Topology is an indispensable object of study in Mathematics with open sets as well as closed sets being the most fundamental concepts in topological spaces. Open sets and closed sets have been generalized by several mathematicians.

In the year 1937, Stone [108] introduced new sets, which are stronger forms of open sets and closed sets called regular open sets and regular closed sets respectively in topological spaces. Velicko [119] has introduced and studied the stronger forms of open sets called $\delta$-open sets. In 1963 Levine [57] introduced and investigated weaker forms of open sets called semi-open sets. Njastad [82] introduced the concept of $\alpha$-open sets (originally called $\alpha$-sets) in topological spaces. Levine [58] introduced and studied the concept of generalized closed sets and a class of topological spaces called $T_{1/2}$-spaces. Dunham [37] and Dunham and Levine [38] further investigated some properties of $T_{1/2}$-spaces.

Mashhour et al [60], Abd El-Monesf et al [1], Andrijevic [2], Bhattacharya and Lahiri [12], Arya and Nour [8], Palaniappan and Rao [92], Nagaveni [81], Pusphalatha [96], have introduced and studied pre-closed, semi-pre-closed, $\beta$-closed, semi generalized closed, generalized semi-closed, regular generalized closed, weakly generalized closed, strongly generalized closed sets in topological spaces respectively, which are also...
weaker than closed sets. Ghanambal [46], Dontchev [28], Veerakumar [116] introduced generalized pre regular closed, generalized semi pre-closed, generalized pre -closed and pre semi-closed sets respectively, which are also weaker forms of closed sets in topological space. The complements of various types of open (closed) sets are called the same type of closed (open) sets.

Maki et al [67, 68] and Nono et al [91] introduced and studied the generalized \( \alpha \)-closed sets and \( \alpha \)-generalized closed sets (briefly \( g_{\alpha} \)-closed and \( \alpha g \)-closed respectively) and \( g^{\#}_{\alpha} \)-closed sets respectively and investigated their basic properties. Recently Sundaram and Sheik John [113] introduced and studied a weaker form of closed sets namely \( \omega \)-closed sets using semi-open sets and which is properly placed between closed sets and \( g \)-closed sets. They have also introduced four new spaces namely, \( T_\omega \)-spaces, \( g T_\omega \)-spaces, \( \alpha T_\omega \)-spaces and \( w_g T_\omega \)-spaces as an application and studied some of their properties.

This chapter contains five sections. In the second section, we introduce a new class of sets, namely \( \omega \alpha \)-closed sets using \( \omega \)-open sets in topological spaces. It is observed that \( \omega \alpha \)-closed sets are properly placed between \( \alpha \)-closed sets and \( \alpha g \)-closed sets. The complement of \( \omega \alpha \)-closed set is called \( \omega \alpha \)-open set. In the third section, we define \( \omega \alpha \)-closure and \( \omega \alpha \)-interior of a set \( A \) which are denoted by \( \omega \alpha \text{cl}(A) \) and \( \omega \alpha \text{int}(A) \) respectively and we prove that the complement of \( \omega \alpha \)-interior of a set \( A \) is the \( \omega \alpha \)-closure of the complement of \( A \). We prove that \( \omega \alpha \)-closure is a Kuratowski closure operator on \( X \). In the fourth section, we introduce \( \omega \alpha \)-
neighbourhood and $\omega \alpha$-limit point and study their properties. Moreover in
the last section we introduce four new spaces namely, $T_{\omega \alpha}$ - spaces, $\omega \alpha T$ -
spaces, $\alpha_8 T_{\omega \alpha}$ - spaces and $\omega \alpha T_{stg}$ - spaces as an application and obtain some
of their properties.

Throughout this thesis $(X, \tau)$, $(Y, \mu)$ and $(Z, \eta)$ (or $X, Y$ and $Z$)
represent non-empty topological spaces on which no separation axioms are
assumed unless explicitly stated. For a subset $A$ of $(X, \tau)$, the closure of $A$, the interior of $A$ with respect to $\tau$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively. The complement of $A$ is denoted by $A^c$. The $\alpha$-closure of $A$ is
the smallest $\alpha$-closed set containing $A$ and is denoted by $\text{acl}(A)$.

Before entering into our work we recall the following definitions from
various authors which are prerequisites for this chapter and we also use these
definitions wherever they are necessary throughout this thesis.

**Definition 1.1.1:** A subset $A$ of a topological space $(X, \tau)$ is called a

i) regular open set [108] if $A = \text{int}(\text{cl}(A))$ and a regular closed set if $A = \text{cl}(\text{int}(A))$.

ii) $\alpha$-open set [82] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and an $\alpha$-closed [61] set if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

iii) pre-open set [60] if $A \subseteq \text{int}(\text{cl}(A))$ and a pre-closed set if $\text{cl}(\text{int}(A)) \subseteq A$.

iv) semi-open set [57] if $A \subseteq \text{int}(\text{cl}(A))$ and a semi-closed if $\text{int}(\text{cl}(A)) \subseteq A$.

v) semi-preopen [2] set (= $\beta$-open set [1]) if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ and a
semi-preclosed (= $\beta$-closed set) if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$. 
For a subset $A$ of $(X, \tau)$, the intersection of all semi-closed (semi-open) subsets of $(X, \tau)$ containing $A$ is called semi-closure (semi-kernel) of $A$ and is denoted by $\text{scl}(A)$ (resp. $\text{sker}(A)$) and semi-interior of $A$ is the union of all semi-open sets contained in $A$ in $(X, \tau)$ and is denoted by $\text{sint}(A)$.

Pre-closure (resp. $\alpha$-closure, semi-preclosure) is the intersection of all pre-closed (resp. $\alpha$-closed, semi-preclosed) containing $A$ and is denoted by $\text{pcl}(A)$ (resp. $\alpha\text{cl}(A)$ or $\alpha\text{cl}(A)$, $\text{spcl}(A)$). The union of all pre-open (resp. $\alpha$-open, semi-preopen) sets contained in $A$ in $(X, \tau)$ is called pre-interior (resp. $\alpha$-interior, semi-preinterior) and is denoted by $\text{pint}(A)$ (resp. $\alpha\text{int}(A)$, $\text{spint}(A)$).

**Definition 1.1.2:** A subset $A$ of a topological space $(X, \tau)$ is called a

i) generalized closed (briefly g-closed) [58] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

ii) semi generalized closed (briefly sg-closed) [12] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.

iii) generalized semi closed (briefly gs-closed) [8] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

iv) generalized $\alpha$-closed (briefly $g\alpha$-closed) [67] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.

v) $\alpha$-generalized closed (briefly $\alpha g$-closed) [68] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

vi) regular generalized closed (briefly rg-closed) [92] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $(X, \tau)$.
vii) generalized semi-preclosed (briefly gsp-closed) [28] if \( \text{spcl}(A) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

viii) generalized pre-closed (briefly gp-closed) [72] if \( \text{pcl}(A) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

ix) generalized pre regular closed (briefly gpr-closed) [46] if \( \text{pcl}(A) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is regular-open in \((X, \tau)\).

x) weakly generalized closed (briefly wg-closed) [81] if \( \text{cl}(\text{int}(A)) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

xi) strongly generalized closed (= g*-closed[115]) [96] if \( \text{cl}(A) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is g-open in \((X, \tau)\).

xii) \( \omega \)-closed [113] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

xiii) \( \alpha \)-generalized regular closed (briefly agr-closed) [118] if \( \alpha \text{cl}(A) \subseteq U \)
whenever \( A \subseteq U \) and \( U \) is regular-open in \((X, \tau)\).

xiv) g*-preclosed (briefly g*p-closed) [117] if \( \text{pcl}(A) \subseteq U \) whenever \( A \subseteq U \)
and \( U \) is g-open in \((X, \tau)\).

xv) g\# \( \alpha \)-closed set [91] if \( \alpha \text{cl}(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is g-open
in \((X, \tau)\).

In 1965, Njastad [82] introduced a new spaces called \( \alpha \)- spaces using \( \alpha \)-closed sets. Later on Levine [58] introduced and studied \( T_{1/2} \)-space and proved that it lies strictly between \( T_0 \) and \( T_1 \) in 1970. Sundaram [110] introduced the concept of semi- \( T_{1/2} \)- spaces and Dontchev [28] introduced and studied semi-pre- \( T_{1/2} \)-spaces. Devi [23], Maki et al [67, 68], Sundaram and Sheik John [113], Pushpalatha [96] and Nôno [91] introduced and studied \( T_b \)-spaces, \( T_d \)- spaces, \( \alpha T_b \)- spaces, \( \alpha T_d \)- spaces, \( \alpha T_{1/2} \)- spaces, \( \alpha_{1/2} T_\alpha \)-
spaces, $T_p$-spaces, $T_s$-spaces, $T_o$-spaces, $aT_m$-spaces, $w_gT_o$-spaces, $\#\alpha T_{1/2}$-spaces and $\#\alpha T_{1/2}$-spaces respectively. Here we give the some definitions of the above mentioned spaces.

**Definition 1.1.3:** A topological space $(X, \tau)$ is said to be a

i) $\alpha$-space [82] if every $\alpha$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

ii) $T_{1/2}$-space [58] if every $g$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

iii) semi-pre- $T_{1/2}$-space [28] if every gsp-closed subset of $(X, \tau)$ is semi-pre closed in $(X, \tau)$.

iv) $aT_b$-space [23] if every $ag$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

v) $aT_d$-space [23] if every $ag$-closed subset of $(X, \tau)$ is g-closed in $(X, \tau)$.

vi) $aT_{1/2}$-space [67] if every $g\alpha$-closed subset of $(X, \tau)$ is $\alpha$-closed in $(X, \tau)$.

vii) $\alpha T_c$-space [26] if every $ag$-closed subset of $(X, \tau)$ is $\alpha$-closed in $(X, \tau)$.

viii) $\alpha T_o$-space [115] if every $ag$-closed subset of $(X, \tau)$ is g-closed in $(X, \tau)$.

ix) $agrT_{1/2}$-space [118] if every $agr$-closed subset of $(X, \tau)$ is $\alpha$-closed in $(X, \tau)$.

x) $T_wg$-space [81] if every $wg$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

xi) $T_o$-space [104] if every $\omega$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

xii) $aT_o$-space [104] if every $ag$-closed subset of $(X, \tau)$ is $\omega$-closed in $(X, \tau)$.

xiii) $\alpha T_{1/2}$-space [91] if every $g\alpha$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$. 
§ 1.2 \( \omega \alpha \)-Closed Sets in Topological Spaces

In this section we introduce \( \omega \alpha \)-closed sets in topological spaces and investigate some of their properties.

**Definition 1.2.1:** A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \omega \alpha \)-closed set if \( \alpha cl(A) \subseteq G \) whenever \( A \subseteq G \) and \( G \) is \( \omega \)-open in \( (X, \tau) \).

The family of all \( \omega \alpha \)-closed sets in a topological space \( (X, \tau) \) is denoted by \( \omega \alpha C(X, \tau) \).

**Theorem 1.2.2:** Every closed set is \( \omega \alpha \)-closed set.

**Proof:** Let \( A \) be closed set in a topological space \( (X, \tau) \). Let \( G \) be any \( \omega \)-open set in \( (X, \tau) \) such that \( A \subseteq G \). Since \( A \) is closed we have, \( cl(A) = A \). But \( \alpha cl(A) \subseteq cl(A) \) is always true, so \( \alpha cl(A) \subseteq cl(A) \subseteq G \). Therefore \( \alpha cl(A) \subseteq G \). Hence \( A \) is \( \omega \alpha \)-closed set in \( (X, \tau) \).

The converse of the above theorem need not be true as seen from the following example.

**Example 1.2.3:** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, X\} \). Then the set \( A = \{a, b\} \) is a \( \omega \alpha \)-closed but not a closed set in \( (X, \tau) \).
Theorem 1.2.4: Every $\alpha$-closed set is $\omega\alpha$-closed.
Proof: Let $A$ be any $\alpha$-closed set in $(X, \tau)$. Let $G$ be any $\omega$-open set in $(X, \tau)$ containing $A$. Since $A$ is $\alpha$-closed, we have $\text{acl}(A) = A \subseteq G$. Therefore $\text{acl}(A) \subseteq G$. Hence $A$ is $\omega\alpha$-closed set.

The converse of the above theorem need not be true as seen from the following example.

Example 1.2.5: In Example 1.2.3, the set $A = \{a, b\}$ is a $\omega\alpha$-closed but not an $\alpha$-closed in $(X, \tau)$.

Theorem 1.2.6: Every $\omega\alpha$-closed set is $\alpha g$-closed but not conversely.
Proof: Let $A$ be $\omega\alpha$-closed set and $G$ be any open set in $(X, \tau)$ such that $A \subseteq G$. Since every open set is $\omega$-open [104] and $A$ is $\omega\alpha$-closed, we have $\text{acl}(A) \subseteq G$ and hence $A$ is $\alpha g$-closed set in $(X, \tau)$.

Example 1.2.7: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then the set $A = \{a, b\}$ is a $\alpha g$-closed but not a $\omega\alpha$-closed set in $(X, \tau)$.

From the Theorem 1.2.6 and from the Theorem 1.2.4, it follows that $\omega\alpha$-closed sets are properly placed between $\alpha$-closed sets and $\omega\alpha$-closed sets.

Remark 1.2.8: From Stone [108] it follows that every regular closed set is closed but not conversely. From Theorem 1.2.2, every closed set is $\omega\alpha$-closed but not conversely. Therefore every regular closed set is $\omega\alpha$-closed set but not conversely.
Theorem 1.2.9: Union of two \( \omega \alpha \)-closed sets is a \( \omega \alpha \)-closed set.

Proof: Let \( A \) and \( B \) be two \( \omega \alpha \)-closed sets in \( (X, \tau) \). Let \( G \) be any \( \omega \)-open set in \( (X, \tau) \) such that \( A \cup B \subseteq G \). Then \( A \subseteq G \) and \( B \subseteq G \). Since \( A \) and \( B \) are \( \omega \alpha \)-closed set, \( \omega \text{cl}(A) \subseteq G \) and \( \omega \text{cl}(B) \subseteq G \). Therefore \( \omega \text{cl}(A) \cup \omega \text{cl}(B) = \omega \text{cl}(A \cup B) \subseteq G \). Hence \( A \cup B \) is \( \omega \alpha \)-closed.

Remark 1.2.10: Intersection of two \( \omega \alpha \)-closed sets need not be a \( \omega \alpha \)-closed set as seen from the following example.

Example 1.2.11: In Example 1.2.3, the sets \( A = \{a, b\} \) and \( B = \{a, c\} \) are \( \omega \alpha \)-closed sets but \( A \cap B = \{a\} \) is not a \( \omega \alpha \)-closed set in \( (X, \tau) \).

Theorem 1.2.12: Every \( \omega \alpha \)-closed set is gpr-closed (resp. \( \alpha \text{gr} \)-closed) set.

Proof: Let \( A \) be \( \omega \alpha \)-closed set in \( (X, \tau) \). Let \( G \) be a regular open set in \( (X, \tau) \) such that \( A \subseteq G \). Since \( G \) is regular open, it is \( \omega \)-open. Again since \( A \) is \( \omega \alpha \)-closed set, \( \omega \text{cl}(A) \subseteq G \). But \( \text{pcl}(A) \subseteq \omega \text{cl}(A) \subseteq G \) and hence \( A \) is gpr-closed set (resp. \( \alpha \text{gr} \)-closed).

The converse of the above theorem need not be true as seen from the following example.

Example 1.2.13: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{a, c\}, X\} \). Then the set \( A = \{a, c\} \) is a gpr-closed but not a \( \omega \alpha \)-closed set in \( (X, \tau) \).

Example 1.2.14: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \). Then the set \( A = \{a, b\} \) is \( \alpha \text{gr} \)-closed but not \( \omega \alpha \)-closed in \( (X, \tau) \).
Theorem 1.2.15: Every $\omega\alpha$-closed set is gsp-closed set.

Proof: Let $A$ be $\omega\alpha$-closed set in $(X, \tau)$. Let $G$ be any open set in $(X, \tau)$ such that $A \subseteq G$. Since every open set is $\omega$-open, $G$ is $\omega$-open. By hypothesis $A$ is $\omega\alpha$-closed. Therefore $\alphacl(A) \subseteq G$. But $spcl(A) \subseteq \alphacl(A)$. So $spcl(A) \subseteq G$ and hence $A$ is gsp-closed in $(X, \tau)$.

The converse of the above theorem need not be true as seen from the following example.

Example 1.2.16: In Example 1.2.14, the set $A = \{a\}$ is a gsp-closed but not a $\omega\alpha$-closed set in $(X, \tau)$.

Theorem 1.2.17: A subset $A$ of $(X, \tau)$ is $\omega\alpha$-closed, then $A$ is gp-closed set in $(X, \tau)$.

Proof: Let $A$ be $\omega\alpha$-closed set of $(X, \tau)$. Let $G$ be an open set in $(X, \tau)$ such that $A \subseteq G$. Since every open set is $\omega$-open, $G$ is $\omega$-open in $(X, \tau)$. Since $A$ is $\omega\alpha$-closed, $\alphacl(A) \subseteq G$. But $pcl(A) \subseteq \alphacl(A)$. So $pcl(A) \subseteq G$. Hence $A$ is gp-closed set in $(X, \tau)$.

The converse of the above theorem need not be true as seen from the following example.

Example 1.2.18: In Example 1.2.7, the set $A = \{a, b\}$ is gp-closed but not $\omega\alpha$-closed set in $(X, \tau)$.

Theorem 1.2.19: Every strongly g-closed set is $\omega\alpha$-closed set.

Proof: Let $A$ be strongly g-closed set in $(X, \tau)$. Let $G$ be any $\omega$-open set and so g-open in $(X, \tau)$ such that $A \subseteq G$. Then $cl(A) \subseteq G$. But $\alphacl(A) \subseteq cl(A)$. Therefore $\alphacl(A) \subseteq G$. Hence $A$ is $\omega\alpha$-closed set in $(X, \tau)$.
The converse of the above theorem need not be true as seen from the following example.

**Example 1.2.20:** In Example 1.2.3, the set \(A = \{b\}\) is a \(\omega\alpha\)-closed but not a strongly g-closed set in \((X, \tau)\).

**Theorem 1.2.21:** Every \(g^\#\alpha\)-closed set is \(\omega\alpha\)-closed set.

**Proof:** Let \(A\) be \(g^\#\alpha\)-closed set in \((X, \tau)\). Let \(G\) be any \(\omega\)-open set and so g-open in \((X, \tau)\) such that \(A \subseteq G\). Then \(\alphacl(A) \subseteq G\). Hence \(A\) is \(\omega\alpha\)-closed set in \((X, \tau)\).

The converse of the above theorem need not be true as seen from the following example.

**Example 1.2.22:** In Example 1.2.3, the set \(A = \{a, c\}\) is a \(\omega\alpha\)-closed but not a \(g^\#\alpha\)-closed set in \((X, \tau)\).

**Remark 1.2.23:** The concept of \(\omega\alpha\)-closed sets is independent of the concepts of sets namely \(\omega\)-closed, g-closed, \(\beta\)-closed, \(g\alpha\)-closed and semi-closed sets as seen from the following examples.

**Example 1.2.24:** In Example 1.2.13, the set \(\{a, b\}\) is \(\omega\alpha\)-closed but not \(\omega\)-closed, \(\beta\)-closed, \(g\alpha\)-closed and semi-closed and the set \(A = \{c\}\) is a \(\omega\alpha\)-closed but not a g-closed set in \((X, \tau)\).

**Example 1.2.25:** In Example 1.2.7, the set \(A = \{b\}\) is \(\omega\)-closed, \(\beta\)-closed, g-closed, \(g\alpha\)-closed but not a \(\omega\alpha\)-closed set in \((X, \tau)\).

**Example 1.2.26:** In Example 1.2.14, the set \(A = \{b\}\) is semi-closed but not a \(\omega\alpha\)-closed set in \((X, \tau)\).
Remark 1.2.27: $\omega\alpha$-closedness is independent of g*p-closedness and sg-closedness are as seen from the following examples.

Example 1.2.28: In Example 1.2.3, the set $A = \{a, b\}$ is a $\omega\alpha$-closed but not a g*p-closed set in $(X, \tau)$.

Example 1.2.29: In Example 1.2.7, the set $A = \{a, b\}$ is a g*p-closed but not a $\omega\alpha$-closed set.

Example 1.2.30: In Example 1.2.13, the set $A = \{a, b\}$ is a $\omega\alpha$-closed but not a sg-closed set in $(X, \tau)$.

Example 1.2.31: In Example 1.2.14, the set $A = \{a\}$ is a sg-closed set but not a $\omega\alpha$-closed set in $(X, \tau)$.

Remark 1.2.32: Since every gsp-closed set is gs-closed [28] but not conversely and by Theorem 1.2.15 every $\omega\alpha$-closed set is gsp-closed but not conversely. Hence every $\omega\alpha$-closed set is gs-closed but not conversely.

Example 1.2.33: In Example 1.2.14, the set $A = \{a\}$ is a gs-closed but not a $\omega\alpha$-closed set in $(X, \tau)$.

Theorem 1.2.34: Every $\omega\alpha$-closed set is wg-closed set.

Proof: Let $A$ be a $\omega\alpha$-closed set and $G$ be an open set and so $\omega$-open in $(X, \tau)$ such that $A \subseteq G$. Then $\alphacl(A) \subseteq G$. But $cl(int(A)) \subseteq \alphacl(A)$, so $cl(int(A)) \subseteq G$. Hence $A$ is $\omega g$-closed in $(X, \tau)$.

The converse of the above theorem need not be true as seen from the following example.
Example 1.2.35: In Example 1.2.7, the set \( A = \{ b \} \) is a \( wg \)-closed but not a \( \omega\alpha \)-closed set in \( (X,\tau) \).

Therefore from the above theorems, it follows that \( \omega\alpha \)-closed sets are contained in \( \alpha g \)-closed sets, \( gpr \)-closed sets, \( gp \)-closed sets, \( gsp \)-closed sets, \( \alpha gr \)-closed sets, \( wg \)-closed sets and \( \omega\alpha \)-closed sets contain \( \alpha \)-closed sets, strongly \( g \)-closed sets and \( g\# \alpha \)-closed sets.

Remark 1.2.36: From the above results we have the following implications.

\[
\begin{align*}
g\text{-closed} & \quad \rightarrow \quad g^*\text{-closed} \quad \rightarrow \quad sg\text{-closed} \quad \rightarrow \quad \omega\alpha\text{-closed} \quad \rightarrow \quad gsp\text{-closed} \quad \rightarrow \quad wg\text{-closed} \\
\text{closed} & \quad \rightarrow \quad \omega\alpha\text{-closed} \quad \rightarrow \quad \alpha g\text{-closed} \\
\alpha\text{-closed} & \quad \rightarrow \quad \omega\alpha\text{-closed} \\
\text{semi-closed} & \quad \rightarrow \quad \omega\alpha\text{-closed} \\
\beta\text{-closed} & \quad \rightarrow \quad g^*p\text{-closed} \quad \rightarrow \quad gp\text{-closed} \quad \leftarrow \quad \text{pre-closed} \\
g\alpha\text{-closed} & \quad \rightarrow \quad \omega\alpha\text{-closed} \\
g\# \alpha\text{-closed} & \quad \rightarrow \quad \omega\alpha\text{-closed}
\end{align*}
\]

Here \( A \rightarrow B \) means "\( A \) implies \( B \)" but not conversely and

\[ A \leftarrow B \text{ means "\( A \) and \( B \) are independent of each other"} \]

Theorem 1.2.37: If a subset \( A \) of \( X \) is \( \omega\alpha \)-closed in \( (X,\tau) \), then \( \alpha cl(A) - A \) contains no non-empty closed set in \( (X,\tau) \).

Proof: Suppose that \( A \) is \( \omega\alpha \)-closed set in \( (X,\tau) \) and \( F \) be non-empty closed subset of \( \alpha cl(A) - A \). Then \( A \subseteq X - F \). Since \( X - F \) is open, it is \( \omega \)-open set
and $A$ is $\omega\alpha$-closed, $\alpha\text{cl}(A) \subseteq (X - F)$. Consequently $F \subseteq (\alpha\text{cl}(A))^c$. Thus $F \subseteq [\alpha\text{cl}(A)] \cap [\alpha\text{cl}(A)]^c = \emptyset$. That is, $F = \emptyset$. Thus $\alpha\text{cl}(A) - A$ contains no non-empty closed set.

The converse of the above theorem need not be true as seen from the following example.

**Example 1.2.38:** In Example 1.2.7, let $A = \{a, b\}$, then $\alpha\text{cl}(A) - A = \{c\}$ does not contain non-empty closed sets. But $A$ is not a $\omega\alpha$-closed in $(X, \tau)$.

**Theorem 1.2.39:** In a $T_1$ - space, $\omega\alpha$-closed sets are $\alpha$-closed.

**Proof:** Let $A$ be a $\omega\alpha$-closed in a $T_1$ - space $(X, \tau)$. If $x \in \alpha\text{cl}(A) - A$, then $\{x\} \subseteq \alpha\text{cl}(A) - A$ and since $(X, \tau)$ is $T_1$, $\{x\}$ is a closed in $(X, \tau)$. Then by Theorem 1.2.37, there exists no element in $\alpha\text{cl}(A) - A = \emptyset$. Therefore $\alpha\text{cl}(A) = A$. Hence $A$ is $\alpha$-closed.

**Theorem 1.2.40:** For each $x \in X$, either $\{x\}$ is $\omega$-closed or $\{x\}^c$ is $\omega\alpha$-closed in $(X, \tau)$.

**Proof:** Suppose that $\{x\}$ is not $\omega$-closed in $(X, \tau)$. Then $\{x\}^c$ is not $\omega$-open and the only $\omega$-open set containing $\{x\}^c$ is the space is $X$ itself. Therefore $\alpha\text{cl}(\{x\}^c) \subseteq X$ and hence $\{x\}^c$ is $\omega\alpha$-closed set in $(X, \tau)$.

**Theorem 1.2.41:** If a subset $A$ of $X$ is $\omega\alpha$-closed in $(X, \tau)$, then $\alpha\text{cl}(A) - A$ does not contain any non-empty $\omega$-closed set in $(X, \tau)$.

**Proof:** Suppose $A$ is $\omega\alpha$-closed set in $(X, \tau)$ and $F$ be a non-empty $\omega$-closed subset of $\alpha\text{cl}(A) - A$. That is, $F \subseteq (\alpha\text{cl}(A) - A)$, then $F \subseteq \alpha\text{cl}(A) \cap (X - A)$.

So $F \subseteq \alpha\text{cl}(A)$ and $F \subseteq X - A$. Since $X - F$ is $\omega$-open set and $A$ is $\omega\alpha$-closed, $\alpha\text{cl}(A) \subseteq (X - F)$.

Thus $F \subseteq X - \alpha\text{cl}(A)$ . Consequently
\( F \subseteq (\text{acl}(A))^c \). Thus \( F \subseteq [\text{acl}(A)] \cap [\text{acl}(A)]^c = \phi \). That is, \( F = \phi \). Thus \( \text{acl}(A) - A \) contains no non-empty \( \omega \)-closed set.

However the converse need not be true as seen from the following example.

**Example 1.2.42:** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{b, c\}, \{b, c, d\}, \{a, b, c\}, X\} \). Let \( A = \{a, b\} \), then \( \text{acl}(A) - A = \{c, d\} \) does not contain non-empty \( \omega \)-closed set. But \( A \) is not \( \omega \alpha \)-closed set in \((X, \tau)\).

**Theorem 1.2.43:** If \( A \) is \( \omega \alpha \)-closed set in \((X, \tau)\) and \( A \subseteq B \subseteq \text{acl}(A) \), then \( B \) is also \( \omega \alpha \)-closed set in \((X, \tau)\).

**Proof:** It is given that \( A \) is \( \omega \alpha \)-closed set in \((X, \tau)\). To prove that \( B \) is also a \( \omega \alpha \)-closed set of \((X, \tau)\). Let \( G \) be an \( \omega \)-open set of \((X, \tau)\) such that \( B \subseteq G \). Since \( A \subseteq B \), we have \( A \subseteq G \). Since \( A \) is \( \omega \alpha \)-closed, we have \( \text{acl}(A) \subseteq G \). Now \( \text{acl}(B) \subseteq \text{acl}((\text{acl}(A))) = \text{acl}(A) \subseteq G \). So \( \text{acl}(B) \subseteq G \). Hence \( B \) is \( \omega \alpha \)-closed set in \((X, \tau)\).

**Remark 1.2.44:** The converse of the above theorem need not be true in general as seen in the following example.

**Example 1.2.45:** In Example 1.2.13, let \( A = \{b\} \) and \( B = \{a, b\} \). Then \( A \) and \( B \) are \( \omega \alpha \)-closed sets but \( A \subseteq B \not\subseteq \text{acl}(A) \).

**Theorem 1.2.46:** If \( A \) is both \( \alpha \)-open and \( \omega \alpha \)-closed in \((X, \tau)\), then it is \( \omega \alpha \)-closed set in \((X, \tau)\).

**Proof:** Let \( A \) be both \( \alpha \)-open and \( \omega \alpha \)-closed in \((X, \tau)\). Let \( U \) be \( \omega \)-open set in \((X, \tau)\) such that \( A \subseteq U \). Since \( A \subseteq A \) and \( A \) is both \( \alpha \)-open and \( \omega \alpha \)-closed,
\[ \text{acl}(A) \subseteq A. \text{ But } \text{acl}(A) \subseteq A \subseteq U. \text{ Therefore } \text{acl}(A) \subseteq U. \text{ Hence } A \text{ is } \omega\alpha\text{-closed in } (X,\tau). \]

**Remark 1.2.47:** If \( A \) is both \( \alpha \)-open and \( \omega\alpha \)-closed in \( (X,\tau) \), then \( A \) need not be a \( g\alpha \)-closed set in \( (X,\tau) \) as seen from the following example.

**Example 1.2.48:** In Example 1.2.3, the set \( A = \{a, b\} \) is both \( \alpha \)-open and \( \omega\alpha \)-closed but not \( g\alpha \)-closed in \( (X,\tau) \).

Now we introduce the following definition.

**Definition 1.2.49:** A subset \( A \) of a topological space \( (X,\tau) \) is called \( \omega\alpha \)-open set if \( X - A \) is \( \omega\alpha \)-closed. The class of all \( \omega\alpha \)-open sets in \( (X,\tau) \) is denoted by \( \tau^{\omega\alpha} \).

**Theorem 1.2.50:** For any topological space \( (X,\tau) \), we have

(i) Every open set is \( \omega\alpha \)-open

(ii) Every \( \alpha \)-open set is \( \omega\alpha \)-open

(iii) Every \( \omega\alpha \)-open set is \( \alpha g \)-open.

(iv) Every \( \omega\alpha \)-open set is gpr-open

(v) Every \( \omega\alpha \)-open set is \( \alpha g r \)-open

(vi) Every \( \omega\alpha \)-open set is gsp-open

(vii) Every \( \omega\alpha \)-open set is gp-open

(viii) Every strongly g-open set is \( \omega\alpha \)-open

(ix) Every \( g^\#\alpha \)-open set is \( \omega\alpha \)-open.

(x) Every \( \omega\alpha \)-open set is wg-open

**Proof:** The proof follows from the Theorems 1.2.2, 1.2.4, 1.2.6, 1.2.12, 1.2.15, 1.2.17, 1.2.19, 1.2.21 and 1.2.34.
Theorem 1.2.51: A subset $A$ of $(X, \tau)$ is $\omega\alpha$-open if and only if $F \subseteq \alpha\text{int}(A)$ whenever $F$ is $\omega$-closed and $F \subseteq A$.

Proof: Assume that $A$ is $\omega\alpha$-open set in $(X, \tau)$ and $F$ be $\omega$-closed set of $(X, \tau)$ such that $F \subseteq A$. Then $X - A$ is an $\omega\alpha$-closed set in $(X, \tau)$. Also $X - A \subseteq X - F$ and $X - F$ is $\omega$-open set of $(X, \tau)$. This implies that $\alpha\text{cl}(X - A) \subseteq X - F$. But $\alpha\text{cl}(X - A) = X - \alpha\text{int}(A)$. Thus $X - \alpha\text{int}(A) \subseteq X - F$. So $F \subseteq \alpha\text{int}(A)$.

Conversely suppose $F \subseteq \alpha\text{int}(A)$ whenever $F$ is $\omega$-closed and $F \subseteq A$. To prove that $A$ is $\omega\alpha$-open. Let $G$ be $\omega$-open set of $(X, \tau)$ such that $X - A \subseteq G$. Then $X - G$ is $\omega$-closed set containing $A$. So $X - G \subseteq \alpha\text{int}(A)$, $X - \alpha\text{int}(A) \subseteq G$. But $\alpha\text{cl}(X - A) = X - \alpha\text{int}(A)$. Thus $\alpha\text{cl}(X - A) \subseteq G$. That is $X - A$ is $\omega\alpha$-closed set and hence $A$ is $\omega\alpha$-open.

Theorem 1.2.52: If $A$ and $B$ are $\omega\alpha$-open sets in $(X, \tau)$, then $A \cap B$ is also $\omega\alpha$-open set in $(X, \tau)$.

Proof: Let $A$ and $B$ be $\omega\alpha$-open sets in $(X, \tau)$. Then $X - A$ and $X - B$ are $\omega\alpha$-closed sets in $(X, \tau)$. By Theorem 1.2.9 $(X - A) \cup (X - B)$ is $\omega\alpha$-closed set. That is $(X - A) \cup (X - B) = X - (A \cap B)$ is $\omega\alpha$-closed set in $(X, \tau)$. Therefore $A \cap B$ is $\omega\alpha$-open set in $(X, \tau)$.

Theorem 1.2.53: If $A$ is a $\omega\alpha$-closed set of $(X, \tau)$, then $\alpha\text{cl}(A) - A$ is a $\omega\alpha$-open set of $(X, \tau)$.

Proof: Let $A$ be a $\omega\alpha$-closed of $(X, \tau)$. Let $G$ be $\omega$-closed set of $(X, \tau)$ such that $G \subseteq \alpha\text{cl}(A) - A$. By the Theorem 1.2.41, we have $\alpha\text{cl}(A) - A$ does not contain any non-empty $\omega$-closed set. Thus $G = \emptyset$. Then $G \subseteq \alpha\text{int}(\alpha\text{cl}(A) - A)$. Therefore by the Theorem 1.2.51, $\alpha\text{cl}(A) - A$ is a $\omega\alpha$-open set of $(X, \tau)$.
Theorem 1.2.54: If \( \alpha \text{int}(A) \subseteq B \subseteq A \) and \( A \) is a \( \omega \alpha \)-open set of \((X, \tau)\) then \( B \) is a \( \omega \alpha \)-open set.

Proof: If \( \alpha \text{int}(A) \subseteq B \subseteq A \), then \( X - A \subseteq X - B \subseteq X - \alpha \text{int}(A) \). That is \( X - A \subseteq X - B \subseteq \alpha \text{cl}(X - A) \). Since \( X - A \) is \( \omega \alpha \)-closed set and then by the Theorem 1.2.43, \( X - B \) is also a \( \omega \alpha \)-closed set. Therefore \( B \) is a \( \omega \alpha \)-open set.

Theorem 1.2.55: If \( A \) is \( \omega \)-open and \( \omega \alpha \)-closed set then \( A \) is \( \alpha \)-closed.

Proof: Since \( A \subseteq A \) and \( A \) is \( \omega \)-open and \( \omega \alpha \)-closed, we have \( \alpha \text{cl}(A) \subseteq A \). Thus \( \alpha \text{cl}(A) = A \). Hence \( A \) is \( \alpha \)-closed set of \((X, \tau)\).

Theorem 1.2.56: In a topological space \((X, \tau)\), \( \omega \text{O}(X, \tau) \subseteq \{F \subseteq X: F^c \in \tau\} \) if and only if every subset of \( X \) is \( \omega \alpha \)-closed.

Proof: Suppose that \( \omega \text{O}(X, \tau) \subseteq \{F \subseteq X: F^c \in \tau\} \). Let \( A \) be a subset of \( X \) such that \( A \subseteq U \) where \( U \) is \( \omega \)-open. Then \( U \in \omega \text{O}(X, \tau) \subseteq \{F \subseteq X: F^c \in \tau\} \). That is \( U \in \{F \subseteq X: F^c \in \tau\} \). Thus \( U \) is closed, it is \( \alpha \)-closed. Then \( \alpha \text{cl}(U) = U \). Also \( \alpha \text{cl}(A) \subseteq \alpha \text{cl}(U) = U \). Hence \( A \) is \( \omega \alpha \)-closed.

Conversely suppose that every subset of \( X \) is \( \omega \alpha \)-closed. Let \( U \in \text{WO}(X, \tau) \). Since \( U \subseteq U \) and \( U \) is \( \omega \alpha \)-closed we have \( \alpha \text{cl}(U) \subseteq U \). Thus \( \alpha \text{cl}(U) = U \) and \( U \in \{F \subseteq X: F^c \in \tau\} \). Thus \( \text{WO}(X, \tau) = \{F \subseteq X: F^c \in \tau\} \).

Remark 1.2.57: From Stone [108] it follows that every regular open set is open but not conversely. By Theorem 1.2.50(i), every open set is \( \omega \alpha \)-open but not conversely and hence every regular open set is \( \omega \alpha \)-open but not conversely.
§ 1.3 \(\omega\alpha\)-Closure and \(\omega\alpha\)-Interior

In this section, the notion of \(\omega\alpha\)-closure is defined and some of its basic properties are studied. For any \(A \subseteq X\), \(\omega\alpha\)-interior of \(A\) is defined and it is proved that the complement of \(\omega\alpha\)-interior of \(A\) is the \(\omega\alpha\)-closure of the complement of \(A\).

**Definition 1.3.1:** For a subset \(A\) of \((X, \tau)\), \(\omega\alpha\)-closure of \(A\), denoted by \(\omega\alpha\text{cl}(A)\) and is defined as \(\omega\alpha\text{cl}(A) = \cap\{G; A \subseteq G, G\) is \(\omega\alpha\)-closed in \((X, \tau)\}\).

**Theorem 1.3.2:** For an \(x \in X\), \(x \in \omega\alpha\text{cl}(A)\) if and only if \(A \cap V \neq \emptyset\) for every \(\omega\alpha\)-open set \(V\) containing \(x\).

**Proof:** Let \(x \in \omega\alpha\text{cl}(A)\). Suppose there exists an \(\omega\alpha\)-open set \(V\) containing \(x\) such that \(V \cap A = \emptyset\). Then \(A \subseteq X - V\). Since \(X - V\) is \(\omega\alpha\)-closed, \(\omega\alpha\text{cl}(A) \subseteq X - V\). This implies \(x \notin \omega\alpha\text{cl}(A)\) which is a contradiction. Hence \(V \cap A \neq \emptyset\) for every \(\omega\alpha\)-open set \(V\) containing \(x\).

Conversely, let \(A \cap V \neq \emptyset\) for every \(\omega\alpha\)-open set \(V\) containing \(x\). To prove that \(x \in \omega\alpha\text{cl}(A)\). Suppose \(x \notin \omega\alpha\text{cl}(A)\). Then there exists a \(\omega\alpha\)-closed set \(G\) containing \(A\) such that \(x \notin G\). Then \(x \in X - G\) and \(X - G\) is \(\omega\alpha\)-open. Also \((X - G) \cap A = \emptyset\) which is a contradiction to the hypothesis. Hence \(x \in \omega\alpha\text{cl}(A)\).

**Theorem 1.3.3:** If \(A \subseteq X\), then \(A \subseteq \omega\alpha\text{cl}(A) \subseteq \text{cl}(A)\).

**Proof:** Since every closed set is \(\omega\alpha\)-closed, the proof follows.

**Remark 1.3.4:** Both containment relations in the Theorem 1.3.3 may be proper as seen from the following example.
Example 1.3.5: In Example 1.2.13, let \( A = \{a\} \). Then \( \omega \alpha \text{cl}(A) = \{a, b\} \), \( \text{cl}(A) = X \) and so \( A \subseteq \omega \alpha \text{cl}(A) \subseteq \text{cl}(A) \).

Theorem 1.3.6: If \( A \subseteq X \), then

(i) \( A \subseteq \omega \alpha \text{cl}(A) \subseteq C^\#(A) \), where \( C^\#(A) = \cap \{G; A \subseteq G, G \text{ is strongly } g\text{-closed set in } (X, \tau)\} [96] \).

(ii) \( A \subseteq \alpha C^\#(A) \subseteq \omega \alpha \text{cl}(A) \subseteq \text{cl}(A) \), where \( \alpha C^\# = \cap \{G; A \subseteq G, G \text{ is } \alpha\text{-closed in } (X, \tau)\} [68] \).

Proof: The proof follows from the Theorems 1.2.19 and 1.2.6.

Remark 1.3.7: Both containment relations in the Theorem 1.3.6 may be proper as seen from the following examples.

Example 1.3.8: In Example 1.2.3, let \( A = \{c\} \) then \( \omega \alpha \text{cl}(A) = \{c\} \) and \( C^\#(A) = \{b, c\} \), so that \( A \subseteq \omega \alpha \text{cl}(A) \subseteq C^\#(A) \).

Example 1.3.9 In Example 1.2.7, let \( A = \{b\} \) then \( \alpha C^\#(A) = \{b\} \) and \( \omega \alpha \text{cl}(A) = \{b, c\} \), so that \( A \subseteq \alpha C^\#(A) \subseteq \omega \alpha \text{cl}(A) \subseteq \text{cl}(A) \).

Theorem 1.3.10: If \( A \) is \( \omega \alpha \text{-closed} \), then \( \omega \alpha \text{cl}(A) = A \).

Proof: Let \( A \) be \( \omega \alpha \text{-closed} \) set in \( (X, \tau) \). Since \( A \subseteq A \) and \( A \) is \( \omega \alpha \text{-closed} \) set, \( A \in \{G; A \subseteq G, G \text{ is } \omega \alpha \text{-closed set}\} \) which implies that \( A = \cap \{G; A \subseteq G, G \text{ is } \omega \alpha \text{-closed set}\} \subseteq A \). That is \( \omega \alpha \text{cl}(A) \subseteq A \). But \( A \subseteq \omega \alpha \text{cl}(A) \) is always true. Hence \( A = \omega \alpha \text{cl}(A) \).

The converse of the above theorem need not be true as seen from the following example.
Example 1.3.11: In Example 1.2.3, the \( \omega \alpha \)-closed sets are \( \phi, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X \). Let \( A = \{a\} \) be a subset of \( X \). Then \( \omega \alpha \text{cl}(\{a\}) = \{a, c\} \cap \{a, b\} \cap X = \{a\} \) which is not a \( \omega \alpha \)-closed set in \((X, \tau)\).

Theorem 1.3.12: Let \( A \) be a subset of \( X \). If \( A \) is \( \omega \alpha \)-closed in \((X, \tau)\), then \( \omega \alpha \text{cl}(A) \) is the smallest \( \omega \alpha \)-closed subset of \( X \) containing \( A \).

Proof: Let \( A \) be \( \omega \alpha \)-closed set in \((X, \tau)\). Then \( \omega \alpha \text{cl}(A) = \bigcap \{ G: A \subseteq G, G \text{ is } \omega \alpha \text{-closed in } (X, \tau) \} \). Since \( A \subseteq A \) and \( A \) is \( \omega \alpha \)-closed set, \( \omega \alpha \text{cl}(A) = A \) is the smallest \( \omega \alpha \)-closed subset of \( X \) containing \( A \).

However the converse is not true in general as can be seen from the following example.

Example 1.3.13: In Example 1.2.13, let \( A = \{a, c\} \). Then \( \omega \alpha \text{cl}(A) = X \) which is the smallest \( \omega \alpha \)-closed in \((X, \tau)\) containing \( A \). But \( A \) is not \( \omega \alpha \)-closed in \((X, \tau)\).

Remark 1.3.14: The following example shows that for any two sub sets \( A \) and \( B \) of \( X \), \( A \subseteq B \) implies that \( \omega \alpha \text{cl}(A) \neq \omega \alpha \text{cl}(B) \).

Example 1.3.15: In Example 1.2.13, let \( A = \{a\} \) and \( B = \{a, c\} \), then \( A \subseteq B \). Now \( \omega \alpha \text{cl}(A) = \{a, b\} \), \( \omega \alpha \text{cl}(B) = X \). Hence \( \omega \alpha \text{cl}(A) \neq \omega \alpha \text{cl}(B) \).

Remark 1.3.16: For a subset \( A \) of \((X, \tau)\), \( \omega \alpha \text{cl}(A) \neq \text{cl}(A) \) as seen from the following example.

Example 1.3.17: In Example 1.2.13, let \( A = \{a\} \subseteq X \), \( \omega \alpha \text{cl}(A) = \{a, b\} \) and \( \text{cl}(A) = X \). Therefore \( \omega \alpha \text{cl}(A) \neq \text{cl}(A) \).
Remark 1.3.18: For any two subsets $A$ and $B$ of $(X, \tau)$, $\omega\text{acl}(A) = \omega\text{acl}(B)$ does not imply that $A = B$. This is shown by the following example.

Example 1.3.19: In Example 1.2.7, let $A = \{a, b\}$ and $B = \{a, c\}$. Then $\omega\text{acl}(A) = \omega\text{acl}(B) = X$. But $A \neq B$.

Theorem 1.3.20: Let $E$ and $F$ be subsets of $(X, \tau)$.

(i) $\omega\text{acl}(\emptyset) = \emptyset$

(ii) $\omega\text{acl}(X) = X$

(iii) $\omega\text{acl}(E)$ is a $\omega\alpha$-closed set in $(X, \tau)$.

(iv) If $E \subseteq F$, then $\omega\text{acl}(E) \subseteq \omega\text{acl}(F)$.

(v) $\omega\text{acl}(E \cup F) = \omega\text{acl}(E) \cup \omega\text{acl}(F)$.

(vi) $\omega\text{acl}(\omega\text{acl}(E)) = \omega\text{acl}(E)$.

Proof: The proof of (i), (ii), (iii) and (iv) follow from the Definition 1.3.1

(v) To prove that $\omega\text{acl}(E) \cup \omega\text{acl}(F) \subseteq \omega\text{acl}(E \cup F)$

We have $\omega\text{acl}(E) \subseteq \omega\text{acl}(E \cup F)$ and $\omega\text{acl}(F) \subseteq \omega\text{acl}(E \cup F)$ Therefore $\omega\text{acl}(E) \cup \omega\text{acl}(F) \subseteq \omega\text{acl}(E \cup F)$ → (1).

Now we prove $\omega\text{acl}(E \cup F) \subseteq \omega\text{acl}(E) \cup \omega\text{acl}(F)$

Let $x$ be any point such that $x \notin \omega\text{acl}(E) \cup \omega\text{acl}(F)$, then there exists $\omega\alpha$-closed sets $A$ and $B$ such that $E \subseteq A$ and $F \subseteq B$, $x \notin A$ and $x \notin B$. Then $x \notin A \cup B$, $E \cup F \subseteq A \cup B$ and $A \cup B$ is $\omega\alpha$-closed set by Theorem 1.2.9. Thus $x \notin \omega\text{acl}(E \cup F)$. Therefore we have

$\omega\text{acl}(E \cup F) \subseteq \omega\text{acl}(E) \cup \omega\text{acl}(F)$ → (2).

Hence from (1) and (2) $\omega\text{acl}(E \cup F) = \omega\text{acl}(E) \cup \omega\text{acl}(F)$. 22
(vi) Let \( A \) be \( \omega\alpha \)-closed set containing \( E \). Then by Definition 1.3.1, \( \omega\alpha\text{cl}(E) \subseteq A \). Since \( A \) is \( \omega\alpha \)-closed set and contains \( \omega\alpha\text{cl}(E) \) and is contained in every \( \omega\alpha \)-closed set containing \( E \), it follows \( \omega\alpha\text{cl}[\omega\alpha\text{cl}(E)] \subseteq \omega\alpha\text{cl}(E) \). Therefore \( \omega\alpha\text{cl}[\omega\alpha\text{cl}(E)] = \omega\alpha\text{cl}(E) \).

**Theorem 1.3.21:** \( \omega\alpha \) - closure is a Kuratowski closure operator on \( X \).

**Proof:** Follows from the Theorem 1.3.20

**Theorem 1.3.22:** Let \( A \) and \( B \) be subsets of \((X, \tau)\). Then \( \omega\alpha\text{cl}(A \cap B) \subseteq \omega\alpha\text{cl}(A) \cap \omega\alpha\text{cl}(B) \).

**Proof:** Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), by Theorem 1.3.20 (iv), \( \omega\alpha\text{cl}(A\cap B) \subseteq \omega\alpha\text{cl}(A) \) and \( \omega\alpha\text{cl}(A\cap B) \subseteq \omega\alpha\text{cl}(B) \). Thus \( \omega\alpha\text{cl}(A \cap B) \subseteq \omega\alpha\text{cl}(A) \cap \omega\alpha\text{cl}(B) \).

In general \( \omega\alpha\text{cl}(A) \cap \omega\alpha\text{cl}(B) \leq \omega\alpha\text{cl}(A \cap B) \) as seen from the following example.

**Example 1.3.23:** In Example 1.2.7, let \( A = \{b\} \) and \( B = \{c\} \). Then \( \omega\alpha\text{cl}(A) = \{b, c\} \), \( \omega\alpha\text{cl}(B) = \{b, c\} \) and \( \omega\alpha\text{cl}(A \cap B) = \emptyset \). Hence \( \omega\alpha\text{cl}(A) \cap \omega\alpha\text{cl}(B) \neq \omega\alpha\text{cl}(A \cap B) \).

**Theorem 1.3.24:** If \( \alpha\text{c}(X, \tau) \) is a closed under finite unions, then \( \omega\alpha\text{c}(X, \tau) \) is closed under finite unions, where \( \alpha\text{c}(X, \tau) \) and \( \omega\alpha\text{c}(X, \tau) \) are the families of \( \alpha \)-closed and \( \omega\alpha \)-closed sets in \((X, \tau)\) respectively.

**Proof:** Let \( \alpha\text{c}(X, \tau) \) be closed under finite unions. Let \( A, B \in \omega\alpha\text{c}(X, \tau) \) and let \( A \cup B \subseteq G \), where \( G \) is \( \omega \)-open in \( X \). Then \( A \subseteq G \) and \( B \subseteq G \). Since \( A \) and \( B \) are \( \omega\alpha \)-closed, \( \alpha\text{cl}(A) \cup \alpha\text{cl}(B) \subseteq G \). By hypothesis \( \alpha\text{cl}(A \cup B) \subseteq G \). Thus \( A \cup B \) is \( \omega\alpha \)-closed set.
**Definition 1.3.25:** Let $\tau^{*}_{\omega\alpha}$ be the topology in $X$ generated by $\omega\alpha$cl in the usual manner that is $\tau^{*}_{\omega\alpha} = \{ G \subset X / \omega\alphacl(X-G) = X-G \}$.

**Definition 1.3.26 [68]:** $\alpha^\# = \{ U / \alphacl^\#(X-U) = X-U \}$.

**Theorem 1.3.27:** For a subset $A$ of $(X, \tau)$ the following statements hold:

(i) $\tau \subseteq \tau^{*}_{\omega\alpha} \subseteq \alpha^\#$

(ii) $\tau \subseteq \tau^\alpha \subseteq \tau^{*}_{\omega\alpha}$

**Proof:** (i) To prove $\tau \subseteq \tau^{*}_{\omega\alpha}$

Let $E \in \tau \Rightarrow E^c$ is closed in $\tau$

$$\Rightarrow E^c \subseteq \omega\alphacl(E^c) \subseteq cl(E^c) = E^c \text{ by Theorem 1.3.3}$$

$$\Rightarrow \omega\alphacl(E^c) \subseteq E^c$$

But $E^c \subseteq \omega\alphacl(E^c)$ holds always. Thus $\omega\alphacl(E^c) = E^c$. Hence $E \in \tau^{*}_{\omega\alpha}$.

Now to prove that $\tau^{*}_{\omega\alpha} \subseteq \alpha^\#$, Let $E \in \tau^{*}_{\omega\alpha}$

$$\Rightarrow \omega\alphacl(E^c) = E^c$$

$$\Rightarrow E^c \subseteq \alpha C^\#(E^c) \subseteq \omega\alphacl(E^c) = E^c \text{ by Theorem 1.3.6 (ii)}$$

$$\Rightarrow \alpha C^\#(E^c) = E^c$$

$$\Rightarrow E \in \alpha^\#$$

Thus $\tau \subseteq \tau^{*}_{\omega\alpha} \subseteq \alpha^\#$.

(ii) Follows from the fact that every $\alpha$-closed set is $\omega\alpha$-closed set.

**Theorem 1.3.28:** The following are equivalent for a space $(X, \tau)$,

(i) Every $\omega\alpha$-closed set is $\alpha$-closed.

(ii) $\tau^\alpha = \tau^{*}_{\omega\alpha}$

(iii) For each $x \in X$, $\{x\}$ is $\omega$-closed or $\alpha$-open.
**Proof:** (i) $\Rightarrow$ (ii) We claim that $\tau^*_{\omega\alpha} \subseteq \tau^\alpha$. Let $G \in \tau^*_{\omega\alpha}$. Then by Theorem 1.2.4 and Definitions 1.3.1 and 1.3.25, $\omega\alpha\text{cl}(A) = \alpha\text{cl}(A)$ for every set $A$ of $X$. Therefore we have $X - G = \omega\alpha\text{cl}(X - G) = \alpha\text{cl}(X - G)$. Therefore $X - G$ is $\alpha$-closed set so $G$ is $\alpha$-open set. That is $G \in \tau^\alpha$. Therefore, if $G \in \tau^*_{\omega\alpha}$ then $G \in \tau^\alpha$. This implies $\tau^*_{\omega\alpha} \subseteq \tau^\alpha$, and by Theorem 1.3.27 (ii) $\tau^\alpha \subseteq \tau^*_{\omega\alpha}$. So $\tau^\alpha = \tau^*_{\omega\alpha}$.

(ii) $\Rightarrow$ (iii) Let $x \in X$. Then by Theorem 1.2.40, $X - \{x\} = \omega\alpha\text{cl}(X - \{x\})$, if $\{x\}$ is not $\omega$-closed in $(X, \tau)$. Therefore $\{x\} \in \omega C(X, \tau)$ or $\{x\} \in \tau^\alpha$.

(iii) $\Rightarrow$ (i) Let $A$ be a $\omega\alpha$-closed set and $x \in \alpha\text{cl}(A)$. We show that $x \in A$ for the following two cases:

Case i) $\{x\}$ is $\omega$-closed. Suppose that $x \notin A$. Then $\alpha\text{cl}(A) - A$ contains a $\omega$-closed set $\{x\}$. This contradicts Theorem 1.2.37. Hence $x \in A$.

Case ii) $\{x\}$ is $\alpha$-open. Since $x \in \alpha\text{cl}(A)$, $\{x\} \cap A \neq \emptyset$. Therefore we have $\alpha\text{cl}(A) = A$. That is $A$ is $\alpha$-closed. Hence we have $\omega\alpha C(X, \tau) \subseteq \alpha C(X, \tau)$.

**Theorem 1.3.29:** Every $\omega\alpha$-closed set is closed in $(X, \tau)$ if and only if $\tau = \tau^*_{\omega\alpha}$.

**Proof:** Suppose that every $\omega\alpha$-closed set is closed. Let $A$ be $\omega\alpha$-closed set. Then by hypothesis $\omega\alpha\text{cl}(A) = \text{cl}(A)$. Therefore $\tau = \tau^*_{\omega\alpha}$.

Conversely, let $A$ be $\omega\alpha$-closed set. Then by Theorem 1.3.10, $A = \omega\alpha\text{cl}(A)$. Hence $X - A \in \tau^*_{\omega\alpha}$ Therefore by hypothesis $A \in (X, \tau)$. That is, $A$ is closed in $(X, \tau)$.

**Theorem 1.3.30:** Every $\omega\alpha$-closed set is $\alpha$-closed if and only if $\tau^\alpha = \tau^*_{\omega\alpha}$.

**Proof:** Follows from the Theorem 1.3.28.
Now we introduce the following definition.

**Definition 1.3.31:** For a subset $A$ of $(X, \tau)$, $\omega\alpha$-interior of $A$, denoted by $\omega\alpha\text{int}(A)$ and is defined as $\omega\alpha\text{int}(A) = \bigcup \{G : G \subseteq A \text{ and } G \text{ is } \omega\alpha\text{-open in } (X, \tau)\}$. That is $\omega\alpha\text{int}(A)$ is the union of all $\omega\alpha$-open sets contained in $A$.

**Theorem 1.3.32:** Let $A$ be a subset of $X$, then $\omega\alpha\text{int}(A)$ is the largest $\omega\alpha$-open subset of $X$ contained in $A$ if $A$ is $\omega\alpha$-open.

**Proof:** Let $A \subseteq X$ be $\omega\alpha$-open. Then $\omega\alpha\text{int}(A) = \bigcup \{G : G \subseteq A \text{ and } G \text{ is } \omega\alpha\text{-open in } (X, \tau)\}$. Since $A \subseteq A$ and $A$ is $\omega\alpha$-open, $A = \omega\alpha\text{int}(A)$ is the largest $\omega\alpha$-open subset of $X$ contained in $A$.

The converse of the above theorem need not be true as seen from the following example.

**Example 1.3.33:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $A = \{b, c\}$, $\omega\alpha\text{int}(A) = \{b\}$ is $\omega\alpha$-open in $(X, \tau)$. But $A$ is not $\omega\alpha$-open in $(X, \tau)$.

**Remark 1.3.34:** For any subset $A$ of $X$, $\text{int}(A) \subseteq \omega\alpha\text{int}(A) \subseteq A$.

**Remark 1.3.35:** For a subset $A$ of $X$, $\omega\alpha\text{int}(A) \neq \text{int}(A)$ as seen from the following example.

**Example 1.3.36:** In Example 1.3.33, let $A = \{b\}$. Then $\omega\alpha\text{int}(A) = \{b\}$ and $\text{int}(A) = \emptyset$. Hence $\omega\alpha\text{int}(A) \neq \text{int}(A)$.

**Remark 1.3.37:** For any two subsets $A$ and $B$ of $X$, $\omega\alpha\text{int}(A) = \omega\alpha\text{int}(B)$, does not imply that $A = B$. This is shown by the following example.
Example 1.3.38: In Example 1.2.7, let $A = \{a, b\}$ and $B = \{a, c\}$. Now $\omega\alpha\text{int}(A) = \{a\}$ and $\omega\alpha\text{int}(B) = \{a\}$. But $A \neq B$.

Theorem 1.3.39: For any subset $A$ of $X$, the following hold:

(i) $\omega\alpha\text{int}(\phi) = \phi$

(ii) $\omega\alpha\text{int}(X) = X$

(iii) If $A \subseteq B$ then $\omega\alpha\text{int}(A) \subseteq \omega\alpha\text{int}(B)$

(iv) $\omega\alpha\text{int}(A \cap B) = \omega\alpha\text{int}(A) \cap \omega\alpha\text{int}(B)$

(v) $\omega\alpha\text{int}(A \cup B) \supseteq \omega\alpha\text{int}(A) \cup \omega\alpha\text{int}(B)$

(vi) $\omega\alpha\text{int}[\omega\alpha\text{int}(A)] = \omega\alpha\text{int}(A)$

Proof: Follows from the Definition 1.3.31.

Remark 1.3.40: If $A$ is $\omega\alpha$-open, then $\omega\alpha\text{int}(A) = A$. But the converse need not be true as seen from the following example.

Example 1.3.41: In Example 1.2.3 the $\omega\alpha$-open sets are $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, c\}$, $\{a, b\}$, $X$. Let $A = \{b, c\}$, $\omega\alpha\text{int}(\{b, c\}) = \{b, c\}$ is not $\omega\alpha$-open.

Remark 1.3.42: For any two subsets $A$ and $B$ of $X$, $\omega\alpha\text{int}(A) \cup \omega\alpha\text{int}(B) \neq \omega\alpha\text{int}(A \cup B)$. This is shown by the following example.

Example 1.3.43: In Example 1.2.3, let $A = \{b\}$ and $B = \{c\}$. Now $\omega\alpha\text{int}(A) = \{b\}$, $\omega\alpha\text{int}(B) = \{c\}$ and $\omega\alpha\text{int}(A \cup B) = \omega\alpha\text{int}(\{b, c\}) = X$. Hence $\omega\alpha\text{int}(A) \cup \omega\alpha\text{int}(B) \neq \omega\alpha\text{int}(A \cup B)$.

Remark 1.3.44: Since every $\alpha$-open set is $\omega\alpha$-open, every $\alpha$-interior point of a subset $A$ of $X$ is $\omega\alpha$-interior point of $A$. Thus $\alpha\text{int}(A) \subseteq \omega\alpha\text{int}(A)$. In general $\alpha\text{int}(A) \neq \omega\alpha\text{int}(A)$, which is shown by the following example.
Example 1.3.45: In Example 1.2.3, let $A = \{a, b\}$ then $\alpha\text{int}(A) = \{a\}$ and $\omega\alpha\text{int}(A) = \{a, b\}$. Therefore $\alpha\text{int}(A) \neq \omega\alpha\text{int}(A)$

Theorem 1.3.46: For any $A \subseteq X$, $[X - \omega\alpha\text{int}(A)] = [\omega\alpha\text{cl}(X - A)]$.

Proof: Let $x \in X - \omega\alpha\text{int}(A)$. Then $x \notin \omega\alpha\text{int}(A)$. That is every $\omega\alpha$-open set $G$ containing $x$ is such that $G \nsubseteq A$. This implies every $\omega\alpha$-open set $G$ containing $x$ intersects $X - A$. That is, $G \cap (X - A) \neq \emptyset$. Then by Theorem 1.3.2, $x \in \omega\alpha\text{cl}(X-A)$ and therefore $[X - \omega\alpha\text{int}(A)] \subseteq [\omega\alpha\text{cl}(X - A)]$.

Conversely let $x \in \omega\alpha\text{cl}(X-A)$. Then every $\omega\alpha$-open set $G$ containing $x$ intersects $X - A$. That is, $G \cap (X - A) \neq \emptyset$. That is, every $\omega\alpha$-open set $G$ containing $x$ is such that $G \nsubseteq A$. Then by Definition 1.3.31, $x \notin \omega\alpha\text{int}(A)$. That is, $x \in [X - \omega\alpha\text{int}(A)]$, and so $[\omega\alpha\text{cl}(X - A)] \subseteq [X - \omega\alpha\text{int}(A)]$. Thus $[X - \omega\alpha\text{int}(A)] = [\omega\alpha\text{cl}(X - A)]$.

Remark 1.3.47: For any $A \subseteq X$, we have

(i) $[X - \omega\alpha\text{cl}(X - A)] = [\omega\alpha\text{int}(A)]$

(ii) $[X - \omega\alpha\text{int}(X - A)] = [\omega\alpha\text{cl}(A)]$

Taking complements in the above Theorem 1.3.46 and by replacing $A$ by $X - A$ in Theorem 1.3.46, the above results follow.
§ 1.4 $\omega\alpha$-Neighbourhoods and $\omega\alpha$-Limit points

In this section, we define the notions of $\omega\alpha$-neighbourhood, $\omega\alpha$-limit point and $\omega\alpha$-derived set of a set and show that some of their properties are analogous to those for open sets.

**Definition 1.4.1:** Let $(X, \tau)$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be $\omega\alpha$-neighbourhood of a point $x \in X$ if there exists a $\omega\alpha$-open set $G$ such that $x \in G \subseteq N$.

**Definition 1.4.2:** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. A subset $N$ of $X$ is said to be $\omega\alpha$-neighbourhood of $A$ if there exists a $\omega\alpha$-open set $G$ such that $A \subseteq G \subseteq N$.

The collection of all $\omega\alpha$-neighbourhood of $x \in X$ is called the $\omega\alpha$-neighbourhood system at ‘$x$’ and shall be denoted by $\omega\alpha N(x)$.

It is evident from the above definition that a $\omega\alpha$-open set is a $\omega\alpha$-neighbourhood of each of its points. But a $\omega\alpha$-neighbourhood of a point need not be a $\omega\alpha$-open set. Also every $\omega\alpha$-open set containing $x$ is a $\omega\alpha$-neighbourhood of $x$.

**Theorem 1.4.3:** A subset of a topological space is $\omega\alpha$-open if it is a $\omega\alpha$-neighbourhood of each of its points.

**Proof:** Let a subset $G$ of a topological space be $\omega\alpha$-open. Then for every $x \in G$, $x \in G \subseteq G$ and therefore $G$ is a $\omega\alpha$-neighbourhood of each of its points.
The converse of the above Theorem need not be true as seen from the following example.

Example 1.4.4: In Example 1.2.3, the set \( A = \{ b, c \} \) is \( \omega_x \)-neighbourhood of each of points its points \( b \) and \( c \) but \( A \) is not \( \omega_x \)-open.

Theorem 1.4.5: Let \( (X, \tau) \) be a topological space. If \( A \) is a \( \omega_x \)-closed subset of \( X \) and \( x \in X - A \), then there exists a \( \omega_x \)-neighbourhood \( N \) of \( x \) such that \( N \cap A = \emptyset \).

Proof: Since \( A \) is \( \omega_x \)-closed, then \( X - A \) is \( \omega_x \)-open set in \( (X, \tau) \). By the above Theorem 1.4.3, \( X - A \) contains a \( \omega_x \)-neighbourhood of each of its points. Hence there exists a \( \omega_x \)-neighbourhood \( N \) of \( x \), such that \( N \subseteq X - A \). That is, no point of \( N \) belongs to \( A \) and hence \( N \cap A = \emptyset \).

Theorem 1.4.6: Let \( (X, \tau) \) be a topological space and \( A \subseteq X \). Then \( x \in \omega_{xcl}(A) \) if and only if for any \( \omega_x \)-neighbourhood \( N \) of \( x \) in \( (X, \tau) \), \( A \cap N \neq \emptyset \).

Proof: Suppose \( x \in \omega_{xcl}(A) \). Let us assume that there is a \( \omega_x \)-neighbourhood \( N \) of the point \( x \) in \( (X, \tau) \) such that \( N \cap A = \emptyset \). Since \( N \) is a \( \omega_x \)-neighbourhood of \( x \) in \( (X, \tau) \) by definition of \( \omega_x \)-neighbourhood there exists an \( \omega_x \)-open set \( G \) of \( x \) such that \( x \in G \subseteq N \). Therefore we have \( G \cap A = \emptyset \) and so \( A \subseteq G^c \). Since \( X - G \) is an \( \omega_x \)-closed set containing \( A \). We have by definition of \( \omega_x \)-closure, \( \omega_{xcl}(A) \subseteq X - G \) and therefore \( x \in \omega_{xcl}(A) \), which is a contradiction to hypothesis \( x \in \omega_{xcl}(A) \). Therefore \( A \cap N \neq \emptyset \).
Conversely, suppose for each $\omega\alpha$-neighbourhood $N$ of $x$ in $(X, \tau)$. $A \cap N \neq \emptyset$. Suppose that $x \notin \omega\alpha\text{cl}(A)$. Then by definition of $\omega\alpha\text{cl}(A)$, there exists a $\omega\alpha$-closed set $G$ of $(X, \tau)$ such that $A \subseteq G$ and $x \notin G$. Thus $x \in X - G$ and $X - G$ is $\omega\alpha$-open in $(X, \tau)$ and hence $X - G$ is a $\omega\alpha$-neighbourhood of $x$ in $(X, \tau)$. But $A \cap (X - G) = \emptyset$ which a contradiction. Hence $x \in \omega\alpha\text{cl}(A)$.

**Theorem 1.4.7:** Let $(X, \tau)$ be a topological space and $p \in X$. Let $\omega\alpha N(p)$ be the collection of all $\omega\alpha$-neighbourhoods of $p$. Then

(i) $\omega\alpha N(p) \neq \emptyset$ and $p \in$ each member of $\omega\alpha N(p)$.

(ii) The intersection of any two members of $\omega\alpha N(p)$ is again a member of $\omega\alpha N(p)$.

(iii) If $N \in \omega\alpha N(p)$ and $M \subseteq N$, then $M \in \omega\alpha N(p)$.

(iv) Each member $N \in \omega\alpha N(p)$ is a superset of a member $G \in \omega\alpha N(p)$ where $G$ is a $\omega\alpha$-open set.

**Proof:**

(i) Since $X$ is a $\omega\alpha$-open set containing $p$, it is a $\omega\alpha$-neighbourhood of every $p \in X$. Hence there exists at least one $\omega\alpha$-neighbourhood namely $X$ for each $p \in X$. Here $\omega\alpha N(p) \neq \emptyset$. Let $N \in \omega\alpha N(p)$, $N$ is a $\omega\alpha$-neighbourhood of $p$. Then there exists a $\omega\alpha$-open set $G$ such that $p \in G \subseteq N$. So $p \in N$. Therefore $p \in$ every member $N$ of $\omega\alpha N(p)$.

(ii) Let $N \in \omega\alpha N(p)$ and $M \in \omega\alpha N(p)$. Then by definition of $\omega\alpha$-neighbourhood, there exists $\omega\alpha$-open sets $G$ and $F$ such that $p \in G \subseteq N$ and $p \in F \subseteq M$. Hence $p \in G \cap F \subseteq M \cap N$. Note that $G \cap F$ is a $\omega\alpha$-open set since intersection of $\omega\alpha$-open sets is $\omega\alpha$-open. Therefore it follows that $N \cap M$ is a $\omega\alpha$-neighbourhood of $p$. Hence $N \cap M \in \omega\alpha N(p)$. 

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(iii) If \( N \in \omega\alpha N(p) \) then there is an \( \omega\alpha \)-open set \( G \) such that \( p \in G \subseteq N \). Since \( M \subseteq N \), \( M \) is a \( \omega\alpha \)-neighbourhood of \( p \). Hence \( M \in \omega\alpha N(p) \).

(iv) Let \( N \in \omega\alpha N(p) \). Then there exist an \( \omega\alpha \)-open set \( G \) such that \( p \in G \subseteq N \). Since \( G \) is \( \omega\alpha \)-open and \( p \in G \), \( G \) is \( \omega\alpha \)-neighbourhood of \( p \). Therefore \( G \in \omega\alpha N(p) \) and also \( G \subseteq N \).

**Definition 1.4.8:** Let \((X,\tau)\) be a topological space and \( A \) be a subset of \( X \). Then a point \( x \in X \) is called a \( \omega\alpha \)-limit point of \( A \) if and only if every \( \omega\alpha \)-neighbourhood of \( x \) contains a point of \( A \) distinct from \( x \). That is \([N - \{x\}] \cap A \neq \phi\) for each \( \omega\alpha \)-neighbourhood \( N \) of \( x \). Also equivalently if and only if every \( \omega\alpha \)-open set \( G \) containing \( x \) contains a point of \( A \) other than \( x \).

In a topological space \((X,\tau)\) the set of all \( \omega\alpha \)-limit points of a given subset \( A \) of \( X \) is called a \( \omega\alpha \)-derived set of \( A \) and it is denoted by \( \omega\alpha d(A) \).

**Theorem 1.4.9:** Let \( A \) and \( B \) be subsets of a topological space \((X,\tau)\). Then

(i) \( \omega\alpha d(\phi) = \phi \),

(ii) If \( A \subseteq B \), then \( \omega\alpha d(A) \subseteq \omega\alpha d(B) \),

(iii) If \( x \in \omega\alpha d(A) \), then \( x \in \omega\alpha d(A - \{x\}) \),

(iv) \( \omega\alpha d(A \cup B) = \omega\alpha d(A) \cup \omega\alpha d(B) \),

(v) \( \omega\alpha d(A \cap B) \subseteq \omega\alpha d(A) \cap \omega\alpha d(B) \).

**Proof:** (i) Let \( x \) be any point of \( X \) and \( x \in \omega\alpha d(\phi) \). That is \( x \) is a \( \omega\alpha \)-limit point of \( \phi \). Then for every \( \omega\alpha \)-open set \( G \) containing \( x \), we should have \([G - \{x\}] \cap \phi \neq \phi\) which is impossible. Hence \( \omega\alpha d(\phi) = \phi \).
(ii) If \( x \in \omega \alpha \text{ad}(A) \), that is if \( x \) is \( \omega \alpha \)-limit point of \( A \), then by Definition 1.4.8 \([G - \{x\}] \cap A \neq \emptyset\) for every \( \omega \alpha \)-open set \( G \) containing \( x \). Since \( A \subseteq B \) implies \([G - \{x\}] \cap A \subseteq [G - \{x\}] \cap B\). Thus if \( x \) is a \( \omega \alpha \)-limit point of \( A \) it is also a \( \omega \alpha \)-limit point of \( B \), that is \( x \in \omega \alpha \text{d}(B) \). Hence \( \omega \alpha \text{d}(A) \subseteq \omega \alpha \text{d}(B) \).

(iii) If \( x \in \omega \alpha \text{d}(A) \), that is \( x \) is a \( \omega \alpha \)-limit point of \( A \). Then by Definition 1.4.8 every \( \omega \alpha \)-open set \( G \) containing \( x \) contains at least one point other than \( x \) of \( A - \{x\} \). That is \( G \cap (A - \{x\}) \neq \emptyset \). Hence \( x \) is a \( \omega \alpha \)-limit point of \( A - \{x\} \) and as such it belongs to \( \omega \alpha \text{d}[A - \{x\}] \). Therefore \( x \in \omega \alpha \text{d}(A) \Rightarrow x \in \omega \alpha \text{d}[A - \{x\}] \).

(iv) Since \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \), it follows from (ii) \( \omega \alpha \text{d}(A) \subseteq \omega \alpha \text{d}(A \cup B) \) and \( \omega \alpha \text{d}(B) \subseteq \omega \alpha \text{d}(A \cup B) \) and hence \( \omega \alpha \text{d}(A) \cup \omega \alpha \text{d}(B) \subseteq \omega \alpha \text{d}(A \cup B) \). To prove the other way that is \( \omega \alpha \text{d}(A \cup B) \subseteq \omega \alpha \text{d}(A) \cup \omega \alpha \text{d}(B) \). If \( x \notin \omega \alpha \text{d}(A) \cup \omega \alpha \text{d}(B) \), then \( x \notin \omega \alpha \text{d}(A) \) and \( x \notin \omega \alpha \text{d}(B) \), that is \( x \) is neither a \( \omega \alpha \)-limit point of \( A \) nor a \( \omega \alpha \)-limit point of \( B \). Hence there exist \( \omega \alpha \)-neighbourhoods \( G_1 \) and \( G_2 \) of \( x \) such that \( G_1 \cap (A - \{x\}) = \emptyset \) and \( G_2 \cap (B - \{x\}) = \emptyset \). Since \( G_1 \cap G_2 \) is a \( \omega \alpha \)-neighbourhood of \( x \), we have \((G_1 \cap G_2) \cap [(A \cup B) - \{x\}] = \emptyset \). Therefore \( x \notin \omega \alpha \text{d}(A \cup B) \). Thus \( \omega \alpha \text{d}(A \cup B) \subseteq \omega \alpha \text{d}(A) \cup \omega \alpha \text{d}(B) \). Hence \( \omega \alpha \text{d}(A \cup B) = \omega \alpha \text{d}(A) \cup \omega \alpha \text{d}(B) \).

(v) Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq A \), by (ii) \( \omega \alpha \text{d}(A \cap B) \subseteq \omega \alpha \text{d}(A) \) and \( \omega \alpha \text{d}(A \cap B) \subseteq \omega \alpha \text{d}(B) \). Consequently \( \omega \alpha \text{d}(A \cap B) \subseteq \omega \alpha \text{d}(A) \cap \omega \alpha \text{d}(B) \).

Theorem 1.4.10: Let \((X, \tau)\) be a topological space and \( A \) be subset of \( X \). If \( A \) is \( \omega \alpha \)-closed, then \( \omega \alpha \text{d}(A) \subseteq A \).
Proof: Let \( A \) be \( \omega\alpha \)-closed, now we will show that \( \omega\alpha d(A) \subseteq A \). Since \( A \) is \( \omega\alpha \)-closed, \( X - A \) is \( \omega\alpha \)-open. To each \( x \in X - A \) there exists \( \omega\alpha \)-neighbourhood \( G \) of \( x \) such that \( G \subseteq X - A \). Since \( A \cap (X - A) = \emptyset \), the \( \omega\alpha \)-neighbourhood \( G \) contains no point of \( A \) and so \( x \) is not a \( \omega\alpha \)-limit point of \( A \). Thus no point of \( X - A \) can be \( \omega\alpha \)-limit point of \( A \) that is, \( A \) contains all its \( \omega\alpha \)-limit points. That is \( \omega\alpha d(A) \subseteq A \).

§ 1.5 \( \omega\alpha \)-Spaces

As applications of \( \omega\alpha \)-closed sets we introduce four new topological spaces namely, \( T_{\omega\alpha} \)-spaces, \( \omega\alpha T \)-spaces, \( \omega a \omega T_{\omega\alpha} \)-spaces and \( \omega a \omega T_{\omega\alpha \omega} \)-spaces in topological spaces and study some of their properties.

Definition 1.5.1: A topological space \((X, \tau)\) is said to be \( T_{\omega\alpha} \)-space if every \( \omega\alpha \)-closed set is closed in \((X, \tau)\).

Example 1.5.2: In Example 1.2.7, the space \((X, \tau)\) is \( T_{\omega\alpha} \)-space.

Definition 1.5.3: A topological space \((X, \tau)\) is said to be \( \omega a \omega T \)-space if every \( \omega\alpha \)-closed set is \( \alpha \)-closed.

Example 1.5.4: Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\} \). Then the space \((X, \tau)\) is \( \omega a \omega T \)-space.
Theorem 1.5.5: A topological space \((X, \tau)\) is \(T_{\omega\alpha}\) - space if and only if every \(\omega\alpha\)-open set in \((X, \tau)\) is open in \((X, \tau)\).

**Proof:** Suppose the space \((X, \tau)\) is \(T_{\omega\alpha}\) - space. Let \(G\) be \(\omega\alpha\)-open set in \((X, \tau)\). Then \(X - G\) is \(\omega\alpha\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(T_{\omega\alpha}\) - space, \(X - G\) is closed in \((X, \tau)\). Therefore \(G\) is open in \((X, \tau)\).

Conversely, assume that every \(\omega\alpha\)-open set is open in \((X, \tau)\). Let \(F\) be \(\omega\alpha\)-closed set in \((X, \tau)\), then \(X - F\) is \(\omega\alpha\)-open set in \((X, \tau)\). By hypothesis, \(X - F\) is open set in \((X, \tau)\). Therefore \(F\) is closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(T_{\omega\alpha}\) - space.

Theorem 1.5.6: A topological space \((X, \tau)\) is \(\omega\alpha T\) - space if and only if every \(\omega\alpha\)-open set in \((X, \tau)\) is \(\alpha\)-open in \((X, \tau)\).

**Proof:** Similar to that of Theorem 1.5.5

Theorem 1.5.7: Every \(T_{\omega\alpha}\) - space is \(\omega\alpha T\)-space.

**Proof:** Let \((X, \tau)\) be \(T_{\omega\alpha}\) - space. Let \(A\) be \(\omega\alpha\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(T_{\omega\alpha}\) - space, \(A\) is closed in \((X, \tau)\). But every closed set is \(\alpha\)-closed. Therefore \(A\) is \(\alpha\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\omega\alpha T\)-space.

The converse of the above theorem need not be true as seen from the following example.

Example 1.5.8: In Example 1.5.4, the space \((X, \tau)\) is \(\omega\alpha T\)-space but not \(T_{\omega\alpha}\) - space. The set \(A = \{a, d\}\) is \(\omega\alpha\)-closed but not closed in \((X, \tau)\).

Theorem 1.5.9: If a space \((X, \tau)\) is \(T_{\omega\alpha}\)-space then every singleton of \((X, \tau)\) is either \(\omega\)-closed or open.
Proof: Suppose \( \{x\} \) is not \( \omega \)-closed set for some \( x \in X \). Then \( X - \{x\} \) is not \( \omega \)-open set and \( X \) is the only \( \omega \)-open set containing \( X - \{x\} \). Therefore \( X - \{x\} \) is \( \omega \alpha \)-closed. Since \( (X, \tau) \) is \( T_{\omega \alpha} \)-space, then \( X - \{x\} \) is closed. Hence \( \{x\} \) is open.

**Theorem 1.5.10:** If \( (X, \tau) \) is \( T_{\omega \alpha} \)-space, then it is \( T_P \)-space. (resp. \( \alpha \)-space)

**Proof:** Let \( A \) be strongly \( g \)-closed set in \( (X, \tau) \). Since every strongly \( g \)-closed set is \( \omega \alpha \)-closed set, \( A \) is \( \omega \alpha \)-closed in \( (X, \tau) \). Since \( (X, \tau) \) is \( T_{\omega \alpha} \)-space, \( A \) is closed. Hence \( (X, \tau) \) is \( T_P \)-space (resp. \( \alpha \)-space).

The converse of the above theorem need not be true as seen from the following example.

**Example 1.5.11:** In Example 1.2.3, the space \( (X, \tau) \) is \( T_P \)-space, but \( (X, \tau) \) is not \( T_{\omega \alpha} \)-space. The set \( A = \{b\} \) is \( \omega \alpha \)-closed but not closed in \( (X, \tau) \).

**Example 1.5.12:** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a, b\}, X\} \). Then the space \( (X, \tau) \) is \( \alpha \)-space but not \( T_{\omega \alpha} \)-space, for the set \( A = \{b, c\} \) is \( \omega \alpha \)-closed but not closed in \( (X, \tau) \).

**Theorem 1.5.13:** If \( (X, \tau) \) is \( 1/2 T_\alpha \)-space then it is \( \omega \alpha T \)-space.

**Proof:** Let \( A \) be \( \omega \alpha \)-closed set in \( (X, \tau) \). Then \( A \) is \( \alpha g \)-closed set in \( (X, \tau) \) as every \( \omega \alpha \)-closed set is \( \alpha g \)-closed. Since \( (X, \tau) \) is \( 1/2 T_\alpha \)-space, \( A \) is \( \alpha \)-closed. Hence \( (X, \tau) \) is \( \omega \alpha T \)-space.

The converse of the above theorem need not be true as seen from the following example.
Example 1.5.14: In Example 1.2.7, the space \((X, \tau)\) is \(\alpha T\)-space but not \(\frac{1}{2} T\alpha\)-space, since the set \(A = \{a, b\}\) is \(\alpha g\)-closed but not a \(\alpha\)-closed set in \((X, \tau)\).

Theorem 1.5.15: If a space \((X, \tau)\) is \(\omega \alpha T\)-space then every singleton of \((X, \tau)\) is either \(\omega\)-closed or \(\alpha\)-open.

Proof: Suppose \(\{x\}\) is not \(\omega\)-closed set for some \(x \in X\). Then \(X - \{x\}\) is not \(\omega\)-open set and \(X\) is the only \(\omega\)-open set containing \(X - \{x\}\). Therefore \(X - \{x\}\) is \(\omega \alpha\)-closed. Since \(X\) is \(\omega \alpha T\)-space, then \(X - \{x\}\) is \(\alpha\)-closed. Hence \(\{x\}\) is \(\alpha\)-open.

Definition 1.5.16: A topological space \((X, \tau)\) is said to be \(\alpha g T\omega \alpha\) - space if every \(\alpha g\)-closed set is \(\omega \alpha\)-closed.

Theorem 1.5.17: A topological space \((X, \tau)\) is \(\alpha g T\omega \alpha\) - space if and only if every \(\alpha g\)-open set in \((X, \tau)\) is \(\omega \alpha\)-open in \((X, \tau)\).

Proof: Suppose the space \((X, \tau)\) is \(\alpha g T\omega \alpha\) - space. Let \(G\) be \(\alpha g\)-open set in \((X, \tau)\). Then \(X - G\) is \(\alpha g\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\alpha g T\omega \alpha\) - space, \(X - G\) is \(\omega \alpha\)-closed in \((X, \tau)\). Therefore \(G\) is \(\omega \alpha\)-open in \((X, \tau)\).

Conversely, assume that every \(\alpha g\)-open set is \(\omega \alpha\)-open in \((X, \tau)\). Let \(F\) be \(\alpha g\)-closed set in \((X, \tau)\), then \(X - F\) is \(\alpha g\)-open set \((X, \tau)\). By hypothesis, \(X - F\) is \(\omega \alpha\)-open set in \((X, \tau)\). Therefore \(F\) is \(\omega \alpha\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\alpha g T\omega \alpha\) - space.

Theorem 1.5.18: Every \(\alpha T_b\) - space is \(\alpha g T\omega \alpha\) - space.
Proof: Let \((X, \tau)\) be \(\alpha T_b\) - space. Let \(A\) be \(\alpha g\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\alpha T_b\) - space, \(A\) is closed and therefore it is \(\omega \alpha\)-closed set in \((X, \tau)\). Hence \((X, \tau)\) is \(\alpha g T_{\omega \alpha}\) - space.

The converse of the above theorem need not be true as seen from the following example.

Example 1.5.19: Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{a, b\}, X\}\). Then the space \((X, \tau)\) is \(\alpha g T_{\omega \alpha}\) - space but not \(\alpha T_b\) - space, since the set \(A = \{b\}\) is \(\alpha g\)-closed but not closed in \((X, \tau)\).

Theorem 1.5.20: If \((X, \tau)\) is \(\alpha g T_{\omega \alpha}\) - space and \(T_{\omega \alpha}\) - space, then \((X, \tau)\) is \(\alpha T_b\)-space (resp. \(\alpha T_c\) - space, \(\alpha T_d\) - space, \(\alpha T_\omega\) - space, \(\alpha\) - space and \(1/2 T_\alpha\) - space.).

Proof: Let \((X, \tau)\) be \(\alpha g T_{\omega \alpha}\) - space. Let \(A\) be \(\alpha g\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\alpha g T_{\omega \alpha}\) - space, \(A\) is \(\omega \alpha\)-closed set in \((X, \tau)\). Again since \((X, \tau)\) is \(T_{\omega \alpha}\) - space, therefore \(A\) is closed in \((X, \tau)\). Hence \((X, \tau)\) is \(\alpha T_b\)-space [From [104], [96], [58],[104], [26], \((X, \tau)\) is \(\alpha T_c\) - space, \(\alpha T_d\) - space, \(\alpha T_\omega\) - space, \(\alpha\) - space and \(1/2 T_\alpha\) - space.].

Theorem 1.5.21: If \((X, \tau)\) is both \(\alpha g T_{\omega \alpha}\) - space and \(\omega \alpha T\)-space, then \((X, \tau)\) is \(1/2 T_\alpha\)-space.

Proof: Let \((X, \tau)\) be both \(\alpha g T_{\omega \alpha}\) - space and \(\omega \alpha T\)-space. Let \(A\) be \(\alpha g\)-closed set in \((X, \tau)\). Since \((X, \tau)\) is \(\alpha g T_{\omega \alpha}\) - space, \(A\) is \(\omega \alpha\)-closed set in \((X, \tau)\). Again since \((X, \tau)\) is \(\omega \alpha T\)-space, \(A\) is \(\alpha\) -closed and hence \((X, \tau)\) is \(1/2 T_\alpha\)-space.
**Theorem 1.5.22:** Every $\alpha T_c$ - space is $\alpha gT_{oa}$ - space but not conversely.

**Proof:** Let $(X, \tau)$ be $\alpha T_c$ - space. Let $A$ be $\alpha g$-closed set in $(X, \tau)$. Since $(X, \tau)$ is $\alpha T_c$ - space, $A$ is strongly $g$-closed in $(X, \tau)$. Then by the Theorem 1.2.19, $A$ is $\omega \alpha$-closed. Hence $(X, \tau)$ is $\alpha g T_{oa}$ - space.

The converse of the theorem need not be true as seen from the following example.

**Example 1.5.23:** In Example 1.2.3, the space $(X, \tau)$ is $\alpha g T_{oa}$ - space but not $\alpha T_c$ - space since the set $A=\{b\}$ is $\alpha g$-closed but not strongly $g$-closed in $(X, \tau)$.

**Theorem 1.5.24:** Every $T_{1/2}$-space is $\alpha g T_{oa}$ - space.

**Proof:** Let $(X, \tau)$ be $T_{1/2}$-space. Let $A$ be $\alpha g$-closed set in $(X, \tau)$. Since $(X, \tau)$ is $T_{1/2}$-space, by Theorem 2.3 of [32], $A$ is $\alpha$-closed set in $(X, \tau)$. From the Theorem 1.2.4 $A$ is $\omega \alpha$-closed in $(X, \tau)$. Hence $(X, \tau)$ is $\alpha g T_{oa}$ - space.

However the converse need not be true in general as seen from the following example.

**Example 1.5.25:** In Example 1.2.13, the space $(X, \tau)$ is $\alpha g T_{oa}$ - space but not $T_{1/2}$-space since the set $A=\{a, c\}$ is $g$-closed but not closed.

**Theorem 1.5.26:** If $(X, \tau)$ is $\alpha gr-T_{1/2}$ - space, then it is $\omega \alpha T$-space.

**Proof:** Let $(X, \tau)$ be $\alpha gr-T_{1/2}$ - space. Let $A$ be $\omega \alpha$-closed set in $(X, \tau)$. By Theorem 1.2.12, $A$ is $\alpha gr$-closed in $(X, \tau)$. Since $(X, \tau)$ is $\alpha gr-T_{1/2}$-space, $A$ is $\alpha$-closed set in $(X, \tau)$ and hence $(X, \tau)$ is $\omega \alpha T$-space.

The converse of the above theorem need not be true as seen from the following example.
**Example 1.5.27:** In Example 1.2.14 the space \((X, \tau)\) is \(\omega\alpha\)T-space but not \(\alpha\text{gr-}T_{1/2}\)space since \(A=\{a, b\}\) is \(\alpha\text{gr}\)-closed but not \(\alpha\)-closed.

**Theorem 1.5.28:** If \((X, \tau)\) is an \(\alpha\text{gr} \ T_{\omega\alpha}\)-space then every singleton subset of \((X, \tau)\) is closed or \(\omega\alpha\)-open.

**Proof:** Suppose for some \(x \in X\), the set \(\{x\}\) is not closed. Then \(\{x\}^c\) is not open and the only open set containing \(\{x\}^c\) is \(X\) itself. Then \(\{x\}^c\) is \(\alpha\text{gr}\)-closed set. Then \(\{x\}^c\) is \(\omega\alpha\)-closed since \((X, \tau)\) is an \(\alpha\text{gr} \ T_{\omega\alpha}\)-space. Hence \(\{x\}\) is \(\omega\alpha\)-open.

The converse of the above theorem need not be true as seen from the following example.

**Example 1.5.29:** In Example 1.2.13, the space \((X,\tau)\) satisfies the conclusion of the above Theorem 1.5.28, but \((X, \tau)\) is not \(\alpha\text{gr} \ T_{\omega\alpha}\)-space.

**Theorem 1.5.30:** Every \(\alpha\text{b}\)-space is \(\alpha\text{b} \ T_{\omega\alpha}\)-space and hence \(\omega\alpha\)T-space.

**Proof:** Let \((X, \tau)\) be \(\alpha\text{b}\)-space. Let \(A\) be \(\omega\alpha\)-closed, so \(\alpha\text{g}\)-closed, then \(A\) is closed as \((X, \tau)\) be \(\alpha\text{b}\)-space, so \(A\) is \(\alpha\)-closed. Hence \((X, \tau)\) is \(T_{\omega\alpha}\)-space and \(\omega\alpha\)T-space.

The converse of the above theorem need not be true as seen from the following example.

**Example 1.5.31:** In Example 1.2.7, the space \((X, \tau)\) is \(T_{\omega\alpha}\)-space but not \(\alpha\text{b}\)-space.

**Theorem 1.5.32:** A space \((X, \tau)\) is \(\alpha\text{b}\)-space if and only if \((X, \tau)\) is both \(\alpha\text{g} \ T_{\omega\alpha}\)-space and \(T_{\omega\alpha}\)-space.
Proof: Suppose \((X, \tau)\) is \(\alpha T_b\)-space. Then from the Theorems 1.5.18 and 1.5.30, \((X, \tau)\) is \(\omega g T_{\omega \alpha}\)-space and \(T_{\omega \alpha}\)-space.

Conversely assume that \((X, \tau)\) is both \(\omega g T_{\omega \alpha}\)-space and \(T_{\omega \alpha}\)-space. Let \(A\) be \(\omega g\)-closed set in \((X, \tau)\). Then \(A\) is \(\omega \alpha\)-closed, since \((X, \tau)\) is \(T_{\omega \alpha}\)-space.

Again \((X, \tau)\) is \(\omega g T_{\omega \alpha}\)-space, \(A\) is closed in \((X, \tau)\). Hence \((X, \tau)\) is \(\alpha T_b\)-space.

**Theorem 1.5.33:** A space \((X, \tau)\) is both \(\omega g T_{\omega \alpha}\)-space and \(T_{\omega \alpha}\)-space then \((X, \tau)\) is \(T_{1/2}\)-space.

**Proof:** Let \(A\) be \(g\)-closed set in \((X, \tau)\). Then \(A\) is \(\omega g\)-closed set in \((X, \tau)\).

Since \((X, \tau)\) is \(\omega g T_{\omega \alpha}\)-space, \(A\) is \(\omega \alpha\)-closed in \((X, \tau)\). Again since \((X, \tau)\) is \(T_{\omega \alpha}\)-space, \(A\) is closed in \((X, \tau)\). Hence \((X, \tau)\) is \(T_{1/2}\)-space.

**Remark 1.5.34:** \(T_{\omega \alpha}\)-space and \(T_{\omega}\)-space are independent as seen from the following examples.

**Example 1.5.35:** In Example 1.2.3, the space \((X, \tau)\) is \(T_{\omega}\)-space, but not \(T_{\omega \alpha}\)-space

**Example 1.5.36:** In Example 1.2.7, the space \((X, \tau)\) is \(T_{\omega \alpha}\)-space but not \(T_{\omega}\)-space.

**Remark 1.5.37:** \(\omega T_{\omega}\)-space and \(\omega g T_{\omega \alpha}\)-space are independent as seen from the following examples.

**Example 1.5.38:** In Example 1.2.7, the space \((X, \tau)\) is \(\omega T_{\omega}\)-space but not \(\omega g T_{\omega \alpha}\)-space.

**Example 1.5.39:** In Example 1.5.19, the space \((X, \tau)\) is \(\omega g T_{\omega \alpha}\)-space but not \(\omega T_{\omega}\)-space.
Theorem 1.5.40: Every $T_{\omega g}$-space is $T_{\omega \alpha}$ -space but not conversely.

Proof: Since every $\omega \alpha$-closed set is $\omega g$-closed, the proof follows.

Example 1.5.41: In Example 1.2.7, the space $(X, \tau)$ is $T_{\omega \alpha}$ -space but not $T_{\omega g}$-space.

Theorem 1.5.42: If $(X, \tau)$ is $\omega \alpha T$ -space, then it is $\alpha^\# T_{1/2}$-space but not conversely.

Proof: Let $A$ be $g^\#\alpha$-closed in $(X, \tau)$. Then it is $\omega \alpha$-closed in $(X, \tau)$ from the Theorem 1.2.19. Since $(X, \tau)$ is $\omega \alpha T$-space, $A$ is $\alpha$-closed. Hence $(X, \tau)$ is $\alpha^\# T_{1/2}$-space.

Example 1.5.43: In Example 1.2.3, the space $(X, \tau)$ $\alpha^\# T_{1/2}$-space, but not $\omega \alpha T$ - space.

Theorem 1.5.44: If $(X, \tau)$ is $\#\alpha T_{1/2}$ -space, then it is $\alpha T_{\omega \alpha}$ - space but not conversely.

Example 1.5.45: In Example 1.2.3, the space $(X, \tau)$ is $\#\alpha T_{1/2}$-space, but not $\alpha T_{\omega \alpha}$ - space.

Remark 1.5.46: $\omega \alpha T$ - space (resp. $T_{\omega \alpha}$ - space) and $\alpha T_{\omega \alpha}$ - space are independent as seen from the following examples.

Example 1.5.47: In Example 1.2.3, the space $(X, \tau)$ is $\alpha T_{\omega \alpha}$ - space but not $\omega \alpha T$ -space (resp. $T_{\omega \alpha}$ -space).

Example 1.5.48: In Example 1.2.7, the space $(X, \tau)$ is $\omega \alpha T$ -space (resp. $T_{\omega \alpha}$ -space) but not $\alpha T_{\omega \alpha}$ - space.
Remark 1.5.49: $agT_{\omega \alpha}$-space and $T_f$-space are independent as seen from the following examples.

Example 1.5.50: In Example 1.2.7, the space $(X, \tau)$ is $T_f$-space but not $agT_{\omega \alpha}$-space.

Example 1.5.51: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then space $(X, \tau)$ is $agT_{\omega \alpha}$-space but not $T_f$-space.

Remark 1.5.52: $agT_{\omega \alpha}$-space and $\alpha T_d$-space are independent as seen from the following examples.

Example 1.5.53: In Example 1.2.7, the space $(X, \tau)$ is $\alpha T_d$-space but not $agT_{\omega \alpha}$-space.

Example 1.5.54: In Example 1.2.13, the space $(X, \tau)$ is $agT_{\omega \alpha}$-space but not $\alpha T_d$-space.

Definition 1.5.55: A space $(X, \tau)$ is said to be $\omega \alpha T_{stg}$-space if every $\omega \alpha$-closed set is strongly $g$-closed.

Example 1.5.56: In Example 1.5.51, the space $(X, \tau)$ is $\omega \alpha T_{stg}$-space.

Remark 1.5.57: A topological space $(X, \tau)$ is $\omega \alpha T_{stg}$-space if and only if every $\omega \alpha$-open set in $(X, \tau)$ is strongly $g$-open in $(X, \tau)$.

Remark 1.5.58: If $(X, \tau)$ is $T_{\omega \alpha}$-space, then it is $\omega \alpha T_{stg}$-space but not conversely.

Example 1.5.59: In Example 1.5.51, the space $(X, \tau)$ is $\omega \alpha T_{stg}$-space but not $T_{\omega \alpha}$-space.
Remark 1.5.60: Every $\alpha T_b$ - space is $\omega \alpha T_{sg}$ - space but not conversely.

Example 1.5.61: In Example 1.2.7, the space $(X, \tau)$ is $\omega \alpha T_{sg}$ - space but not $\alpha T_b$ - space.

Remark 1.5.62: The concept of $\omega \alpha T_{stg}$ - spaces and $\alpha g T_{\omega \alpha}$ - spaces are different as seen from the following examples.

Example 1.5.63: In Example 1.2.3, the space $(X, \tau)$ is $\alpha g T_{\omega \alpha}$ - space but not $\omega \alpha T_{sg}$ - space.

Example 1.5.64: In Example 1.2.7, the space $(X, \tau)$ is $\omega \alpha T_{stg}$ - space but not $\alpha g T_{\omega \alpha}$ - space.

Remark 1.5.65: From the above observations we get the following diagram.

Here $A \rightarrow B$ means "A implies B" but not conversely and $A \leftrightarrow B$ means "A and B are independent of each other".
Theorem 1.5.66: If \((X, \tau)\) is \(\omega_\alpha T_{stg}\) - space then for each \(x \in X\), \(\{x\}\) is \(\omega_\alpha\)-closed or strongly g-open.

Proof: Suppose \(\{x\}\) is not \(\omega_\alpha\)-closed for some \(x \in X\). Then \(\{x\}^c\) is not \(\omega_\alpha\)-open so \(\{x\}^c\) is not open since every open set is \(\omega_\alpha\)-open. Thus the only open set containing \(\{x\}^c\) is \(X\) itself and hence \(\{x\}^c\) is \(\omega_\alpha\)-closed. By hypothesis \(\{x\}^c\) is strongly g-closed. Hence \(\{x\}\) is strongly g-open.

Remark 1.5.67: The converse of the Theorem 1.5.66 need not be true as seen from the following example.

Example 1.5.68: In Example 1.2.3, the space \((X, \tau)\) satisfies the conclusion of the Theorem 1.5.66 but \((X, \tau)\) is not \(\omega_\alpha T_{stg}\) - space.

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