \(X\) which is not \(rw\)-closed in \(X\).

3.2.20 Remark: From the above discussions and known results we have the following implications

In the following diagram, by

\[ A \longrightarrow B \] we mean \(A\) implies \(B\) but not conversely and

\[ A \iff B \] means \(A\) and \(B\) are independent of each other.

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3.2.21 Remark: The composition of two rw-continuous maps need not be rw-continuous and this is shown by the following example.

3.2.22 Example: Let $X=Y=Z=\{a, b, c,\}$, $x=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\sigma=\{Y, \emptyset, \{a\}, \{a, b\}\}$ and $\eta=\{Z, \emptyset, \{a, c\}\}$. Define $f:(X, x)\rightarrow(Y, \sigma)$ by $f(a)=f(c)=c$, $f(b)=b$ and $g:(Y, \sigma)\rightarrow(Z, \eta)$ be the identity map. Then $f$ and $g$ are rw-continuous, but their composition $gof:(X, x)\rightarrow(Z, \eta)$ is not rw-continuous, because $F=\{b\}$ is closed in $(Z, \eta)$, but $(gof)^{-1}(F)=\{b\}$ which is not rw-closed in $(X, x)$.

3.2.23 Theorem: If $f:(X, \tau)\rightarrow(Y, \sigma)$ is a rw-continuous and $g:(Y, \sigma)\rightarrow(Z, \eta)$ is continuous, then their composition $gof:(X, \tau)\rightarrow(Z, \eta)$ is rw-continuous.

Proof: Let $F$ be any closed set in $(Z, \eta)$. Since $g$ is continuous, $g^{-1}(F)$ is closed in $(Y, \sigma)$. Since $f$ is rw-continuous, $f^{-1}(g^{-1}(F))=(gof)^{-1}(F)$ is a rw-closed set in $X$ and hence $gof$ is rw-continuous.
3.2.24 **Theorem:** Let \((X, \tau), (Z, \eta)\) be any topological spaces and \((Y, \sigma)\) be topological space "where every rw-closed subset is closed". Then the composition \(gof:(X, \tau) \rightarrow (Z, \eta)\) of the rw-continuous maps \(f: (X, \tau) \rightarrow (Y, \sigma)\) and \(g: (Y, \sigma) \rightarrow (Z, \eta)\) is rw-continuous.

**Proof:** Let \(F\) be any closed set of \((Z, \eta)\). As \(g\) is rw-continuous, \(g^{-1}(F)\) is a rw-closed set in \((Y, \sigma)\). By hypothesis, every rw-closed set in \((Y, \sigma)\) is closed, \(g^{-1}(F)\) is a closed set in \((Y, \sigma)\). Since \(f\) is rw-continuous, \(f^{-1}(g^{-1}(F))\) is rw-closed in \((X, \tau)\). But \(f^{-1}(g^{-1}(F))=(gof)^{-1}(F)\) and so \(gof\) is rw-continuous.

3.2.25 **Theorem:** If a map \(f:(X, \tau) \rightarrow (Y, \sigma)\) is a rw-continuous, then \(/(rw-cl(A)) \subseteq cl(/(A))\) for every subset \(A\) of \(X\).

**Proof:** Let \(A\) be subset of \((X, \tau)\). Then \(cl(/(A))\) is closed in \(Y\). Since \(f\) is rw-continuous, \(f^{-1}(cl(/(A)))\) is rw-closed in \(Y\) and \(A \subseteq f^{-1}(/(A)) \subseteq f^{-1}(cl(/(A)))\) that is \(f^{-1}(cl(/(A)))\) is rw-closed subset of \(X\) containing \(A\). By Theorem 2.2.16, \(rw-cl(A) \subseteq f^{-1}(cl(/(A)))\). Hence \(f(rw-cl(A)) \subseteq cl(/(A))\).

3.2.26 **Remark:** Converse of the above Theorem 3.2.25 is not true in general as seen from the following example.

3.2.27 **Example:** Let \(X=\{1, 2, 3\}\), \(Y=\mathbb{R}\), \(\tau = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}\) and \(\sigma = \) usual topology on \(\mathbb{R}\). Define \(f: (X, \tau) \rightarrow (Y, \sigma)\) by \(f(x)=x\) for every \(x \in X\). In \((X, \tau)\), \(rw-cl(A)=A\) for every \(A \subseteq X\). So \(f(rw-cl(A))=f(A) \subseteq cl(/(A))\) for every \(A \subseteq X\). But \(f\) is not rw-continuous, as \(f^{-1}(\{0, 1\})=\{1\}\) is not rw-closed in \(X\).

3.2.28 **Theorem:** Let \(f:(X, \tau) \rightarrow (Y, \sigma)\) be a map. If for each point \(x \in X\) and each open set \(V\) containing \(f(x)\), there exists a rw-open set \(U\)
containing $x$ such that $f(U) \subseteq V$, then for every subset $A$ of $X$, $f(\text{rw-cl}(A)) \subseteq \text{cl}(f(A))$ holds.

**Proof:** Suppose that for each point $x \in X$ and each open set $V$ containing $f(x)$, there exists a rw-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. Let $A$ be any subset of $X$ and $y \in f(\text{rw-cl}(A))$. Let $V$ be an open set containing $y$. Then by hypothesis, there exists an $x \in X$ such that $f(x) = y$ and an rw-open set $U$ containing $x$ such that $f(U) \subseteq V$ and $x \in \text{rw-cl}(A)$. Therefore by Theorem 2.4.23, $U \cap A \neq \emptyset$. This implies $V \cap f(A) \neq \emptyset$. Hence $y \in \text{cl}(A)$. That is $f(\text{rw-cl}(A)) \subseteq \text{cl}(f(A))$.

**3.2.29 Theorem:** Let $f:(X, \tau) \rightarrow (Y, \sigma)$ be rw-continuous map and let $H$ be a regular open subset of $(X, \tau)$. Then restriction $f|_H : (H, \tau_H) \rightarrow (Y, \sigma)$ is also rw-continuous.

**Proof:** Let $U$ be any open set in $(Y, \sigma)$. Since $f$ is rw-continuous map, $f^{-1}(U)$ is a rw-open subset of $(X, \tau)$. Let $f^{-1}(U) \cap H = H_1$. Then $H_1$ is a rw-open subset in $X$, as a finite intersection of rw-open sets is rw-open. Now $H_1 \subseteq H \subseteq X$ and $H$ is a regular open subspace of $X$. By Theorem 2.3.26, $H_1$ is an rw-open set in $(H, \tau_H)$. Also $(f|_H)^{-1}(U) = f^{-1}(U) \cap H = H_1$. That is $(f|_H)^{-1}(U)$ is rw-open in $(H, \tau_H)$. Hence $f|_H : (H, \tau_H) \rightarrow (Y, \sigma)$ is rw-continuous.

**3.2.30 Lemma:** Let $f:(X, \tau) \rightarrow (Y, \sigma)$ be irresolute and $F$ be a regular semiopen in $(Y, \sigma)$. Then $f^{-1}(F)$ is regular semiopen in $(X, \tau)$.

**Proof:** Let $F$ be a regular semiopen in $(Y, \sigma)$. To prove $f^{-1}(F)$ is regular semiopen in $(X, \tau)$. That is to prove $f^{-1}(F)$ is both semiopen and semi-closed in $(X, \tau)$. Now $F$ is semiopen in $(Y, \sigma)$. Since $f$ is irresolute, $f^{-1}(F)$ is semiopen in $(X, \tau)$.

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Now F is semi-closed in \((Y, \sigma)\), as regular semiopen is semi-
closed. Then \(Y - F\) is semiopen in \((Y, \sigma)\). Since \(f\) is irresolute, \(f^{-1}(Y - F)\) is
semiopen in \((X, \tau)\). That is \(f^{-1}(Y - F) = X - f^{-1}(F)\) is semiopen and so
\(f^{-1}(F)\) is semi-closed in \((X, \tau)\). Thus \(f^{-1}(F)\) is both semiopen and semi-
closed in \((X, \tau)\) and hence \(f^{-1}(F)\) is regular semiopen in \((X, \tau)\).

3.2.31 Theorem: If \(A\) is rw-closed in \((X, \tau)\) and if \(f: (X, \tau) \rightarrow (Y, \sigma)\) is
irresolute and closed, then \(f(A)\) is rw-closed in \((Y, \sigma)\).

Proof: Let \(U\) be any regular semiopen set in \((Y, \sigma)\) such that \(f(A) \subseteq U\). Then \(A \subseteq f^{-1}(U)\). Since \(f\) is irresolute, by above Lemma 3.2.30, \(f^{-1}(U)\) is
regular semiopen in \((X, \tau)\). Since \(A\) is rw-closed in \((X, \tau)\), \(cl(A) \subseteq f^{-1}(U)\).
Thus \(f(cl(A)) \subseteq U\) and \(f(cl(A))\) is a closed set in \((Y, \sigma)\), as \(f\) is closed.
Now \(cl(f(A)) \subseteq cl(f(cl(A))) = f(cl(A)) \subseteq U\). That is \(cl(f(A)) \subseteq U\) and hence \(f(A)\) is rw-closed in \((X, \tau)\).

3.2.32 Theorem: If \(A\) is rw-open in \((X, \tau)\) and if \(f: (X, \tau) \rightarrow (Y, \sigma)\) is
irresolute and open, then \(f(A)\) is rw-open in \((Y, \sigma)\).

Proof: Let \(U\) be any regular semiopen set in \((Y, \sigma)\) such that \(f(A) \subseteq U\). Then \(A \subseteq f^{-1}(U)\). Since \(f\) is irresolute, by above Lemma 3.2.30, \(f^{-1}(U)\) is
regular semiopen in \((X, \tau)\). Since \(A\) is rw-open in \((X, \tau)\), \(A \subseteq int(f^{-1}(U))\).
Now \(f(A) \subseteq f(int(f^{-1}(U)))\) and \(f(int(f^{-1}(U)))\) is open in \((Y, \sigma)\). Also
\(int(f(int(f^{-1}(U)))) = f(int(f^{-1}(U)))\). Then \(f(int(f^{-1}(U))) = int(f(int(f^{-1}(U)))) \subseteq int(int(f^{-1}(U))) = int(U)\). That is \(f(A) \subseteq int(U)\) and hence \(f(A)\) is rw-open in
\((Y, \sigma)\).

3.3 rw-irresolute maps and strongly rw-continuous maps.

Irresolute functions are introduced and studied by Crossely and
Hildebrand [20]. Sundaram [89] and Devi [22] have investigated gc-
irresolute, \(\alpha g\)-irresolute and gs-irresolute functions. Recently Sheik Jhon
introduced and studied w-irresolute, strongly w-continuous and perfectly w-continuous maps. In this section we introduce the concepts of rw-irresolute maps and strongly rw-continuous maps in topological spaces and investigate some of their properties.

3.3.1 **Definition:** A map \( f:(X, \tau) \to (Y, \sigma) \) is called a rw-irresolute map if the inverse image of every rw-closed set in \((Y, \sigma)\) is rw-closed in \((X, \tau)\).

3.3.2 **Theorem:** A map \( f:(X, \tau) \to (Y, \sigma) \) is rw-irresolute if and only if the inverse image of an rw-open set in \((Y, \sigma)\) is rw-open in \((X, \tau)\).

**Proof:** Let \( f:(X, \tau) \to (Y, \sigma) \) be rw-irresolute and \( U \) be an rw-open set in \((Y, \sigma)\). Then \( U^c \) is rw-closed in \((Y, \sigma)\). Since \( f \) is rw-irresolute, \( f^{-1}(U^c) \) is rw-closed in \((X, \tau)\). But \( f^{-1}(U^c)=(f^{-1}(U))^c \) and so \( f^{-1}(U) \) is rw-open in \((X, \tau)\).

Conversely, assume that \( f^{-1}(U) \) is rw-open in \((X, \tau)\) for each rw-open set \( U \) in \((Y, \sigma)\). Let \( F \) be a rw-closed set in \((Y, \sigma)\). Then \( F^c \) is rw-open in \((Y, \sigma)\) and by assumption, \( f^{-1}(F^c) \) is rw-open in \((X, \tau)\). Since \( f^{-1}(F^c)=(f^{-1}(F))^c \), we have \( f^{-1}(F) \) is rw-closed in \((X, \tau)\) and so \( f \) is rw-irresolute.

3.3.3 **Theorem:** If a map \( f:(X, \tau) \to (Y, \sigma) \) is rw-irresolute, then it is rw-continuous.

**Proof:** Let \( f:(X, \tau) \to (Y, \sigma) \) be rw-irresolute and \( F \) be a closed set in \((Y, \sigma)\). Then \( F^c \) is rw-open in \((Y, \sigma)\) and by assumption, \( f^{-1}(F^c) \) is rw-open in \((X, \tau)\). Since \( f^{-1}(F^c)=(f^{-1}(F))^c \), we have \( f^{-1}(F) \) is rw-closed in \((X, \tau)\) and so \( f \) is rw-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.4 **Example:** Let \( X=Y=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma=\{Y, \phi, \{a\}, \{a, b\}\} \). Then the identity map \( f:(X, \tau) \to (Y, \sigma) \) is rw-
continuous but not rw-irresolute, as the inverse image of rw-closed set \{b\} in (Y, \sigma) is \{b\} which is not rw-closed set in (X, \tau).

**3.3.5 Remark:** The following examples show that the notion of irresolute maps and rw-irresolute maps are independent.

**3.3.6 Example:** Let X=\{a, b, c\}, \tau=\{X, \phi, \{a\}\} and \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\}. Then the identity map on X is rw-irresolute but it is not irresolute, as the inverse image of the semiopen set \{b\} in (Y, \sigma) is \{b\} which is not semiopen in (X, \tau).

**3.3.7 Example:** Let X=\{a, b, c\}, \tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\} and \sigma=\{Y, \phi, \{a\}, \{b, c\}\}. Then the identity map f:(X, \tau)\rightarrow(Y, \sigma) is irresolute but it is not rw-irresolute, as the inverse image of the rw-closed set \{a\} in (Y, \sigma) is \{a\} which is not rw-closed in (X, \tau).

**3.3.8 Remark:** The following examples show that the notion of w-irresolute maps and rw-irresolute maps are independent.

**3.3.9 Example:** Let X=\{a, b, c\}, \tau=\{X, \phi, \{a\}\} and \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\}. Then the identity map on X is rw-irresolute but it is not w-irresolute, as the inverse image of the w-closed set \{a, c\} in (Y, \sigma) is \{a, c\} which is not w-closed in (X, \tau).

**3.3.10 Example:** Let X=\{a, b, c\}, \tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\} and \sigma=\{Y, \phi, \{a\}\}. Then the map f: (X, \tau)\rightarrow(Y, \sigma) defined by f(a)=b and f(b)=f(c)=c is w-irresolute but it is not rw-irresolute, as the inverse image of the rw-closed set \{b\} in (Y, \sigma) is \{a\} which is not rw-closed in (X, \tau).

**3.3.11 Theorem:** If f:(X, \tau)\rightarrow(Y, \sigma) and g:(Y, \sigma)\rightarrow(Z, \eta) are both rw-irresolute, then gof: (X, \tau)\rightarrow(Z, \eta) is rw-irresolute.

**Proof:** Let F be a rw-closed set in (Z, \eta). Since g is a rw-irresolute, g^{-1}(F) is rw-closed in (Y, \sigma). As f is rw-irresolute, f^{-1}(g^{-1}(F)) is rw-closed in
(X, τ). Thus \((gof)^{-1}(F)=f^{-1}(g^{-1}(F))\) is rw-closed in \((X, τ)\) and \(gof\) is rw-irresolute.

3.3.12 Theorem: If \(f:(X, τ)→(Y, σ)\) is rw-irresolute and \(g:(Y, σ)→(Z, η)\) is rw-continuous, then their composition \(gof:(X, τ)→(Z, η)\) is rw-continuous.

Proof: Let \(F\) be a closed set in \((Z, T|)\). Since \(g\) is rw-continuous, \(g^{-1}(F)\) is rw-closed in \((Y, a)\). As \(f\) is rw-irresolute, \(f^{-1}(g^{-1}(F))\) is rw-closed in \((X, τ)\). Thus \((gof)^{-1}(F)=f^{-1}(g^{-1}(F))\) is rw-closed in \((X, τ)\) and \(gof\) is rw-continuous.

3.3.13 Theorem: Let \((X, τ)\) be any topological space, \((Y, σ)\) be a topological space where “every rw-closed subset is closed” and \(f:(X, τ)→(Y, σ)\) be a map. Then the following are equivalent.

(i) \(f\) is rw-irresolute (ii) \(f\) is rw-continuous.

Proof: (i)⇒(ii) Follows from Theorem 3.3.3.

(ii)⇒(i) Let \(F\) be a rw-closed set in \((Y, σ)\). Then \(F\) is a closed set in \((Y, σ)\) by hypothesis. Since \(f\) is rw-continuous, \(f^{-1}(F)\) is a rw-closed set in \((X, τ)\). Therefore \(f\) is rw-irresolute.

3.3.14 Theorem: If a map \(f:(X, τ)→(Y, σ)\) is bijective, closed and irresolute then the inverse map \(f^{-1}:(Y, σ)→(X, τ)\) is rw-irresolute.

Proof: Let \(A\) be a rw-closed set in \((X, τ)\). Let \((f^{-1})^{-1}(A) = f(A)⊂ U\) where \(U\) is regular semiopen in \((Y, σ)\). Then \(A⊂f^{-1}(U)\) holds. Since \(f\) is irresolute, \(f^{-1}(U)\) is regular semiopen in \((X, τ)\) by the Lemma 3.2.30. As \(A\) is rw-closed in \((X, τ)\), \(cl(A)⊂f^{-1}(U)\) and hence \(f(cl(A))⊂U\). Since \(f\) is closed and \(cl(A)\) is closed \((X, τ)\), \(f(cl(A))\) is closed in \((Y, σ)\). Therefore
cl(f(cl(A)))⊂U and hence cl(f(A))⊂U. Thus f(A) is rw-closed in (Y, σ) and so f⁻¹ is rw-irresolute.

3.3.15 Definition: A map \( f: (X, \tau) \to (Y, \sigma) \) is called a strongly rw-continuous if the inverse image of every rw-open set in \( (Y, \sigma) \) is open in \( (X, \tau) \).

3.3.16 Theorem: A map \( f: (X, \tau) \to (Y, \sigma) \) is a strongly rw-continuous if and only if the inverse image of every rw-closed set in \( (Y, \sigma) \) is closed in \( (X, \tau) \).

Proof: Suppose a map \( f: (X, \tau) \to (Y, \sigma) \) is strongly rw-continuous. Let \( F \) be a rw-closed set in \( Y \). Then \( F^c \) is a rw-open set in \( Y \). Since \( f \) is strongly rw-continuous, \( f^{-1}(F^c) \) is an open set in \( X \). But \( f^{-1}(F^c) = X - f^{-1}(F) \) and so \( f^{-1}(F) \) is closed in \( X \).

Conversely assume that the inverse image of every rw-closed set in \( Y \) is a closed set in \( X \). Let \( U \) be a rw-open set in \( Y \). Then \( U^c \) is a rw-closed in \( Y \). By hypothesis, \( f^{-1}(U^c) = X - f^{-1}(U) \) is closed in \( X \). That is \( f^{-1}(U) \) is an open set in \( X \). Thus \( f \) is strongly rw-continuous.

3.3.17 Theorem: If \( f: (X, \tau) \to (Y, \sigma) \) is strongly rw-continuous, then it is continuous but not conversely.

Proof: Suppose \( f: (X, \tau) \to (Y, \sigma) \) is strongly rw-continuous. Let \( U \) be an open set in \( Y \). As every open set is rw-open, \( U \) is rw-open in \( Y \). Since \( f \) is strongly rw-continuous, \( f^{-1}(U) \) is an open set in \( X \). Thus \( f \) is continuous.

3.3.18 Example: Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{p, q\}\} \). Define a map \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = p, f(b) = q \) and \( f(c) = r \). Then \( f \) is continuous but it is not strongly rw-continuous, since for rw-open set \( \{r\} \) in \( (Y, \sigma) \), \( f^{-1}(\{r\}) = \{c\} \) which is not open in \( (X, \tau) \).
3.3.19 Theorem: If a map \( f: (X, \tau) \to (Y, \sigma) \) is strongly rw-continuous, then it is strongly w-continuous but not conversely.

**Proof:** Suppose \( f: (X, \tau) \to (Y, \sigma) \) is strongly rw-continuous. Let \( U \) be w-open set in \( Y \). As every w-open set is rw-open, \( U \) is rw-open in \( Y \). Since \( f \) is strongly rw-continuous, \( f^{-1}(U) \) is an open set in \( X \). Thus \( f \) is strongly w-continuous.

3.3.20 Example: Let \( X = \{a, b, c\} \), \( Y = \{p, q, r\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{p\}, \{q\}, \{p, q\}\} \). Define a map \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = p \), \( f(b) = q \) and \( f(c) = r \). Then \( f \) is strongly w-continuous but it is not strongly rw-continuous, since for the rw-open set \( U = \{r\} \) in \( (Y, \sigma) \), \( f^{-1}(U) = \{c\} \) which is not open in \( (X, \tau) \).

3.3.21 Theorem: Let \( (X, \tau) \) be any topological space, \( (Y, \sigma) \) be a topological space where "every rw-closed subset is closed" and \( f: (X, \tau) \to (Y, \sigma) \) be a map. Then the following are equivalent.

(i) \( f \) is strongly rw-continuous

(ii) \( f \) is continuous.

**Proof:** (i)\( \Rightarrow \) (ii) Follows from Theorem 3.3.17

(ii) \( \Rightarrow \) (i) Let \( F \) be any rw-closed set in \( (Y, \sigma) \). Then \( F \) is closed set in \( (Y, \sigma) \), by hypothesis. Since \( f \) is continuous, \( f^{-1}(F) \) is closed set in \( (X, \tau) \). By Theorem 3.3.16, \( f \) is strongly rw-continuous.

3.3.22 Theorem: If a map \( f: (X, \tau) \to (Y, \sigma) \) is strongly continuous, then it is strongly rw-continuous.

**Proof:** Proof is easy consequence from definitions.
3.3.23 **Theorem:** If \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) are both strongly \( rw \)-continuous, then their composition \( g \circ f: (X, \tau) \to (Z, \eta) \) is strongly \( rw \)-continuous.

**Proof:** Let \( U \) be a \( rw \)-open set in \( (Z, \eta) \). Since \( g \) is a strongly \( rw \)-continuous, \( g^{-1}(U) \) is an open set in \( (Y, \sigma) \). As every open set is \( rw \)-open, \( g^{-1}(U) \) is a \( rw \)-open set in \( (Y, \sigma) \). Since \( f \) is a strongly \( rw \)-continuous, \( f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \). Thus \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \) and hence \( g \circ f \) is strongly \( rw \)-continuous.

3.3.24 **Definition:** A topological space \( (X, \tau) \) is called \( rw \)-space if every subset in it is \( rw \)-closed.

3.3.25 **Theorem:** Every \( w \)-space is \( rw \)-space but not conversely.

**Proof:** Proof follows from the two definitions and fact that every \( w \)-closed set is \( rw \)-closed.

3.3.26 **Example:** Let \( X = \{a, b, c\} \) be with topology \( \tau = \{X, \emptyset, \{a\}\} \). Then the space \( (X, \tau) \) is \( rw \)-space but it is not \( w \)-space.

3.3.27 **Theorem:** Let \( (X, \tau) \) be a discrete topological space, \( (Y, \sigma) \) be a \( rw \)-space and \( f: (X, \tau) \to (Y, \sigma) \) be a map. Then the following are equivalent.

(i) \( f \) is strongly continuous

(ii) \( f \) is strongly \( rw \)-continuous

**Proof:** (i) \( \Rightarrow \) (ii) Follows from Theorem 3.3.22.

(ii) \( \Rightarrow \) (i) Let \( U \) be any subset in \( (Y, \sigma) \). Since \( (Y, \sigma) \) is a \( rw \)-space, \( U \) is a \( rw \)-open subset in \( (Y, \sigma) \) and by hypothesis, \( f^{-1}(U) \) is open in \( (X, \tau) \). But \( (X, \tau) \) is discrete topological space and so \( f^{-1}(U) \) is also a closed set in \( (X, \tau) \). That is \( f^{-1}(U) \) is both open and closed set in \( (X, \tau) \) and so \( f \) is strongly continuous.
3.3.28 **Theorem:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map and \( \text{RWO}(X, \tau) = \tau \), \( \text{RWO}(Y, \sigma) = \sigma \). Then the following are equivalent.

(i) \( f \) is strongly rw-continuous  
(ii) \( f \) is continuous  
(iii) \( f \) is rw-irresolute  
(iv) \( f \) is rw-continuous.

**Proof:** Proof follows from Theorems 3.3.21 and 3.3.13

3.3.29 **Theorem:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be any two maps. Then their composition \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) is

(i) strongly rw-continuous if \( g \) is strongly rw-continuous and \( f \) is continuous,

(ii) rw-irresolute if \( g \) is strongly rw-continuous and \( f \) is rw-continuous,

(iii) continuous if \( g \) is rw-continuous and \( f \) is strongly rw-continuous.

**Proof:** (i) To prove that \( \text{gof} \) is strongly rw-continuous. Let \( U \) be a rw-open set in \((Z, \eta)\). Since \( g \) is strongly rw-continuous, \( g^{-1}(U) \) is an open set in \((Y, \sigma)\). Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) = (\text{gof})^{-1}(U) \) is an open set in \((X, \tau)\). Therefore \( \text{gof} \) is a strongly rw-continuous.

(ii) To prove that \( \text{gof} \) is rw-irresolute. Let \( U \) be a rw-open set in \((Z, \eta)\). Since \( g \) is strongly rw-continuous, \( g^{-1}(U) \) is an open set in \((Y, \sigma)\). Since \( f \) is rw-continuous, \( f^{-1}(g^{-1}(U)) = (\text{gof})^{-1}(U) \) is a rw-open set in \((X, \tau)\). Therefore \( \text{gof} \) is rw-irresolute.

(iii) To prove that \( \text{gof} \) is continuous. Let \( U \) be an open set in \((Z, \eta)\). Since \( g \) is rw-continuous, \( g^{-1}(U) \) is a rw-open set in \((Y, \sigma)\). Since \( f \) is strongly rw-continuous, \( f^{-1}(g^{-1}(U)) = (\text{gof})^{-1}(U) \) is an open set in \((X, \tau)\). Therefore \( \text{gof} \) is continuous.
3.4 rw-closed maps and rw-open maps.

Generalized closed mappings were introduced and studied by Malghan [57]. wg-closed maps and rwg-closed maps were introduced and studied by Nagaveni [65]. Regular closed maps, gpr-closed maps and rg-closed maps have been introduced and studied by Long [51], Gnanambal [40] and Arockiarani [3] respectively. In this section, new classes of maps called rw-closed maps are introduced and their relations with various generalized closed maps are studied. We prove that the composition of two rw-closed maps need not be rw-closed map. We also discuss some properties of rw-closed maps.

3.4.1 Definition: A map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is said to be rw-closed if the image of every closed set in \((X, \tau)\) is rw-closed in \((Y, \sigma)\).

3.4.2 Theorem: Every closed map is a rw-closed map.

Proof: The proof follows from the two definitions and the fact that every w-closed set is rw-closed.

The converse of the above theorem need not be true in general as seen from the following example.

3.4.3 Example: Consider \( X=Y=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma=\{Y, \phi, \{a\}\} \). Let a map \( f:(X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a)=a \), \( f(b)=b \) and \( f(c)=c \). Then this function is rw-closed but not closed, as the image of the closed set \( \{c\} \) in \( X \) is \( \{c\} \) which is not closed in \( Y \).

3.4.4 Theorem: Every w-closed map is rw-closed map but not conversely.

Proof: The proof follows from the two definitions and the fact that every w-closed set is rw-closed.
The converse of the above theorem need not be true in general as seen from the following example.

3.4.5 Example: Consider \( X=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma=\{Y, \phi, \{a\}\} \). Let a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a)=f(b)=b \) and \( f(c)=c \). Then this function is rw-closed but not w-closed, as the image of the closed set \( \{c\} \) in \( X \) is \( \{c\} \) which is not w-closed in \( Y \).

3.4.6 Theorem: Every rw-closed map is rg-closed map but not conversely.

Proof: The proof follows from the two definitions and the fact that every rw-closed set is rg-closed.

The converse of the above theorem need not be true in general as seen from the following example.

3.4.7 Example: Consider \( X=\{a, b, c\} \), \( Y=\{a, b, c, d\} \), \( \tau=\{X, \phi, \{a\}, \{c\}, \{a, c\}\} \) and \( \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Let a map \( f:(X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a)=a \), \( f(b)=c \) and \( f(c)=d \). Then this function is rg-closed but not rw-closed, as the image of the closed set \( \{b\} \) in \( X \) is \( \{c\} \) which is not rw-closed in \( Y \).

3.4.8 Theorem: Every rw-closed map is rwg-closed map but not conversely.

Proof: The proof follows from the two definitions and the fact that every rw-closed set is rwg-closed.

The converse of the above theorem need not be true in general as seen from the following example.

3.4.9 Example: Consider \( X=\{a, b, c\} \), \( Y=\{a, b, c, d\} \), \( \tau=\{X, \phi, \{b, c\}\} \) and \( \sigma=\{Y, \phi, \{d\}, \{a, c\}, \{a, c, d\}\} \). Let a map \( f:(X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a)=a \), \( f(b)=b \) and \( f(c)=d \). Then this function is rwg-closed.
but not rw-closed, as the image of the closed set \{a\} in X is \{a\} which is not rw-closed in Y.

3.4.10 **Theorem:** Every rw-closed map is gpr-closed map but not conversely.

**Proof:** The proof follows from the two definitions and the fact that every rw-closed set is gpr-closed.

The converse of the above theorem need not be true in general as seen from the following example.

3.4.11 **Example:** Consider \(X=\{a, b, c\}\), \(Y=\{a, b, c, d, e\}\), \(\tau=\{X, \emptyset, \{b, c\}\}\) and \(\sigma=\{Y, \emptyset, \{a, b\}, \{c, d\}\}\). Let a map \(f:(X, \tau)\rightarrow(Y, \sigma)\) be defined by \(f(a)=a, f(b)=b\) and \(f(c)=e\). Then this function is gpr-closed but not rw-closed, as the image of the closed set \{a\} in X is \{a\} which is not rw-closed in Y.

3.4.12 **Remark:** The following examples show that the regular closed maps and rw-closed maps are independent.

3.4.13 **Example:** Let \(X=Y=\{a, b, c\}\), and a map \(f:(X, \tau)\rightarrow(Y, \sigma)\) be the identity map with \(\tau=\{X, \emptyset, \{a\}, \{b\}\}\) and \(\sigma=\{Y, \emptyset, \{a\}, \{a, b\}\}\). Then \(f\) is rw-closed but not regular closed, as the image of the regular closed set \{a\} in X is \{a\} which is not closed in Y.

3.4.14 **Example:** Let \(X=Y=\{a, b, c\}\), \(\tau=\{X, \emptyset, \{a, b\}\}\) and \(\sigma=\{Y, \emptyset, \{a\}, \{a, b\}\}\). Let a map \(f:(X, \tau)\rightarrow(Y, \sigma)\) be defined by \(f(a)=c, f(b)=b\) and \(f(c)=a\). Then \(f\) is regular closed but not rw-closed, as the image of the closed set \{c\} in X is \{a\} which is not rw-closed in Y.

3.4.15 **Remark:** The following examples show that the g-closed maps and rw-closed maps are independent.
3.4.16 Example: Let $X=Y=\{a, b, c\}$, and a map $f:(X, \tau)\to(Y, \sigma)$ be the identity map with $\tau=\{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma=\{Y, \phi, \{a\}\}$. Then $f$ is rw-closed but not g-closed, as the image of the closed set $\{a\}$ in $X$ is $\{a\}$ which is not g-closed in $Y$.

3.4.17 Example: Consider $X=\{a, b, c\}$, $Y=\{a, b, c, d\}$, $\tau=\{X, \phi, \{c\}\}$ and $\sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let a map $f:(X, \tau)\to(Y, \sigma)$ be defined by $f(a)=a$, $f(b)=d$ and $f(c)=c$. Then this function is g-closed but not rw-closed, as the image of the closed set $\{a, b\}$ in $X$ is $\{a, d\}$ which is not rw-closed in $Y$.

3.4.18 Remark: The following examples show that the wg-closed maps and rw-closed maps are independent.

3.4.19 Example: Consider $X=Y=\{a, b, c\}$, $\tau=\{X, \phi, \{b, c\}\}$ and $\sigma=\{Y, \phi, \{a\}\}$. Let a map $f:(X, \tau)\to(Y, \sigma)$ be the identity map. Then this function is rw-closed but not wg-closed, as the image of the closed set $\{a\}$ in $X$ is $\{a\}$ which is not wg-closed in $Y$.

3.4.20 Example: Consider $X=\{a, b, c\}$, $Y=\{a, b, c, d\}$, $\tau=\{X, \phi, \{a\}\}$ and $\sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let a map $f:(X, \tau)\to(Y, \sigma)$ be defined by $f(a)=c$, $f(b)=b$ and $f(c)=d$. Then this function is wg-closed but not rw-closed, as the image of the closed set $\{b, c\}$ in $X$ is $\{b, d\}$ which is not rw-closed in $Y$.

3.4.21 Remark: From the above discussions and known results we have the following implications:

In the following diagram, by

A \[\longrightarrow\] B we mean A implies B but not conversely and

A \[\longleftrightarrow\] B means A and B are independent of each other.
3.4.22 **Theorem:** If a mapping \( f:(X, \tau) \rightarrow (Y, \sigma) \) is \( rw \)-closed, then \( \text{rw-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \) of \( (X, \tau) \).

**Proof:** Suppose that \( f \) is \( rw \)-closed and \( A \subseteq X \). Then \( \text{cl}(A) \) is closed in \( X \) and so \( f(\text{cl}(A)) \) is \( rw \)-closed in \( (Y, \sigma) \). We have \( f(A) \subseteq f(\text{cl}(A)) \), by Theorem 2.4.16 (iv), \( \text{rw-cl}(f(A)) \subseteq \text{rw-cl}(f(\text{cl}(A))) \)---(i). Since \( f(\text{cl}(A)) \) is \( rw \)-closed in \( (Y, \sigma) \), \( \text{rw-cl}(f(\text{cl}(A))) = f(\text{cl}(A)) \)---(ii), by the Theorem 2.4.17. From (i) and (ii), we have \( \text{rw-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \) of \( (X, \tau) \).

3.4.23 **Remark:** The converse of the above theorem need not be true in general as seen from the following example.

3.4.24 **Example:** Let \( X=Y=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{c\}, \{a, c\}\} \) and \( \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Define \( f:(X, \tau) \rightarrow (Y, \sigma) \) by \( f(x)=x \) for every \( x \in X \). Then \( \text{rw-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \) of \( (X, \tau) \). But \( f \) is not \( rw \)-closed, since \( f(\{b\})=\{b\} \) is not \( rw \)-closed in \( (Y, \sigma) \).

3.4.25 **Corollary:** If a mapping \( f:(X, \tau) \rightarrow (Y, \sigma) \) is \( rw \)-closed, then the image \( f(A) \) of a closed set \( A \) in \( (X, \tau) \) is \( \tau_{rw} \)-closed in \( (Y, \sigma) \).

**Proof:** Let \( A \) be a closed set in \( (X, \tau) \). Since \( f \) is \( rw \)-closed, by above Theorem 3.4.22, \( \text{rw-cl}(f(A)) \subseteq f(\text{cl}(A)) \)---(i). Also \( \text{cl}(A)=A \), as \( A \) is a
closed set and so \( f(\text{cl}(A)) = f(A) \) —— (ii). From (i) and (ii), we have \( \text{rw-cl}(f(A)) \subseteq f(A) \). We know that \( f(A) \subseteq \text{rw-cl}(f(A)) \) and so \( \text{rw-cl}(f(A)) = f(A) \). Therefore \( f(A) \) is \( \tau_{\text{rw}} \)-closed in \((Y, \sigma)\).

3.4.26 Theorem: Let \((X, \tau)\) be any topological space and \((Y, \sigma)\) be a topological space where "\( \text{rw-cl}(A) = \text{w-cl}(A) \) for every subset \( A \) of \( Y \)" and \( f:(X, \tau) \rightarrow (Y, \sigma) \) be a map, then the following are equivalent.

(i) \( f \) is a \( \text{rw-closed} \) map.

(ii) \( \text{rw-cl}(f(A)) \subseteq f(\text{cl}(A)) \) for every subset \( A \) of \((X, \tau)\).

Proof: (i) \( \Rightarrow \) (ii) Follows from the Theorem 3.4.22.

(ii) \( \Rightarrow \) (i) Let \( A \) be any closed set in \((X, \tau)\). Then \( A = \text{cl}(A) \) and so \( f(A) = f(\text{cl}(A)) \) \( \supseteq \) \( \text{rw-cl}(f(A)) \) by hypothesis. We have \( f(A) \subseteq \text{rw-cl}(f(A)) \), by Theorem 2.4.16 (ii). Therefore \( f(A) = \text{rw-cl}(f(A)) \). Also \( f(A) = \text{rw-cl}(f(A)) = \text{w-cl}(f(A)) \), by hypothesis. That is \( f(A) = \text{w-cl}(f(A)) \) and so \( f \) is w-closed set in \((Y, \sigma)\). Thus \( f(A) \) is \( \text{rw-closed} \) set in \((Y, \sigma)\) and hence \( f \) is a \( \text{rw-closed} \) map.

3.4.27 Theorem: A map \( f:(X, \tau) \rightarrow (Y, \sigma) \) is \( \text{rw-closed} \) if and only if for each subset \( S \) of \((Y, \sigma)\) and for each open set \( U \) containing \( f^{-1}(S) \subseteq U \), there is a \( \text{rw-open} \) set \( V \) of \((Y, \sigma)\) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Proof: Suppose \( f \) is \( \text{rw-closed} \). Let \( S \subseteq Y \) and \( U \) be an open set of \((X, \tau)\) such that \( f^{-1}(S) \subseteq U \). Now \( X-U \) is a closed set in \((X, \tau)\). Since \( f \) is \( \text{rw-closed} \), \( f(X-U) \) is a \( \text{rw-closed} \) set in \((Y, \sigma)\). Then \( V = Y-f(X-U) \) is a \( \text{rw-open} \) set in \((Y, \sigma)\). Note that \( f^{-1}(S) \subseteq U \) implies \( S \subseteq V \) and \( f^{-1}(V) = X-f^{-1}(f(X-U)) \subseteq X-(X-U) = U \). That is \( f^{-1}(V) \subseteq U \).

For the converse, let \( F \) be a closed set of \((X, \tau)\). Then \( f^{-1}(f(F)^C) \subseteq F^C \) and \( F^C \) is an open in \((X, \tau)\). By hypothesis, there exists a \( \text{rw-open} \) set \( V \)
in \((Y, \sigma)\) such that \(f(F) \subseteq V\) and \(f^{-1}(V) \subseteq F\) and so \(F \subseteq (f^{-1}(V))^C\). Hence \(V^C \subseteq f(F) \subseteq f((f^{-1}(V))^C) \subseteq V^C\) which implies \(f(F) \subseteq V^C\). Since \(V^C\) is rw-closed, \(f(F)\) is rw-closed. That is \(f(F)\) is rw-closed in \((Y, \sigma)\) and therefore \(f\) is rw-closed.

**3.4.28 Remark:** The composition of two rw-closed maps need not be rw-closed map in general and this is shown by the following example.

**3.4.29 Example:** Let \(X=Y=Z=\{a, b, c\}\), \(\tau=\mathcal{P}(X)\), \(\sigma=\{Y, \phi, \{c\}, \{a, b\}\}\) and \(\eta=\{Z, \phi, \{a\}, \{b\} \{a, b\}\}\). Define \(f:(X, \tau)\to(Y, \sigma)\) by \(f(a)=a\), \(f(b)=b\) and \(f(c)=c\) and \(g:(Y, \sigma)\to(Z, \eta)\) be the identity map. Then \(f\) and \(g\) are rw-closed maps, but their composition \(gof:(X, \tau)\to(Z, \eta)\) is not rw-closed map, because \(F=\{a\}\) is closed in \((X, \tau)\), but \(gof(F)=gof(\{a\})=g(f(\{a\}))=g(\{a\})=\{a\}\) which is not rw-closed in \((Z, \eta)\).

**3.4.30 Theorem:** If \(f:(X, \tau)\to(Y, \sigma)\) is a closed map and \(g:(Y, \sigma)\to(Z, \eta)\) is a rw-closed map, then the composition \(gof:(X, \tau)\to(Z, \eta)\) is a rw-closed map.

**Proof:** Let \(F\) be any closed set in \((X, \tau)\). Since \(f\) is a closed map, \(f(F)\) is a closed set in \((Y, \sigma)\). Since \(g\) is a rw-closed map, \(g(f(F))\) is a rw-closed set in \((Z, \eta)\). That is \(gof(F) = g(f(F))\) is rw-closed and hence \(gof\) is a rw-closed map.

**3.4.31 Remark:** If \(f:(X, \tau)\to(Y, \sigma)\) is a rw-closed map and \(g:(Y, \sigma)\to(Z, \eta)\) is a closed map, then their composition need not be a rw-closed map as seen from the following example.

**3.4.32 Example:** Consider \(X=Y=Z=\{a, b, c\}\), \(\tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}\), \(\sigma=\{Y, \phi, \{a\}, \{b, c\}\}\) and \(\eta=\{Z, \phi, \{b\}, \{c\} \{b, c\}\}\). Let \(f:(X, \tau)\to(Y, \sigma)\) be the identity map and \(g:(Y, \sigma)\to(Z, \eta)\) is defined by \(g(a)=g(b)=a\) and \(g(c)=b\). Then \(f\) is a rw-closed map and \(g\) is a closed map. But their...
composition \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) is not a \( \text{rw-closed} \) map, since for the closed set \( \{c\} \) in \( (X, \tau) \), \( \text{gof}(\{c\}) = \text{g}(f(\{c\})) = \{b\} \), which is not \( \text{rw-closed} \) in \( (Z, \eta) \).

3.4.33 Theorem: If a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is irresolute, \( \text{rw-closed} \) and \( A \) is a \( \text{rw-closed} \) subset of \( X \), then \( f(A) \) is a \( \text{rw-closed} \) set in \( Y \).

Proof: Let \( f(A) \subseteq O \), where \( O \) is a regular semiopen set in \( Y \). Since \( f \) is irresolute, \( f^{-1}(O) \) is also regular semiopen in \( (X, \tau) \), by Lemma 3.3.30, and \( A \subseteq f^{-1}(O) \). Since \( A \) is a \( \text{rw-closed} \) set in \( X \), \( \text{cl}(A) \subseteq f^{-1}(O) \). Since \( f \) is \( \text{rw-closed} \), \( f(\text{cl}(A)) \) is a \( \text{rw-closed} \) set contained in the regular semiopen set \( O \), which implies \( \text{cl}(f(\text{cl}(A))) \subseteq O \) and so \( \text{cl}(f(A)) \subseteq O \). Hence \( f(A) \) is \( \text{rw-closed} \) in \( (Y, \sigma) \).

3.4.34 Corollary: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a \( \text{rw-closed} \) map and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be \( \text{rw-closed} \) and irresolute, then their composition \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) is \( \text{a \text{rw-closed} map} \).

Proof: Let \( A \) be a closed set of \( (X, \tau) \). Since \( f \) is a \( \text{rw-closed} \) map, \( f(A) \) is \( \text{rw-closed} \) in \( (Y, \sigma) \). Since \( g \) is both \( \text{rw-closed} \) and irresolute, \( g(f(A)) \) is \( \text{rw-closed} \) in \( (Z, \eta) \) by Theorem 3.4.33. Also \( g(f(A)) = \text{gof}(A) \). Therefore \( \text{gof} \) is \( \text{rw-closed} \).

3.4.35 Theorem: Let \( (X, \tau) \) and \( (Z, \eta) \) be topological spaces and \( (Y, \sigma) \) be topological space where “every \( \text{rw-closed} \) subset is closed”. Then the composition \( \text{gof}: (X, \tau) \rightarrow (Z, \eta) \) of the \( \text{rw-closed} \) maps \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) is \( \text{rw-closed} \).

Proof: Let \( A \) be a closed set of \( (X, \tau) \). Since \( f \) is \( \text{rw-closed} \), \( f(A) \) is \( \text{rw-closed} \) in \( (Y, \sigma) \). Then by hypothesis, \( f(A) \) is closed. Since \( g \) is \( \text{rw-closed} \), \( g(f(A)) \) is \( \text{rw-closed} \) in \( (Z, \eta) \) and \( g(f(A)) = \text{gof}(A) \). Therefore \( \text{gof} \) is \( \text{rw-closed} \).
3.4.36 Theorem: If \( f:(X, \tau) \rightarrow (Y, \sigma) \) is g-closed, \( g:(Y, \sigma) \rightarrow (Z, \eta) \) is rw-closed and \((Y, \sigma)\) is \(T_{\Sigma}\)-space then their composition \( g \circ f:(X, \tau) \rightarrow (Z, \eta) \) is rw-closed map.

Proof: Let \( A \) be a closed set of \((X, \tau)\). Since \( f \) is g-closed, \( f(A) \) is g-closed in \((Y, \sigma)\). Since \((Y, \sigma)\) is \(T_{\Sigma}\)-space, \( f(A) \) is closed in \((Y, \sigma)\). Since \( g \) is rw-closed, \( g(f(A)) \) is rw-closed in \((Z, \eta)\) and \( g(f(A)) = g \circ f(A) \). Therefore \( g \circ f \) is rw-closed.

3.4.37 Theorem: Let \( f:(X, \tau) \rightarrow (Y, \sigma) \) and \( g:(Y, \sigma) \rightarrow (Z, \eta) \) be two mappings such that their composition \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) be rw-closed mapping. Then the following statements are true.

(i) If \( f \) is continuous and surjective, then \( g \) is rw-closed.

(ii) If \( g \) is rw-irresolute and injective, then \( f \) is rw-closed.

(iii) If \( f \) is g-continuous, surjective and \((X, \tau)\) is a \(T_{\Sigma}\)-space, then \( g \) is rw-closed.

(iv) If \( g \) is strongly rw-continuous and injective, then \( f \) is rw-closed.

Proof: (i) Let \( A \) be a closed set of \((Y, \sigma)\). Since \( f \) is continuous, \( f^{-1}(A) \) is closed in \((X, \tau)\) and since \( g \circ f \) is rw-closed, \((g \circ f)(f^{-1}(A)) \) is rw-closed in \((Z, \eta)\). That is \( g(A) \) is rw-closed in \((Z, \eta)\), since \( f \) is surjective. Therefore \( g \) is rw-closed.

(ii) Let \( B \) be a closed set of \((X, \tau)\). Since \( g \circ f \) is rw-closed, \( g \circ f(B) \) is rw-closed in \((Z, \eta)\). Since \( g \) is rw-irresolute, \( g^{-1}(g \circ f(B)) \) is a rw-closed set in \((Y, \sigma)\). That is \( f(B) \) is rw-closed in \((Y, \sigma)\), since \( f \) is injective. Therefore \( f \) is rw-closed.
(iii) Let $C$ be a closed set of $(Y, \sigma)$. Since $f$ is $g$-continuous, $f^{-1}(C)$ is $g$-closed set in $(X, \tau)$. Since $(X, \tau)$ is a $T_X$-space, $f^{-1}(C)$ is a closed set in $(X, \tau)$. Since $gof$ is rw-closed, $(gof)(f^{-1}(C))$ is rw-closed in $(Z, \eta)$. That is $g(C)$ is rw-closed in $(Z, \eta)$, since $f$ is surjective. Therefore $g$ is rw-closed.

(iv) Let $D$ be a closed set of $(X, \tau)$. Since $gof$ is rw-closed, $(gof)(D)$ is rw-closed in $(Z, \eta)$. Since $g$ is strongly rw-continuous, $g^{-1}((gof)(D))$ is a closed set in $(Y, \sigma)$. That is $f(D)$ is a closed set in $(Y, \sigma)$, since $g$ is injective. Therefore $f$ is closed.

3.4.38 Theorem: If $f:(X, \tau)\to(Y, \sigma)$ is an open, continuous, rw-closed surjection and $\text{cl}(F)=F$ for every rw-closed set in $(Y, \sigma)$, where $X$ is regular, then $Y$ is regular.

Proof: Let $U$ be an open set in $Y$ and $p\in U$. Since $f$ is surjection, there exists a point $x\in X$ such that $f(x)=p$. Since $X$ is regular and $f$ is continuous, there is an open set $V$ in $X$ such that $x\in V\subset \text{cl}(V)\subset f^{-1}(U)$. Here $p\in f(V)\subset f(\text{cl}(V))\subset U$—(i). Since $f$ is rw-closed, $f(\text{cl}(V))$ is a rw-closed set contained in the open set $U$. By hypothesis, $\text{cl}(f(\text{cl}(V)))=f(\text{cl}(V))$ and $\text{cl}(f(V))\subset \text{cl}(f(\text{cl}(V)))$—(ii). From (i) and (ii), we have $p\in f(V)\subset \text{cl}(f(V))\subset U$ and $f(V)$ is open, since $f$ is open. Hence $Y$ is regular.

3.4.39 Theorem: If a map $f:(X, \tau)\to(Y, \sigma)$ is rw-closed and $A$ is a closed set of $X$, then $f_A: (A, \tau_A)\to(Y, \sigma)$ is rw-closed.

Proof: Let $F$ be a closed set of $A$. Then $F=A\cap E$ for some closed set $E$ of $(X, \tau)$ and so $F$ is closed set of $(X, \tau)$. Since $f$ is rw-closed, $f(F)$ is a rw-closed set in $(Y, \sigma)$. But $f(F)=f_A(F)$ and therefore $f_A: (A, \tau_A)\to(Y, \sigma)$ is rw-closed.
Analogous to rw-closed maps, we define rw-open map as follows.

3.4.40 **Definition:** A map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is called a rw-open map if the image \( f(A) \) is rw-open in \( (Y, \sigma) \) for each open set \( A \) in \( (X, \tau) \).

From the definitions we have the following results.

3.4.41 **Theorem:** (i) Every open map is rw-open but not conversely.

(ii) Every w-open map is rw-open but not conversely.

(iii) Every rw-open map is rg-open but not conversely.

(iv) Every rw-open map is rwg-open but not conversely.

(v) Every rw-open map is gpr-open but not conversely.

3.4.42 **Theorem:** For any bijection map \( f: (X, \tau) \rightarrow (Y, \sigma) \), the following statements are equivalent:

(i) \( f^{-1}: (Y, \sigma) \rightarrow (X, \tau) \) is rw-continuous.

(ii) \( f \) is rw-open map

(iii) \( f \) is rw-closed map.

**Proof:** (i)\( \Rightarrow \) (ii) Let \( U \) be an open set of \( (X, \tau) \). By assumption, \( (f^{-1})^{-1}(U) = f(U) \) is rw-open in \( (Y, \sigma) \) and so \( f \) is rw-open.

(ii)\( \Rightarrow \) (iii) Let \( F \) be a closed set of \( (X, \tau) \). Then \( F^C \) is an open set in \( (X, \tau) \). By assumption, \( f(F^C) \) is rw-open in \( (Y, \sigma) \). That is \( f(F^C) = f(F)^C \) is rw-open in \( (Y, \sigma) \) and therefore \( f(F) \) is rw-closed in \( (Y, \sigma) \). Hence \( f \) is rw-closed.

(iii) \( \Rightarrow \) (i) Let \( F \) be a closed set in \( (X, \tau) \). By assumption, \( f(F) \) is rw-closed in \( (Y, \sigma) \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is continuous.

3.4.43 **Theorem:** If a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is rw-open, then \( f(\text{int}(A)) \subseteq \text{rw-int}(f(A)) \) for every subset \( A \) of \( (X, \tau) \).
**Proof:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be an open map and \( A \) be any subset of \((X, \tau)\). Then \( \text{int}(A) \) is open in \((X, \tau)\) and so \( f(\text{int}(A)) \) is rw-open in \((Y, \sigma)\). We have \( f(\text{int}(A)) \subseteq f(A) \). Therefore by Theorem 2.4.3 (iii), \( f(\text{int}(A)) \subseteq \text{rw-int}(f(A)) \).

3.4.44 **Remark:** The converse of the Theorem 3.4.43 need not be true in general as seen from the following example.

3.4.45 **Example:** Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Define a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = a, f(b) = b \) and \( f(c) = c \). In \((Y, \sigma)\), \( \text{rw-int}(f(A)) = f(A) \) for every subset \( A \) of \((X, \tau)\). So \( f(\text{int}(A)) \subseteq f(A) = \text{rw-int}(f(A)) \) for every subset \( A \) of \( X \). But \( f \) is not a rw-open map, since for the open set \( \{a, c\} \) of \((X, \tau)\), \( f(\{a, c\}) = \{a, c\} \) which is not rw-open in \((Y, \sigma)\).

3.4.46 **Theorem:** If a map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is rw-open, then for each neighbourhood \( U \) of \( x \) in \((X, \tau)\), there exists a rw-neighbourhood \( W \) of \( f(x) \) in \((Y, \sigma)\) such that \( W \subseteq f(U) \).

**Proof:** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a rw-open map. Let \( x \in X \) and \( U \) be an arbitrary neighbourhood of \( x \) in \((X, \tau)\). Then there exists an open set \( G \) in \((X, \tau)\) such that \( x \in G \subseteq U \). Now \( f(x) \in f(G) \subseteq f(U) \) and \( f(G) \) is a rw-open set in \((Y, \sigma)\), as \( f \) is a rw-open map. By Theorem 2.3.29, \( f(G) \) is a rw-neighbourhood of each of its points. Taking \( f(G) = W \), \( W \) is a rw-neighbourhood of \( f(x) \) in \((Y, \sigma)\) such that \( W \subseteq f(U) \).

3.4.47 **Theorem:** A map \( f: (X, \tau) \rightarrow (Y, \sigma) \) is rw-open if and only if for any subset \( S \) of \((Y, \sigma)\) and any closed set of \((X, \tau)\) containing \( f^{-1}(S) \), there exists a rw-closed set \( K \) of \((Y, \sigma)\) containing \( S \) such that \( f^{-1}(K) \subseteq F \).

**Proof:** Suppose \( f \) is a rw-open map. Let \( S \subseteq Y \) and \( F \) be a closed set of \((X, \tau)\) such that \( f^{-1}(S) \subseteq F \). Now \( X - F \) is an open set in \((X, \tau)\). Since \( f \) is a
rw-open map, $f(X-F)$ is a rw-open set in $(Y, \sigma)$. Then $K=Y-f(X-F)$ is a rw-closed set in $(Y, \sigma)$. Note that $f^{-1}(S)\subseteq F$ implies $S \subseteq K$ and $f^{-1}(K)=X-f^{-1}(X-F)\subseteq X- (X-F)=F$. That is $f^{-1}(K) \subseteq F$.

For the converse, let $U$ be an open set of $(X, \tau)$. Then $f^{-1}((f(U))^\text{c}) \subseteq U^\text{c}$ and $U^\text{c}$ is a closed set in $(X, \tau)$. By hypothesis, there exists a rw-closed set $K$ of $(Y, \sigma)$ such that $(f(U))^\text{c} \subseteq K$ and $f^{-1}(K) \subseteq U^\text{c}$ and so $U \subseteq (f^{-1}(K))^\text{c}$. Hence $K^\text{c} \subseteq f(U) \subseteq f((f^{-1}(K))^\text{c}) \subseteq K^\text{c}$ which implies $f(U) = K^\text{c}$. Since $K^\text{c}$ is a rw-open, $f(U)$ is rw-open in $(Y, \sigma)$ and therefore $f$ is a rw-open map.

**3.4.48 Theorem:** If a function $f:(X, \tau) \rightarrow (Y, \sigma)$ is rw-open, then $f^{-1}(\text{rw-cl}(B)) \subseteq \text{cl}(f^{-1}(B))$ for each subset of $B$ of $(Y, \sigma)$.

**Proof:** Let $f:(X, \tau) \rightarrow (Y, \sigma)$ be a rw-open map and $B$ be any subset of $(Y, \sigma)$. Then $f^{-1}(B) \subseteq \text{cl}(f^{-1}(B))$ and $\text{cl}(f^{-1}(B))$ is a closed set in $(X, \tau)$. By above Theorem 3.4.47, there exists a rw-closed set $K$ of $(Y, \sigma)$ such that $B \subseteq K$ and $f^{-1}(K) \subseteq \text{cl}(f^{-1}(B))$. Now $\text{rw-cl}(B) \subseteq \text{rw-cl}(K)=K$, by Theorems 2.4.16 and 2.4.17, as $K$ is a rw-closed set of $(Y, \sigma)$. Therefore $f^{-1}(\text{rw-cl}(B)) \subseteq f^{-1}(K)$ and so $f^{-1}(\text{rw-cl}(B)) \subseteq f^{-1}(K) \subseteq \text{cl}(f^{-1}(B))$. Thus $f^{-1}(\text{rw-cl}(B)) \subseteq \text{cl}(f^{-1}(B))$ for each subset of $B$ of $(Y, \sigma)$.

**3.4.49 Remark:** The converse of the Theorem 3.4.48 need not be true in general as seen from the following example.

**3.4.50 Example:** Let $X=Y=\{a, b, c\}$, $\tau=\{X, \phi, \{b\}, \{b, c\}\}$ and $\sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define a map $f:(X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=a$, $f(b)=b$ and $f(c)=c$. In $(Y, \sigma)$, $\text{rw-cl}(B)=B$ for every subset $B$ of $(Y, \sigma)$. So $f^{-1}(\text{rw-cl}(B))=f^{-1}(B) \subseteq \text{cl}(f^{-1}(B))$ for every subset $B$ of $(Y, \sigma)$. But $f$ is not a rw-open map, since for the open set $\{b, c\}$ of $(X, \tau)$, $f(\{b, c\})=\{b, c\}$ which is not rw-open in $(Y, \sigma)$. 

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We define another new class of maps called $\text{rw}^*$-closed maps which are stronger than $\text{rw}$-closed maps.

3.4.50 Definition: A map $f: (X, \tau) \to (Y, \sigma)$ is said to be a $\text{rw}^*$-closed map if the image $f(A)$ is $\text{rw}$-closed in $(Y, \sigma)$ for every $\text{rw}$-closed set $A$ in $(X, \tau)$.

3.4.51 Theorem: Every $\text{rw}^*$-closed map is a $\text{rw}$-closed map but not conversely.

Proof: The proof follows from the two definitions and fact that every closed set is $\text{rw}$-closed.

The converse of the above Theorem is not true in general as seen from the following example.

3.4.52 Example: Let $X=Y=\{a, b, c\}, \tau=\{X, \phi, \{a\}, \{a, b\}\} \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $f:(X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is a $\text{rw}$-closed map but not a $\text{rw}^*$-closed map. Since $\{a\}$ is a $\text{rw}$-closed set in $(X, \tau)$, but its image under $f$ is $\{a\}$, which is not $\text{rw}$-closed in $(Y, \sigma)$.

3.4.53 Theorem: If a map $f:(X, \tau) \to (Y, \sigma)$ is irresolute and $\text{rw}$-closed, then it is a $\text{rw}^*$-closed map.

Proof: Let $f$ be a irresolute, $\text{rw}$-closed map and $A$ be any $\text{rw}$-closed set in $(X, \tau)$. By Theorem 3.4.33, $f(A)$ is $\text{rw}$-closed in $(Y, \sigma)$. Therefore $f$ is a $\text{rw}^*$-closed map.

3.4.54 Theorem: If $f:(X, \tau) \to (Y, \sigma)$ and $g:(Y, \sigma) \to (Z, \eta)$ are $\text{rw}^*$-closed maps, then their composition $g \circ f: (X, \tau) \to (Z, \eta)$ is also $\text{rw}^*$-closed.

Proof: Let $F$ be a $\text{rw}$-closed set in $(X, \tau)$. Since $f$ is $\text{rw}^*$-closed map, $f(F)$ is $\text{rw}$-closed set in $(Y, \sigma)$. Since $g$ is a $\text{rw}^*$-closed map, $g(f(F))$ is a $\text{rw}$-closed set in $(Z, \eta)$. Therefore $g \circ f$ is a $\text{rw}^*$-closed map.
Analogous to rw*-closed map, we define another new class of maps called rw*-open maps which are stronger than rw-open maps.

3.4.55 Definition: A map \( f:(X, \tau) \to (Y, \sigma) \) is said to be rw*-open map if the image \( f(A) \) is a rw-open set in \((Y, \sigma)\) for every rw-open set \( A \) in \((X, \tau)\).

3.4.56 Remark: Since every open set is a rw-open set, we have every rw*-open map is a rw-open map. The converse is not true in general as seen from the following example.

3.4.57 Example: Let \( X=Y=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}\} \) and \( \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Let \( f:(X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f \) is rw-open map but not rw*-open map, since for the rw-open set \( \{a, c\} \) in \((X, \tau)\), \( f(\{a, c\})=\{a, c\} \) which is not a rw-open set in \((Y, \sigma)\).

3.4.58 Theorem: If \( f:(X, \tau) \to (Y, \sigma) \) and \( g:(Y, \sigma) \to (Z, \eta) \) are rw*-open maps, then their composition \( gof:(X, \tau) \to (Z, \eta) \) is also rw*-open.

Proof: proof is similar to that of the Theorem 3.4.54.

3.4.59 Theorem: For any bijection map \( f:(X, \tau) \to (Y, \sigma) \), the following statements are equivalent:

(i) \( f^{-1}:(Y, \sigma) \to (X, \tau) \) is rw-irresolute

(ii) \( f \) is rw*-open map

(iii) \( f \) is rw*-closed map.

Proof: Proof is similar to that of Theorem 3.4.42.

3.5 \textit{rw-homeomorphisms in topological spaces.}

In this section, we introduce the concept of \textit{rw-homeomorphism} and study the relationship between homeomorphisms, w-homeomorphisms, g-homeomorphisms and rwg-homeomorphisms. Also
we introduce new class of maps rwc-homeomorphisms which form a sub class of rw-homeomorphisms. This class of maps is closed under composition of maps.

3.5.1 Definition: A bijection \( f:(X, \tau)\to(Y, \sigma) \) is called rw-homeomorphism if \( f \) and \( f^{-1} \) are rw-continuous.

We denote the family of all rw-homeomorphisms of a topological space \((X, \tau)\) onto itself by \( \text{rw-h}(X, \tau) \).

3.5.2 Example: Let \( X=Y=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \). Let \( f:(X, \tau)\to(Y, \sigma) \) be the identity map. Then \( f \) is bijective, rw-continuous and \( f^{-1} \) is rw-continuous. Therefore \( f \) is a rw-homeomorphism.

3.5.3 Theorem: Every homeomorphism is an rw-homeomorphism but not conversely.

Proof: Let \( f:(X, \tau)\to(Y, \sigma) \) be a homeomorphism. Then \( f \) and \( f^{-1} \) are continuous and \( f \) is bijection. As every continuous function is rw-continuous, we have \( f \) and \( f^{-1} \) are rw-continuous. Therefore \( f \) is a rw-homeomorphism.

The converse of the above Theorem is not true in general as seen from the following example.

3.5.4 Example: Let \( X=Y=\{a, b, c\} \) with \( \tau=\{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma=\{Y, \phi, \{a, b\}\} \). Define a map \( f:(X, \tau)\to(Y, \sigma) \) by \( f(a)=a \), \( f(b)=c \) and \( f(c)=b \). Then \( f \) is a rw-homeomorphism but it is not a homeomorphism, since the inverse image of the open set \( \{a\} \) in \((X, \tau)\) is \( \{a\} \), which is not a rw-open set in \((Y, \sigma)\).

3.5.5 Theorem: Every w-homeomorphism is an rw-homeomorphism but not conversely.
Proof: Let \( f:(X, \tau) \to (Y, \sigma) \) be a w-homeomorphism. Then \( f \) and \( f^{-1} \) are w-continuous and \( f \) is bijection. As every w-continuous function is rw-continuous, we have \( f \) and \( f^{-1} \) are rw-continuous. Therefore \( f \) is a rw-homeomorphism.

The converse of the above Theorem is not true in general as seen from the following example.

3.5.6 Example: Let \( X=Y=\{a, b, c\} \) with \( \tau=\{X, \phi, \{a\}\} \) and \( \sigma=\{Y, \phi, \{b\}\} \). Define a map \( f:(X, \tau) \to (Y, \sigma) \) by \( f(a)=c \), \( f(b)=b \) and \( f(c)=a \). Then \( f \) is a rw-homeomorphism but it is not a w-homeomorphism, since the inverse image of the open set \( \{a\} \) in \( (X, \tau) \) is \( \{c\} \), which is not a w-open set in \( (Y, \sigma) \).

3.5.7 Corollary: Every w*-homeomorphism is an rw-homeomorphism but not conversely.

Proof: From Sheik John [83], it follows that every w*-homeomorphism is a w-homeomorphism but not conversely. By Theorem 3.5.5, every w-homeomorphism is a rw-homeomorphism but not conversely and hence w*-homeomorphism is a rw-homeomorphism but not conversely.

3.5.8 Theorem: Every rw-homeomorphism is a rwg-homeomorphism but not conversely.

Proof: Let \( f:(X, \tau) \to (Y, \sigma) \) be a rw-homeomorphism. Then \( f \) and \( f^{-1} \) are rw-continuous and \( f \) is a bijection. Since every rw-continuous function is rwg-continuous, we have \( f \) and \( f^{-1} \) are rwg-continuous. Therefore \( f \) is a rwg-homeomorphism.

The converse of the above Theorem is not true in general as seen from the following example.
3.5.9 Example: Let $X=Y=\{a, b, c, d\}$ with $\tau=\{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\sigma=\{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Define a map $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=c$, $f(b)=b$, $f(c)=a$, and $f(d)=d$. Then $f$ is a rwg-homeomorphism but it is not a rw-homeomorphism, since the inverse image of the open set $\{a, b\}$ in $(X, \tau)$ is $\{b, c\}$, which is not a rw-open set in $(Y, \sigma)$.

3.5.10 Remark: rw-homeomorphism and g-homeomorphism are independent as seen from the following example.

3.5.11 Example: Let $X=Y=\{a, b, c\}$ with $\tau=\{X, \phi, \{a\}\}$ and $\sigma=\{Y, \phi, \{a, b\}\}$. Define a map $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=c$, $f(b)=b$ and $f(c)=a$. Then $f$ is a rw-homeomorphism but it is not a g-homeomorphism, since the inverse image of the open set $\{a\}$ of $(X, \tau)$ is $\{c\}$, which is not a g-open set in $(Y, \sigma)$.

3.5.12 Example: Let $X=Y=\{a, b, c, d\}$ and $\tau=\sigma=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define a map $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=c$, $f(b)=b$, $f(c)=a$, and $f(d)=d$. Then $f$ is a g-homeomorphism but it is not a rw-homeomorphism, since the inverse image of the open set $\{a, b\}$ of $(X, \tau)$ is $\{b, c\}$, which is not a rw-open set in $(Y, \sigma)$.

3.5.13 Theorem: Let $f:(X, \tau)\rightarrow(Y, \sigma)$ be a bijective rw-continuous map. Then the following are equivalent.

(i) $f$ is a rw-open map

(ii) $f$ is a rw-homeomorphism

(iii) $f$ is a rw-closed map.

Proof: Proof follows from Theorem 3.4.42.
3.5.14 **Remark:** The composition of two rw-homeomorphisms need not be a rw-homeomorphism in general as seen from the following example.

3.5.15 **Example:** Consider \( X=\{a, b, c\} \), \( \tau=\{X, \phi, \{a\}, \{a, c\}\} \), \( \sigma=\{Y, \phi, \{a, b\}\} \) and \( \eta=\{Z, \phi, \{a\}, \{a, b\}\} \). Let \( f:(X, \tau)\rightarrow(Y, \sigma) \) and \( g:(Y, \sigma)\rightarrow(Z, \eta) \) be the identity maps. Then both \( f \) and \( g \) are rw-homeomorphisms but their composition \( g\circ f:(X, \tau)\rightarrow(Z, \eta) \) is not a rw-homeomorphism, because for the open set \( \{a, c\} \) of \( (X, \tau) \), \( g\circ f(\{c\})=g(f(\{a, c\}))=g(\{c\})=\{a, c\} \), which is not rw-closed in \( (Z, \eta) \). Therefore \( g\circ f \) is not rw-open and so \( g\circ f \) is not a rw-homeomorphism.

3.5.16 **Definition:** A bijection \( f:(X, \tau)\rightarrow(Y, \sigma) \) is said to be a rwc-homeomorphism if both \( f \) and \( f^{-1} \) are rw-irresolute. We say that spaces \( (X, \tau) \) and \( (Y, \sigma) \) are rwc-homeomorphic if there exists a rwc-homeomorphism from \( (X, \tau) \) onto \( (Y, \sigma) \).

We denote the family of all rwc-homeomorphisms of a topological space \( (X, \tau) \) onto itself by rwc-h\((X, \tau)\).

3.5.17 **Theorem:** Every rwc-homeomorphism is a rw-homeomorphism but not conversely.

**Proof:** Let \( f:(X, \tau)\rightarrow(Y, \sigma) \) be an rwc-homeomorphism. Then \( f \) and \( f^{-1} \) are rw-irresolute and \( f \) is a bijection. By Theorem 3.3.3, \( f \) and \( f^{-1} \) are rw-continuous. Therefore \( f \) is a rw-homeomorphism.

The converse of the above Theorem is not true in general as seen from the following example.

3.5.18 **Example:** Let \( X=\{a, b, c\} \) with \( \tau=\{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma=\{Y, \phi, \{a, b\}\} \). Let \( f:(X, \tau)\rightarrow(Y, \sigma) \) be the identity map. Then \( f \) is a rw-homeomorphism but it is not a rwc-homeomorphism, since \( f \) is not rw-irresolute.
3.5.19 **Theorem:** Every rwc-homeomorphism is a rwg-homeomorphism but not conversely.

**Proof:** Proof follows from Theorems 3.5.17 and 3.5.8.

3.5.20 **Example:** The map $f$ in Example 3.5.9 is a rwg-homeomorphism but it is not a rwc-homeomorphism.

3.5.21 **Remark:** rwc-homeomorphism and $w^*$-homeomorphism are independent as seen from the following example.

3.5.22 **Example:** Let $X=Y=\{a, b, c\}$ with $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma=\{Y, \emptyset, \{a, b\}\}$. Let $f:(X, \tau)\rightarrow(Y, \sigma)$ be the identity map. Then $f$ is a $w^*$-homeomorphism but it is not a rwc-homeomorphism, since $f$ is not $w$-irresolute.

3.5.23 **Example:** Let $X=Y=\{a, b, c\}$ with topologies $\tau=\{X, \emptyset, \{a\}\}$ and $\sigma=\{Y, \emptyset, \{a, b\}\}$. Define a map $f:(X, \tau)\rightarrow(Y, \sigma)$ by $f(a)=c$, $f(b)=b$ and $f(c)=c$. Then $f$ is a rwc-homeomorphism but it is not a $w^*$-homeomorphism, since $f$ is not $w$-irresolute.

3.5.24 **Remark:** From the above discussions and known results we have the following implications

In the following diagram, by

- $A \rightarrow B$ we mean $A$ implies $B$ but not conversely and
- $A \leftrightarrow B$ means $A$ and $B$ are independent of each other.

- **Diagram:**
  
  $\xymatrix{w$-homeomorphism & $g$-homeomorphism \ar[r] & \text{homeomorphism} \ar[r] & \text{rw-homeomorphism} \ar[r] & \text{rwg-homeomorphism} \ar[r] & \text{rgw-homeomorphism} \ar[r] & \text{w*-homeomorphism} \ar[r] & \text{rwc-homeomorphism} \ar[r] & \text{homeomorphism} \ar[r] & w$-homeomorphism}
3.5.25 Theorem: Let \( f:(X, \tau)\rightarrow(Y, \sigma) \) and \( g:(Y, \sigma)\rightarrow(Z, \eta) \) be rwc-homeomorphisms. Then their composition \( gof:(X, \tau)\rightarrow(Z, \eta) \) is also a rwc-homeomorphism.

Proof: Let \( U \) be a rw-open set in \((Z, \eta)\). Since \( g \) is rw-irresolute, \( g^{-1}(U) \) is rw-open in \((Y, \sigma)\). Since \( f \) is rw-irresolute, \( f^{-1}(g^{-1}(U))=(gof)^{-1}(U) \) is a rw-open set in \((X, \tau)\). Therefore \( gof \) is rw-irresolute.

Also for a rw-open set \( G \) in \((X, \tau)\), we have \((gof)(G)=g(f(G))=g(W)\), where \( W=f(G) \). By hypothesis, \( f(G) \) is rw-open in \((Y, \sigma)\) and so again by hypothesis, \( g(f(G)) \) is a rw-open set in \((Z, \eta)\). That is \((gof)(G)\) is a rw-open set in \((Z, \eta)\) and therefore \((gof)^{-1}\) is rw-irresolute. Also \( gof \) is a bijection. Hence \( gof \) is a rwc-homeomorphism.

3.5.26 Theorem: The set \( \text{rwc-h}(X, \tau) \) is a group under the composition of maps.

Proof: Define a binary operation \(*: \text{rwc-h}(X, \tau)\times\text{rwc-h}(X, \tau)\rightarrow\text{rwc-h}(X, \tau)\) by \( f*g=gof \) for all \( f, g \in \text{rwc-h}(X, \tau) \) and \( o \) is the usual operation of composition of maps. Then by Theorem 3.5.25, \( gof \in \text{rwc-h}(X, \tau) \). We know that the composition of maps is associative and the identity map \( I:(X, \tau)\rightarrow(X, \tau) \) belonging to \( \text{rwc-h}(X, \tau) \) serves as the identity element.

If \( f \in \text{rwc-h}(X, \tau) \), then \( f^{-1} \in \text{rwc-h}(X, \tau) \) such that \( fof^{-1}=f^{-1}of=I \) and so inverse exists for each element of \( \text{rwc-h}(X, \tau) \). Therefore \( \text{rwc-h}(X, \tau), o \) is a group under the operation of composition of maps.

3.5.27 Theorem: Let \( f:(X, \tau)\rightarrow(Y, \sigma) \) be a rwc-homeomorphism. Then \( f \) induces an isomorphism from the group \( \text{rwc-h}(X, \tau) \) on to the group \( \text{rwc-h}(Y, \sigma) \).

Proof: Using the map \( f \), we define a map \( \psi_f: \text{rwc-h}(X, \tau)\rightarrow\text{rwc-h}(Y, \sigma) \) by \( \psi_f(h)=foho^{-1} \) for every \( h \in \text{rwc-h}(X, \tau) \). Then \( \psi_f \) is a bijection.
Further, for all $h_1, h_2 \in \text{rwc-h}(X, \tau)$, $\psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \psi_f(h_1) \circ \psi_f(h_2)$. Therefore $\psi_f$ is a homeomorphism and so it is an isomorphism induced by $f$. 

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