CHAPTER -6

SEMI- $\delta$ - PREOPEN SETS
6.1 INTRODUCTION

In 1968 N.V. Velicko [73] defined and studied the notions of $\delta$-open sets, Raychoudhuri [65] and J.H. Park et.al. [61] introduced and studied $\delta$-preopen sets and $\delta$-semiopen sets respectively. Analogous to these sets, in this chapter, we introduce and study a new class of sets called semi-$\delta$-pre-open sets in a topological space by using $\delta$-open sets.

This chapter has three sections, in the first section we define and study semi-$\delta$-pre-interior, semi-$\delta$-pre-closure and some properties of them.

In the second section we define and study semi-$\delta$-pre-axioms $T_0$, $T_1$, and $T_2$ and theorems related to them. Also studied comparison of these axioms with $spT_0$, $spT_1$, and $spT_2$.

Third section of this chapter deals with semi-$\delta$-pre-continuous functions, semi-$\delta$-pre-open and semi-$\delta$-pre-closed functions and results on them.

We recall the following definitions.

A subset $A$ of $X$ is said to be $\delta$-open [73] if for $x \in A$ there exists regular open set $G$ such that $x \in G \subseteq A$. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $\text{IntCl}(V) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called $\delta$-closure of $A$ and is denoted by $\delta\text{Cl}(A)$. The set of points $x \in X$ such that $x \in U \subseteq A$ for some regular open set $U$ of is called the $\delta$-interior of $A$ and is denoted by $\delta\text{-Int}(A)$.

From this definition we have $A \subseteq \delta\text{Cl}(A)$ and $\delta\text{-Int}(A) \subseteq A$. 
6.2 Semi-δ-pre-open sets

In this section we define and study a new class of weak forms of open sets, semi-δ-pre-open sets.

**DEFINITION 6.2.1:** A subset \( A \) of \( X \) is said to semi-δ-preopen if there exists δ-preopen set \( U \) such that \( U \subset A \subset \text{Cl}(U) \). Equivalently, \( A \subset \text{ClInt}(\text{Cl}(A)) \).

The complement of semi-δ-preopen set is called semi-δ-pre closed. The family of all semi-δ-preopen sets of \( X \) is denoted by \( S\delta\text{PO}(X) \) and that of all semi-δ-preclosed sets of \( X \) is denoted by \( S\delta\text{PF}(X) \).

**DEFINITION 6.2.2:** The union of all semi-δ-preopen sets of \( X \) which are contained in \( A \) is called semi-δ-pre interior of \( A \) and is denoted \( s\delta\text{pInt}(A) \).

**DEFINITION 6.2.3:** The intersection of all semi-δ-preclosed sets containing \( A \) is called semi-δ-preclosure of \( A \) and is denoted by \( s\delta\text{pCl}(A) \).

**THEOREM 6.2.4:** In topological space \( X \) ( a) every open set is semi-δ-preopen, by (b) every δ-preopen set is semi-δ-preopen and (iii) every semipre open set is semi-δ-preopen.
PROOF: (a) Let $A$ be an open set $\Rightarrow A = \text{Int}(A) \subset \text{Int}(\text{Cl}(A)) = \text{Int}(\delta\text{Cl}(A)) \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$ by Lemma 1.1.48 if $A$ is open. Thus we have $A \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$, therefore $A$ is semi-$\delta$-preopen.

(b) Let $A$ be $\delta$-preopen set then $A \subset \text{Int}(\delta\text{Cl}(A)) \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$

Implies $A$ is semi-$\delta$-preopen.

(c) Let $A$ be semi-preopen set then $A \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A))) \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$.

Therefore $A$ is semi-$\delta$-preopen.

But converse of above are not true in general. For,

EXAMPLE 6.2.5: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be a topology on $X$. $\text{SPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\text{S\deltaPO}(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{c\}, \{b, c\}\}$ since set $\{c\}$ is semi-$\delta$-preopen but not semi-preopen, and not open.

Recall the definition $\delta$-semipreopen sets

DEFINITION 6.2.6: A set $A$ in $X$ is said to be $\delta$-semipreopen set [72] if $A \subset \text{Cl}_{\delta}(\text{Int}(\text{Cl}_{\delta}(A)))$ where $\text{Cl}_{\delta}(A)$ denote $\delta$-closure of $A$.

THEOREM 6.2.7: Every semi-$\delta$-preopen set is $\delta$-semipreopen.

PROOF: Let $A$ be semi-$\delta$-preopen then $A \subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$

$\subset \text{Cl}(\text{Int}(\delta\text{Cl}(A)))$. That implies $A$ is $\delta$-semipreopen.

LEMMA 6.2.8: In a topological space $(X, \tau)$ $x \in s\text{pCl}(A)$ iff $U \cap A \neq \emptyset$ for every semi-$\delta$-preopen set $U$ containing $x$. 

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PROOF: Let $x \in s\delta pCl(A)$. Let us assume that $U \cap A = \emptyset$ for some semi-$\delta$-pre-open set $U$ containing $x$. Then $A \subseteq X-U$ and $X-U$ is semi-$\delta$-pre-closed, i.e. $X-U$ is semi-$\delta$-pre-closed not containing $x$ and hence $x \notin s\delta pCl(A)$. Therefore $U \cap A \neq \emptyset$.

Conversely: Let $U \cap A \neq \emptyset$ for every semi-$\delta$-preopen set $U$ containing $x$. Let us suppose that $x \notin s\delta pCl(A)$ then $x \notin \cap F$, where $F$ is semi-$\delta$-preclosed with $A \subseteq F$. So $x \in X - \cap F$ and $X - \cap F$ is semi-$\delta$-pre-open set containing $x$, and hence $(X - \cap F) \cap A \subseteq (X - \cap F) \cap (\cap F) = \emptyset$ i.e there exists semi-$\delta$-pre-open set $X - \cap F$ such that $(X - \cap F) \cap A = \emptyset$ contradicts the hypothesis. Hence $x \in s\delta pCl(A)$.

THEOREM 6.2.9: In a topological space $(X, \tau)$ if $A$ is subset of $X$ then,

(a) $s\delta pCl(A) \subseteq spCl(A)$ .

(b) $spInt(A) \subseteq s\delta pInt(A)$

PROOF: Let $x \in s\delta pCl(A)$ implies $U \cap A \neq \emptyset$ for every semi-$\delta$-pre-open set $U$ containing $x$, implies $U \cap A \neq \emptyset$ for every semi-pre open set $U$, as every semi-open set is semi-$\delta$-pre open. Therefore $x \in spCl(A)$ and hence $s\delta pCl(A) \subseteq spCl(A)$.

ii) Let $x \in spInt(A)$ implies there exists semi-preopen set $U$ such that $x \in U \subseteq A$ and $U$ is semi-$\delta$-preopen hence $x \in s\delta pInt(A)$. Therefore $spInt(A) \subseteq s\delta pInt(A)$.

However, we observed the following:

(a) $spCl(A) \not\subset s\delta pCl(A)$
EXAMPLE 6.2.10: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$ be a topology on $X$. $\text{SPO}(X) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $\text{SSPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{c\}, \{b, c\}\}$. If $A = \{a, b\}$ then $\text{spCl}(A) = X$ and $\text{sdpCl}(A) = A$, therefore $\text{spCl}(A) \subset \text{sdpCl}(A)$.

Similarly $\text{spInt}(a, c) = \emptyset$ and $\text{sdpInt}(a, c) = \{a, c\}$, therefore $\text{sdpInt}(A) \subset \text{spInt}(A)$

THEOREM 6.2.11: In a topological space $(X, \tau)$ following statements are true w.r.t subsets $A$ and $B$ of $X$:

(a) $A \subset \text{sdpCl}(A)$

(b) $A \subset B \Rightarrow \text{sdpCl}(A) \subset \text{sdpCl}(B)$

(c) $\text{sdpCl}(A) = A$ iff $A$ is semi-$\delta$-preclosed

(d) $\text{sdpCl}(A) = \text{sdpInt}(X-A)$

(e) $\text{sdpCl}[\text{sdpCl}(A)] = \text{sdpCl}(A)$

PROOF: (a) Obvious

(b) Let $A \subset B$, we have $B \subset \text{sdpCl}(B) \Rightarrow A \subset \text{sdpCl}(B)$, as $\text{sdpCl}(A)$ is the smallest semi $\delta$-preclosed set containing $A$, $\text{sdpCl}(A) \subset \text{sdpCl}(B)$?

THEOREM 6.2.12: In a topological space $(X, \tau)$ every semipre-regular set is semipre-$\theta$-open and is semi-$\delta$-pre open.

PROOF: A be a semipre-regular set then $A$ is semipreopen and semipreclosed $\Rightarrow \text{spCl}(A) = \text{spCl}_0(A) = A$ [57] implies $A$ is semipre-$\theta$-closed. Similarly $X-A$ is semipre-$\theta$-closed implies $A$ is semipre-$\theta$-open.
Let A be semipre-θ-open therefore for each $x \in A$ there exists semipreopen set $U_x$ containing $x$ such that $x \in U_x \subseteq \text{spCl}(U_x) \subseteq A$.

Since $\text{spCl}(U_x)$ is semipre-regular. Therefore $A = \bigcup \text{spCl}(U_x)$ is semipreopen which then semi-δ-pre open.

Converse of the above Theorem need not be true in general:

i.e  (i) semipre-δ-open set need not be semipre-θ-open

(ii) semipre-θ-open set need not be semipre regular. For,

**EXAMPLE 6.2.13:** Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ a topology on $X$. $\text{SPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\text{sSPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{c\}, \{b,c\}\}$ clearly set $\{b,c\}$ is semipre-δ-open but not semipre-θ-open as its complement $\{a\}$ is not semipre-θ-closed because $\text{spCl}_{\theta}(\{a\}) = \{a,c\} \neq \{a\}$ and set $\{a,b\}$ is semipre-θ-open but not semipre regular.

### 6.3 SEMI-δ-PRE SEPARATION AXIOMS

In this section we define and study the separation axioms via semi-δ-preopen sets.

**DEFINITION 6.3.1:** A topological space $(X, \tau)$ is said be to semi-δ-pre T₀ (in brief, sδp-T₀) space if for each pair of distinct points $x$ and $y$ there exists a semi-δ-preopen set $G$ such that $x \in G$ and $y \notin G$.

**THEOREM 6.3.2:** Every sp-T₀ is sδp-T₀.

**PROOF:** Let $X$ be sp-T₀ and $x$ and $y$ be any two distinct points of $X$, therefore there exist a semipre-open set $G$ such that $x \in G$ and $y \notin G$. 

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Since every semipre-open set is semi-δ-pre open set. Hence G is semi-δ-preopen set such that x ∈ G and y ∉ G. Therefore X is $\delta p-T_0$.

Converse above theorem is not true in general. For,

**EXAMPLE 6.3.3:** Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}\}$ therefore $\text{SPO}(X) = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ and $\text{SδPO}(X) = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{b\}, \{c\}, \{b,c\}\}$. Clearly $(X, \tau)$ is $\delta p-T_0$ but not $p-T_0$.

**THEOREM 6.3.4:** A topological space $(X, \tau)$ is $\delta p-T_0$ iff $\delta p\text{Cl}$ of distinct points are distinct.

**PROOF:** Let $x, y \in X$ with $x \neq y$ and $(X, \tau)$ is $\delta p-T_0$, we have to show that $\delta p\text{Cl}\{(x)\} \neq \delta p\text{Cl}\{(y)\}$. Since $(X, \tau)$ is $\delta p-T_0$, there exists $\delta p$ open set $G$ such that $x \in G$ and $y \notin G$. Also $x \notin X-G$ and $y \in X-G$ where $X-G$ is $\delta p$ closed in $(X, \tau)$. Clearly $\delta p\text{Cl}\{(y)\}$ is intersection of all $\delta p$ closed sets containing $y$ implies $y \in \delta p\text{Cl}\{(y)\}$ but $x \notin \delta p\text{Cl}\{(y)\}$ as $x \notin X-G$. Therefore $\delta p\text{Cl}\{(x)\} \neq \delta p\text{Cl}\{(y)\}$.

Conversely: For any pair of distinct points $x, y \in X$, let $\delta p\text{Cl}\{(x)\} \neq \delta p\text{Cl}\{(y)\}$. Then there exists at least one point $z$ such that $z \in \delta p\text{Cl}\{(x)\}$ and $z \notin \delta p\text{Cl}\{(y)\}$, clearly $x \notin \text{spCl}_S\{(y)\}$ (because if $x \in \text{spCl}_S\{(y)\}$ then $\{x\} \subset \text{spCl}_S\{(y)\}$ implies $\delta p\text{Cl}\{(x)\} \subset \delta p\text{Cl}\{(y)\}$ which is wrong. Hence $x \in X-\delta p\text{Cl}\{(y)\}$ which is semi-δ-preopen set containing $x$ but not $y$. Hence $(X, \tau)$ is $\delta p-T_0$ space.

We define semi-δ-pre $T_1$ spaces in the following:
DEFINITION 6.3.5: A topological space \((X, \tau)\) is said to be semi-\(\delta\)-pre\(T_1\) (in brief, \(s\delta p-T_1\)) if for each pair of distinct points \(x, y\) there exits to \(s\delta p\) open sets \(U \& V\) containing \(x\) and \(y\) resly but \(x \not\in V, y \not\in U\).

Clearly
1) Every \(sp\ T_1\) is \(s\delta p\ T_1\) space
2) Every \(s\delta p\ T_0\) is \(s\delta p\ T_0\) space

THEOREM 6.3.6: A topological space \((X, \tau)\) is \(s\delta p\ T_1\) if every singleton subset \(\{x\}\) of \(X\) is \(s\delta p\) closed.

PROOF: Let every singleton set \(\{x\}\) be \(s\delta p\) closed and let \(x, y\) be any two distinct points of \(X\), so that \(\{x\}\) and \(\{y\}\) are \(s\delta p\)-closed and as such \(\{x\}^c\) and \(\{y\}^c\) be \(s\delta p\) open. Thus \(y \in \{x\}^c\) but \(x \not\in \{x\}^c\) and \(x \in \{y\}^c\) but \(y \in \{y\}^c\) \(\Rightarrow (X, \tau)\) be \(s\delta p\ T_1\) space.

Conversely, let \((X, \tau)\) be a \(s\delta p\ T_1\) space and \(x\) be any arbitrary point of \(X\). If \(y \in \{x\}^c\) then \(y \neq x\). As \((X, \tau)\) is \(s\delta p\ T_1\) there exists \(s\delta p\) open set \(G\) such that \(y \in G\) and \(x \not\in G\). Thus corresponding to each \(y \in \{x\}^c\) there exists \(s\delta p\) open set \(G\) such that \(y \in G \subseteq \{x\}^c\). therefore \(\cup \{y : y \neq x\} \subseteq \cup \{G : \neq x\} \subseteq \{x\}^c\). Since \(G\) is \(s\delta p\) open and the union of \(s\delta p\) open is \(s\delta p\) open and hence \(\{x\}^c\) is \(s\delta p\) open \(\Rightarrow \{x\}\) is \(s\delta p\) closed.

Next, we and study the semi-\(\delta\)-pre \(T_2\) spaces in the following:

DEFINITION 6.3.7: A topological space \((X, \tau)\) is said to be semi-\(\delta\)-pre \(\langle T_2, (in brief, s\delta p-T_2)\) if for each pair of distinct points \(x, y\) there exist two disjoint \(s\delta p\) open sets \(U\) and \(V\) such that \(x \in U, y \in V\).
Clearly 1) Every sp-T₂ is sδp-T₂. 2) Every sδp-T₂ is sδp-T₁.

### 6.4 SEMI-δ-PRECONTINUOUS FUNCTION

In this section we define and study the semi-δ-pre continuous function

**DEFINITION 6.4.1:** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be semi-δ-pre continuous if for each \( x \in X \) and each \( \sigma \)-open set \( V \) containing \( f(x) \) there exists \( U \in SδPO(X, x) \) such that \( f(U) \subseteq V \).

**THEOREM 6.4.2:** A function \( f : (X, \tau) \to (Y, \sigma) \) is semi-δ-pre continuous if the inverse image of each open set \( V \) of \( Y \), \( f^{-1}(V) \) is semi-δ-pre open.

**PROOF:** Let \( f : (X, \tau) \to (Y, \sigma) \) is semi-δ-pre continuous therefore for each \( x \in X \) and for each \( \sigma \)-open set \( V \) containing \( f(x) \) there exists semi-δ-pre open set \( U_x \) containing \( x \) such that \( f(U_x) \subseteq V \) implies \( U_x \subseteq f^{-1}(V) \) i.e \( x \in U_x \subseteq f^{-1}(V) \), therefore \( \bigcup (U_x) = f^{-1}(V) \) implies \( f^{-1}(V) \) is semi-δ-pre open.

**Conversely**, For a \( \sigma \)-open set \( V \), \( f^{-1}(V) \) is semi-δ-pre open . Let \( U = f^{-1}(V) \). For \( x \in X \), \( V \) be an arbitrary \( \sigma \)-open set containing \( f(x) \) i.e \( f(x) \in V \) implies \( x \in f^{-1}(V) \) and \( f^{-1}(V) \) is semi-δ-pre open that is \( U \) is a semi-δ-pre open set containing \( x \), therefore \( f(x) \in f(U) \subseteq f(f^{-1}(V)) \subseteq V \) i.e \( f(x) \in f(U) \subseteq V \) implies \( V \) is semi-δ-pre continuous.

**THEOREM 6.4.3:** Every semipre continuous is semi-δ-pre continuous.
PROOF: Let \( f : (X, \tau) \to (Y, \sigma) \) is semipre continuous therefore for \( \sigma \)-open set \( V \), \( f^{-1}(V) \) is semipre open implies \( f^{-1}(V) \) is semi-\( \delta \)-preopen and hence it is semi-\( \delta \)-pre continuous.

Converse in Theorem 5.3 is not true in general. For,

EXAMPLE 6.4.4: Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \) be topologies on \( X \) and \( Y \) respectively.

\[
\text{SPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \quad \text{and} \quad \text{SPO}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c\}\}.
\]

Define an identity function \( f : (X, \tau) \to (Y, \sigma) \) then clearly \( f \) is not semipre open as for \( \sigma \)-open set \( \{b, c\} \), \( f^{-1}(\{b, c\}) = \{b, c\} \) is not semipre open in \( X \) but it is semi-\( \delta \)-preopen in \( X \).

Therefore \( f \) is semi-\( \delta \)-pre continuous but not semipre continuous.

THEOREM 6.4.5: Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces. The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \)

(a) \( f \) be semi-\( \delta \)-pre-continuous
(b) for every open set \( V \) of \( Y \), \( f^{-1}(V) \) is semi-\( \delta \)-preopen
(c) for every closed set \( V \) of \( Y \), \( f^{-1}(V) \) is semi-\( \delta \)-preclosed.

PROOF. (a) \( \Rightarrow \) (b): Obvious
(b) \( \Rightarrow \) (c): Let \( V \) be closed subset of \( Y \), therefore \( Y - V \) be open subset of \( Y \)

\[
f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V) \text{ is semi-\( \delta \)-preopen implies } f^{-1}(V) \text{ is semi-\( \delta \)-preclosed.}
\]

(c) \( \Rightarrow \) (a): Obvious
**DEFINITION 6.4.6:** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be semi-$\delta$-pre irresolute if $f^{-1}(V) \in S\delta PO(X)$ for each $V \in S\delta PO(Y)$.

**DEFINITION 6.4.7:** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be weakly semi-$\delta$-pre irresolute (resp. strongly semi-$\delta$-pre irresolute) if for each point $x \in X$ and for each $\sigma$ semi-pre-open set $V$ containing $f(x)$ there exists a $U \in SPO(X, x)$ such that $f(U) \subseteq s\delta pCl(V)$ (resp. $f(s\delta pCl(U)) \subseteq V$).

**DEFINITION 6.4.8:** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be semi-$\delta$-pre open mapping if $f(U) \in S\delta PO(Y)$ for $U \in S\delta PO(X)$

**THEOREM 6.4.9:** Weakly semi-$\delta$-pre irresolute (resp. strongly semi-pre irresolute) is weakly semi-pre irresolute (resp. strongly semi-$\delta$-pre irresolute).

**PROOF:** Let $f: X \to Y$ be weakly semi-$\delta$-pre irresolute then for each point $x \in X$ and for each $(Y, \sigma)$ semi-pre-open set $V$ containing $f(x)$ there exists a $U \in SPO(X, x)$ such that $f(U) \subseteq s\delta pCl(V)$ but $s\delta pCl(A) \subseteq spCl(A)$ Therefore $f(U) \subseteq spCl(A) \Rightarrow f$ is weakly semi-pre-irresolute.

Similarly strongly semi-pre-irresolute is strongly semi-$\delta$-pre irresolute.

**THEOREM 6.4.10:** If a function $f: (X, \tau) \to (Y, \sigma)$ is weakly semi-$\delta$-pre irresolute then $f^{-1}(V) \subseteq spInt(f^{-1}(spCl(V)))$ for every $V \in SPO(Y)$.  

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PROOF: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be weakly semi-\( \delta \)-pre irresolute and \( V \in \text{SPO}(Y) \), let \( x \in f^{-1}(V) \). As \( f \) is weakly semi-\( \delta \)-pre irresolute there exists \( U \in \text{SPO}(X) \) containing \( x \) such that \( f(U) \subset \text{scl}(V) \subset \text{spcl}(V) \) i.e \( U \subset f^{-1}(\text{spcl}(V)) \) i.e \( x \in U \subset f^{-1}(\text{spcl}(V)) \) implies \( x \in \text{spint}(f^{-1}(\text{spcl}(V))) \) therefore \( x \in f^{-1}(V) \) implies \( x \in \text{spint}(f^{-1}(\text{spcl}(V))) \) implies \( f^{-1}(V) \subset \text{spint}(f^{-1}(\text{spcl}(V))) \).

**THEOREM 6.4.11:** The property of being \( s\delta \)-\( T_0 \) space is preserved under one-one, onto, and \( s\delta \) open mapping.

PROOF: Let \((X, \tau)\) be \( s\delta \)-\( T_0 \) space and \((Y, \sigma)\) be any other topological space. Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a one-one, onto, and \( s\delta \) open mapping.

Let \( y_1, y_2 \in Y \) with \( y_1 \neq y_2 \) and since \( f \) is one-one, onto, there exist distinct points \( x_1, x_2 \) in \( X \) such that \( f(x_1) = y_1, f(x_2) = y_2 \). Since \((X, \tau)\) is \( s\delta \)-\( T_0 \) space, therefore there exists semi-\( \delta \)-preopen set \( G \) such that \( x_1 \notin G, x_2 \in G \).

As \( f \) is \( s\delta \) open mapping, there exists \( f(G) \in \text{SPO}(Y, f(x_1)) \) but not containing \( f(x_2) \). Thus there exists \( s\delta \) open set \( f(G) \) containing \( y_1 \) but not \( y_2 \). \( \Rightarrow (Y, \sigma) \) is \( s\delta \)-\( T_0 \).

**THEOREM 6.4.12:** Let \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) be functions then following are true,

(a) If \( f \) is semi-\( \delta \)-pre irresolute and \( g \) is semi-\( \delta \)-pre continuous then \( \text{gof}: X \rightarrow Z \) is semi-\( \delta \)-pre continuous.

(b) if \( f \) is semi-\( \delta \)-pre continuous and \( g \) is completely continuous then \( \text{gof}: X \rightarrow Z \) is semi-\( \delta \)-pre continuous.

(c) if \( f \) is semi-\( \delta \)-pre continuous and \( g \) is perfectly continuous then \( \text{gof}: X \rightarrow Z \) is semi-\( \delta \)-pre continuous.
PROOF: (a) Let $V$ be any open set in $Z$. Since $g$ is semi-\(\delta\)-pre continuous $g^{-1}(V)$ is semi-\(\delta\)pre open in $X$. Since $f$ is semi-\(\delta\)-pre irresolute therefore $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is semi-\(\delta\)-precontinuous.

(b) Let $f : X \rightarrow Y$ be semi-\(\delta\)-pre continuous, $g: Y \rightarrow Z$ be completely continuous and let $V$ be open subset of $Z$. Since $g$ is completely continuous therefore $g^{-1}(V)$ is regular open and hence it is open in $Y$. As $f$ is semi-\(\delta\)-pre continuous $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is semi-\(\delta\)-preopen. Implies $gof$ is semi-\(\delta\)-pre continuous.

(c) Let $f : X \rightarrow Y$ be semi-\(\delta\)-pre continuous, $g: Y \rightarrow Z$ be perfectly continuous and let $V$ be open subset of $Z$. Since $g$ is perfectly continuous therefore $g^{-1}(V)$ is clopen in $Y$ implies open. As $f$ is semi-\(\delta\)-pre continuous $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is semi-\(\delta\)-preopen. Implies $gof$ is semi-\(\delta\)-pre continuous.

THEOREM 6.4.13: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and if $f$ is semi-\(\delta\)-pre irresolute and surjective and $gof : X \rightarrow Z$ is semi-\(\delta\)-pre open then $g$ is semi-\(\delta\)-pre continuous.

PROOF: Let $V$ be any open set in $Z$, since $gof$ is semi - \(\delta\) - pre open, $$(gof)^{-1}(V) = f^{-1}g^{-1}(V)$$ is semi-\(\delta\)-pre open in $X$. As $f$ semi-\(\delta\)-pre continuous, then $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is semi-\(\delta\)-pre open in $Y$ for open set $V$ of $Z$. $\Rightarrow$ $g$ is semi-\(\delta\)-pre continuous.

THEOREM 6.4.14: Let $f : X \rightarrow Y$ be semi-\(\delta\)-pre irresolute, semi-\(\delta\)-pre open and surjective and $g : Y \rightarrow Z$ be a function then $gof : X \rightarrow Z$ is
semi-δ-pre continuous iff g is semi-δ-pre continuous.

**PROOF:** Let \( f: X \rightarrow Y \) be semi-δ-pre irresolute, semi-δ-pre open and surjective and \( g: Y \rightarrow Z \) be semi-δ-pre continuous. We have to prove that \( \text{gof}: X \rightarrow Z \) is semi-δ-pre continuous. Let \( V \) be semi-δ-pre open subset of \( Z \) as \( g \) is semi-δ-pre continuous \( g^{-1}(V) \) is semi-δ-pre open. Since \( f \) is semi-δ-pre irresolute therefore \( f^{-1}(g^{-1}(V)) = (\text{gof})^{-1}(V) \) is semi-δ-pre open in \( X \). \( \Rightarrow \) \( \text{gof} \) is semi-δ-pre continuous.

**Converse** part is same as Theorem 6.4.16

**THEOREM 6.4.15:** The property of being \( s\delta p-T_0 \) space is preserved under one-one, onto, and \( s\delta p \) open mapping and hence is a topological property.

**PROOF:** Let \((X,\tau)\) be \( s\delta p-T_0 \) space and \((Y,\sigma)\) be any other topological space. Let \( f: (X,\tau) \rightarrow (Y,\sigma) \) be a one-one, onto and \( s\delta p \) open mapping. Let \( y_1,y_2 \in Y \) with \( y_1 \neq y_2 \) and since \( f \) is one-one, onto, there exist distinct points \( x_1, x_2 \) in \( X \) such that \( f(x_1) = y_1, f(x_2) = y_2 \). Since \((X,\tau)\) is \( s\delta p-T_0 \) space, therefore there exists semi-δ-preopen set \( G \) such that \( x_1 \in G, x_2 \notin G \). As \( f \) is \( s\delta p \) open mapping, \( f(G) \in S\delta PO(Y,f(x_1)) \) but not containing \( f(x_2) \). Thus there exists \( s\delta p \) open set \( f(G) \) containing \( y_1 \) but not \( y_2 \). \( \Rightarrow \) \((Y,\sigma)\) is \( s\delta p-T_0 \).

**THEOREM 6.4.16:** If \( f: X \rightarrow Y \) be semi-δ-pre continuous injection and \( Y \) is \( T_1 \) then \( X \) is semi-δ-pre \( T_1 \).
PROOF: Suppose that $Y$ is $T_1$. For any pair of distinct points $x$ and $y$ of $X$, $f(x)$ and $f(y)$ be distinct points of $Y$, Hence there exists open set $U$ of $Y$ such that $f(x) \in U$ and $f(y) \notin U$.

Since $f$ is semi-$\delta$-pre continuous therefore $f^{-1}(U)$ is semi-$\delta$-pre open in $X$ such that $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. Thus $X$ is $s\delta p$-$T_1$.

**Theorem 6.4.17**: If $f : X \rightarrow Y$ be semi-$\delta$-pre continuous injection and $Y$ is $T_2$ then $X$ is semi-$\delta$-pre $T_2$.

**Proof**: Let $f : X \rightarrow Y$ be semi-$\delta$-pre continuous, injection and $Y$ be $T_2$. Let $x_1$ and $x_2$ be two distinct points of $X$, as $f$ is injective $f(x_1)$ and $f(x_2)$ are two distinct points of $Y$. Since $Y$ is $T_2$ there exist open sets $U$ and $V$ in $Y$ such that $U \cap V = \emptyset$ and $f(x_1) \in U$, $f(x_2) \in V \Rightarrow x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. As $f$ is semi-$\delta$-pre continuous $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint semi-$\delta$-pre open sets of $X$ such $x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(V)$ implies $X$ is semi-$\delta$-pre $T_2$.

**Theorem 6.4.18**: If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is semi-$\delta$-pre irresolute and one-one, then the following are true,

(a) If $(Y, \sigma)$ is $s\delta p$-$T_1$ then $(X, \tau)$ is also $s\delta p$-$T_1$

(b) If $(Y, \sigma)$ is $s\delta p$-$T_2$ then $(X, \tau)$ is also $s\delta p$-$T_2$

**Proof**: (a) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be semi-$\delta$-pre irresolute, one-one and $(Y, \sigma)$ be $s\delta p$-$T_1$. Let $x_1$ and $x_2$ be two distinct points of $X$, as $f$ is injective $f(x_1)$ and $f(x_2)$ are two distinct points of $Y$. Since $Y$ is $s\delta p$-$T_1$ there exist semi-$\delta$-pre open sets $U$ and $V$ such that $f(x_1) \in U$, $f(x_2) \notin U$ and $f(x_2) \in V$, $f(x_1) \in V \Rightarrow x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, $x_1 \notin f^{-1}(V)$. As
f is semi-δ-pre irresolute $f^{-1}(U)$ and $f^{-1}(V)$ are semi-δ-pre open sets in $X$ such that $x_1 \in f^{-1}(U)$, $x_2 \not\in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, $x_1 \not\in f^{-1}(V)$. $\Rightarrow$ $X$ is $s\delta p-T_1$.

Similarly (b) can be proved.