Chapter 4

A New Solution Concept in Fuzzy Bi-Matrix Games

4.1 Introduction

Fuzziness, in Bi-Matrix Games, was studied by various authors (Bector and chandra 2000), (Nishizaki and Sakawa 1996), (Maeda 2000). They considered the payoff values as fuzzy numbers in a bi-matrix game. Nishizaki and Sakawa (Nishizaki and Sakawa 2001) presented an approach for solving a bi-matrix game with fuzzy payoffs. (A Two Person Zero-Sum Game is a particular case of a Bi-Matrix Game.)

In chapter 3, we have discussed a ranking function approach to matrix games with fuzzy payoffs. Here, in this chapter we extend the same to bi-matrix games with fuzzy payoffs. Here we consider fuzzy bi-matrix games namely, the games with fuzzy payoff where the number of play-
ers are two. In the literature there are many models of matrix games with fuzzy payoffs, but attempts to solve bi-matrix games with fuzzy payoffs are comparatively very less. However, some remarkable works in this respect are done by Bector C. R and S. Chandra (Bector and Chandra 2005).

In this section we solve a fuzzy bi-matrix game with fuzzy payoff by using fuzzy non linear programming method.

4.2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}_{+}^n$ be its non negative orthant. Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real matrix and $e$ be the column matrix with all entries equal 1.

A (crisp) Two Person Non-Zero Sum (Bi-matrix) Game can be expressed as $BG = (A, B, S_m, S_n)$ where $S_m$ and $S_n$ are the strategy space for player I and player II respectively and $A$ and $B$ are $m \times n$ real matrices representing the payoffs to player I and player II respectively.

Let $N(\mathbb{R})$ be the set of all fuzzy numbers. Let $\tilde{A}, \tilde{b}$ and $\tilde{c}$ respectively be $m \times n$ matrix, $m \times 1$ and $n \times 1$ vectors having entries from $N(\mathbb{R})$ and the double fuzzy constraints are given by $\tilde{A}x \preceq \tilde{b}$ and $\tilde{A}^T y \succeq \tilde{c}$ with adequacies $\tilde{p}$ and $\tilde{q}$ respectively. Based on a resolution method proposed in (Yager 1981), the constraints $\tilde{A}x \preceq \tilde{b}$ is expressed as $\tilde{A}x(\leq) \tilde{b} + \tilde{p}(1 - \lambda)$, $\lambda \in [0, 1]$ where for $i = 1, 2, \ldots, m$ the $i^{th}$ component of the fuzzy vector
\( \tilde{p}_i \), namely \( \tilde{p}_i \), measures the adequacy between the fuzzy numbers \( \tilde{A}_i x \) and \( \tilde{b}_i \) which are the \( i \)-th component of the fuzzy vectors \( \tilde{A}x \) and \( \tilde{b} \) respectively. Similarly, the constraints \( \tilde{A}^T y \preceq_{\eta} \tilde{c} \) is expressed as \( \tilde{A}^T y(\preceq_{\eta})\tilde{c} - \tilde{q}(1 - \eta) \), \( \eta \in [0, 1] \) where for \( j = 1, 2, \ldots, n \), the \( j \)-th component of the fuzzy vector \( \tilde{q}_j \), namely \( \tilde{q}_j \), measures the adequacy between the fuzzy number \( \tilde{A}_j^T y \) and \( \tilde{c}_j \) which are the \( j \)-th component of fuzzy vectors \( \tilde{A}^T y \) and \( \tilde{c} \) respectively. Here \( (\preceq) \) and \( (\succeq) \) are relations between fuzzy numbers which preserve the ranking when fuzzy numbers are multiplied by positive scalars.

This could be with respect to any ranking function \( F : N(\mathbb{R}) \rightarrow \mathbb{R} \) such that \( a(\preceq)b \) implies \( F(a) \leq F(b) \). The function \( F \) is used to defuzzify the given fuzzy linear programming problems (Campos 1989). Therefore the double fuzzy constraints of the type \( \tilde{A}x \preceq_{\eta} \tilde{b} \) and \( \tilde{A}^T y \succeq_{\eta} \tilde{c} \) are to be understood as:

\[
\tilde{A}_i x(\preceq)\tilde{b}_i + (1 - \lambda)\tilde{p}_i \text{ for } 0 \leq \lambda \leq 1; \quad i = 1, 2, \ldots, m;
\]

and

\[
\tilde{A}_j^T y(\succeq)\tilde{c}_j - (1 - \eta)\tilde{q}_j \text{ for } 0 \leq \eta \leq 1 \text{ and } j = 1, 2, \ldots, n
\]

This means,

\[
F(\tilde{A}_i x) \leq F(\tilde{b}_i) + (1 - \lambda)F(\tilde{p})
\]

and

\[
F(\tilde{A}_j^T y) \geq F(\tilde{c}_j) - (1 - \eta)F(\tilde{q}).
\]
Definition 4.1. (Bector and Chandra 2005)

A pair \((x^*, y^*) \in S^m \times S^n\) is said to be an equilibrium solution of the bi-matrix game \(BG\) if
\[ x^T A y^* \leq x^T y^* \quad \text{and} \quad x^T B y \leq x^T By^* \]
for all \(x \in S^m\) and \(y \in S^n\).

Theorem 4.1. (Nash 1951)

Every bi-matrix game \(BG = (S^m, S^n, A, B)\) has at least one equilibrium solution.

A trapezoidal fuzzy number is defined as follows:

Definition 4.2.

A fuzzy number \(\tilde{A}\) is called a trapezoidal fuzzy number (TrFN), if its membership function is given by
\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0 & \text{if } x < a_l, x > a_u \\
\frac{x - a_l}{a - a_l} & \text{if } a_l \leq x < a \\
1 & \text{if } a \leq x \leq a \\
\frac{a_u - x}{a_u - a} & \text{if } a < x \leq a_u
\end{cases}
\]

The TrFN \(\tilde{A}\) is denoted by the quadruplet \(\tilde{A} = (a_l, a, a, a_u)\).

Now, let \(a_{i,j}, \tilde{b}_i, \tilde{c}_j, \tilde{p}_i\) and \(\tilde{q}_j\) are trapezoidal fuzzy numbers:
\[
a_{i,j} = ((a_{i,j})_l, a_{i,j}, (a_{i,j})_u); \quad \tilde{b}_i = ((b_i)_l, \tilde{b}_i, (b_i)_u); \quad \tilde{c}_j = ((c_j)_l, \tilde{c}_j, (c_j)_u); \quad \tilde{p}_i = ((p_i)_l, \tilde{p}_i, (p_i)_u); \quad \tilde{q}_j = ((q_j)_l, \tilde{q}_j, (q_j)_u); \]
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\[ \bar{p}_i = ((p_i)_l, p_i, (p_i)_u); \text{ and } \bar{q}_j = ((q_j)_l, q_j, (q_j)_u) \]

Let \( F \) is the Yagers (Yager 1981) first index given by:

\[
\frac{\int x \mu_D(x) dx}{\int \mu_D(x) dx}; \quad \frac{\int y \mu_D(x) dx}{\int \mu_D(x) dx},
\]

where \( dl \) and \( du \) are the lower and upper limits of the support of the fuzzy number \( D \).

Then the constraints \( \bar{A} x \preceq_{\bar{p}} \bar{b} \) and \( \bar{A}^T y \succeq_{\bar{q}} \bar{c} \) respectively means:

\[
\sum_{j=1}^{n} \left( \frac{a_{ij} + \overline{a_{ij}}}{2} + (a_{ij})_u \right) x_j \leq \left( \frac{b_i + \overline{b_i}}{2} + (b_i)_u \right) + (1-\lambda)(p_i)_l + \frac{p_i + \overline{p_i}}{2} + (p_i)_u
\]
and

\[
\sum_{i=1}^{m} \left( \frac{a_{ij} + \overline{a_{ij}}}{2} + (a_{ij})_u \right) y_i \geq \left( \frac{c_j + \overline{c_j}}{2} + (c_j)_u \right) + (1-\eta)(q_j)_l + \frac{q_j + \overline{q_j}}{2} + (q_j)_u
\]

for \( \lambda \in [0, 1] \), \( \eta \in [0, 1] \), \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

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Definition 4.3. (Bector and Chandra 2005)

Let \( \bar{v}, \bar{w} \in N(\mathbb{R}) \). Then \( (\bar{v}, \bar{w}) \) is called a reasonable solution of the bi-matrix game with fuzzy payoff (BGFP), if there exist \( x^* \in S^m \), \( y^* \in S^n \) such that

\[
x^T \bar{A} y^* \leq_{\bar{p}} \bar{v} \quad \forall x \in S^m
\]
\[
(x^*)^T B y \leq_{\bar{q}} \bar{w} \quad \forall y \in S^n
\]
(x*)^T A y* \succeq_{q'} \tilde{v}
(x*)^T B y* \succeq_{q'} \tilde{w}
If (\tilde{v}, \tilde{w}) is called a reasonable solution of BGFP then \tilde{v} (respectively (\tilde{w})) is called a reasonable value for player I (respectively player II).

Definition 4.4.

Let \( T_1 \) and \( T_2 \) be the set of all reasonable values \( \tilde{v} \) and \( \tilde{w} \) for player I and player II respectively, where \( \tilde{v}, \tilde{w} \in N(\mathbb{R}) \). Let there exist \( \tilde{v}^* \in T_1, \tilde{w}^* \in T_2 \) such that \( F(\tilde{v}^*) \geq F(\tilde{v}); \forall \tilde{v} \in T_1 \) and \( F(\tilde{w}^*) \geq F(\tilde{w}); \forall \tilde{w} \in T_2 \), where \( F : N(\mathbb{R}) \to \mathbb{R} \) is the chosen defuzzification function. Then the pair \((x^*, y^*)\) is called an equilibrium solution of the game BGFP. Also \( \tilde{v}^* \) (respectively \( \tilde{w}^* \)) is called the value of the game BGFP for player I (respectively player II).

Using the definition of the bi-matrix game with fuzzy payoffs namely BGFP, we construct the fuzzy non-linear programming problems as follows:

\( P_1: \)

\[ \max \ F(\tilde{v}) + F(\tilde{w}) \]
subject to
\[ x^T A y \leq_p \tilde{v} \quad \forall x \in S^m \]
\[ x^T B y \leq_q \tilde{w} \quad \forall y \in S^n \]
\[ x^T A y \geq_{p'} \tilde{v} \]
\[ x^T B y \geq_{q'} \tilde{w} \]
\[ x \in S^m; \]
\[ y \in S^n; \]
\[ \tilde{v}, \tilde{w} \in N(\mathbb{R}) \]
Now, applying the meaning of the double fuzzy constraints proposed in (Yager 1981), the relations \((\leq)\) and \((\geq)\) preserve the ranking when fuzzy numbers are multiplied by positive scalars, consider only the extreme points of sets \(S^m\) and \(S^n\) in the constraints of \(P_1\). The problem \(P_1\) will be converted into \(P_2\) given as follows:

\[
P_2:
\]

\[
\max \quad F(\tilde{v}) + F(\tilde{w})
\]

subject to

\[
\tilde{A}_i y \preceq \tilde{q}_i \tilde{v} \quad i = 1, 2, \ldots, m
\]
\[
x^T \tilde{B}_j \preceq \tilde{q}_j \tilde{w} \quad j = 1, 2, \ldots, n
\]
\[
x^T \tilde{A}_i y \succeq \tilde{q}_i \tilde{v}
\]
\[
x^T \tilde{B}_j y \succeq \tilde{q}_j \tilde{w}
\]
\[
x \in S^m;
\]
\[
y \in S^n;
\]
\[
\tilde{v}, \tilde{w} \in N(\mathbb{R})
\]

Here \(A_i\) denotes the \(i^{th}\) row of \(\tilde{A}\) and \(B_j\) denotes the \(j^{th}\) column of \(\tilde{B}\), \((i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n)\)

Again, applying the resolution procedure for the double fuzzy constraints in \(P_2\), we get \(P_3\) as:
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$P_3$:

\[ \max F(\tilde{v}) + F(\tilde{w}) \]

subject to

\[
\begin{align*}
\sum_{j=1}^{n} a_{ij} y_j (\leq) \tilde{v} + (1 - \lambda) \tilde{p}; & \quad i = 1, 2, \ldots, m \\
\sum_{i=1}^{m} b_{ij} x_i (\geq) \tilde{w} + (1 - \eta) \tilde{q}; & \quad j = 1, 2, \ldots, n \\
x^T \tilde{A} y (\geq) \tilde{v} - (1 - \lambda) \tilde{p} \\
x^T \tilde{B} y (\geq) \tilde{w} - (1 - \eta) \tilde{q} \\
x \in S^m; \\
y \in S^n; \\
\lambda, \eta \in [0, 1]; \\
\tilde{v}, \tilde{w} \in N(\mathbb{R})
\end{align*}
\]

where

\[
\begin{align*}
\tilde{a}_{ij} &= ((a_{ij})_l, \frac{a_{ij} + \bar{a}_{ij}}{2}, (a_{ij})_u) \\
\tilde{v} &= (v_l, \frac{v + \bar{v}}{2}, v_u) \\
\tilde{p} &= (p_l, \frac{p + \bar{p}}{2}, p_u) \\
\tilde{b}_{ij} &= ((b_{ij})_l, \frac{b_{ij} + \bar{b}_{ij}}{2}, (b_{ij})_u) \\
\tilde{w} &= (w_l, \frac{w + \bar{w}}{2}, w_u) \\
\tilde{q} &= (q_l, \frac{q + \bar{q}}{2}, q_u)
\end{align*}
\]
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\[ q' = \left( q'_0, \frac{q'_0 + q'_u}{2}, q'_u \right) \]

\[ p' = \left( p'_0, \frac{p'_0 + p'_u}{2}, p'_u \right) \]

Now, by utilizing the chosen defuzzification function for the constraints in \( P_3 \), the problem can be further written as non-linear programming problem as \( NP_1 \) as:

\[ NP_1: \]

\[ \max \quad F(\tilde{v}) + F(\tilde{w}) \]

subject to

\[ \sum_{j=1}^{n} F(a_{ij})y_j \leq F(\tilde{v}) + (1 - \lambda)F(\tilde{p}); \quad i = 1, 2, \ldots, m \]

\[ \sum_{i=1}^{m} F(b_{ij})x_i \leq F(\tilde{w}) + (1 - \eta)F(\tilde{q}); \quad j = 1, 2, \ldots, n \]

\[ F(x^T \tilde{A} y) \geq F(\tilde{v}) - (1 - \lambda)F(\tilde{p}') \]

\[ F(x^T \tilde{B} y) \geq F(\tilde{w}) - (1 - \eta)F(\tilde{q}') \]

\[ x \in S^m; \]

\[ y \in S^n; \]

\[ \lambda, \eta \in [0, 1]; \]

\[ \tilde{v}, \tilde{w} \in N(\mathbb{R}) \]

This method proves that, for solving the fuzzy bi-matrix game \( BGFP \), we have to solve only the crisp non-linear programming problem \( NP_1 \). Also, if \((x^*, \lambda^*, v^*, y^*, \eta^*, w^*)\) is an optimal solution of the crisp non-linear programming problem \( NP_1 \) then \((x^*, y^*)\) is an equilibrium solution of the game \( BGFP \).

These findings can be stated in the form of a theorem as:
Theorem 4.2.

The fuzzy bi-matrix game BGFP described by $BGFP = (S^m, S^n, \bar{A}, \bar{B})$ is equivalent to the crisp non-linear programming problem $(NP_1)$ in which the objective as well as the constraint functions are linear except two constraint functions, which are quadratic.

Remark 4.1.

If all the fuzzy numbers are to be taken as crisp numbers; i.e., $a_{ij} = \check{a}_{ij}$, $\check{b}_{ij} = b_{ij}$, $\check{\upsilon} = \upsilon$, $\check{\omega} = \omega$; and if we choose in the optimal solution $NP_1$, $\lambda^* = \eta^* = 1$, then the fuzzy game $BGFP$ reduces to the crisp bi-matrix game $BG$. Thus if $\bar{A}$, $\bar{B}$, $\check{\upsilon}$ and $\check{\omega}$ are crisp and $\lambda^* = \eta^* = 1$, then $BGFP$ reduces to $BG$ and the crisp non-linear programming $NP_1$ reduces to the non-linear programming problem $NP_2$.

\[ NP_2 \]

\[
\max \quad v + w \\
\text{subject to} \\
Ay \leq \upsilon e \\
B^T x \leq \omega e \\
x^T Ay \geq v \\
x^T By \geq w \\
x \in S^m, \\
y \in S^n \\
v, w \in \mathbb{R}
\]
For the case when \( B = -A \) the bi-matrix game \( BG \) reduces to the matrix game \( G = (S^m, S^n, A) \) and the problem \( NP_2 \) reduces to the system:

\[
Ay \leq ve \\
A^T x \geq -we \\
v + w = 0 \\
x \in S^m; \\
y \in S^n \\
v, w \in \mathbb{R}
\]

This is equivalent to the usual primal-dual pair of linear programming problem corresponding to the matrix game \( G = (S^m, S^n, A) \). It is generally difficult to find a solution to the problem \( (NP_1) \) and obtain exact membership for fuzzy values \( \tilde{v}^* \) and \( \tilde{w}^* \) because the decision variables \( \tilde{v} \) and \( \tilde{w} \) are fuzzy and their representation will involve large number of parameters. For example, if \( \tilde{v} \) is a TrFN \( (v_l, v, v_u) \), then to determine \( \tilde{v} \) completely we need all these four variables. Therefore from the computational point of view, it is necessary to take \( F(\tilde{v}) \) and \( F(\tilde{w}) \) as real variables \( V \) and \( W \) and modify the problem \( NP_1 \) as
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\[ \text{max } V + W \]

subject to

\[ \sum_{j=1}^{n} F(\tilde{a}_{ij})y_j \leq V + (1 - \lambda)F(\tilde{p}); \quad i = 1, 2, \ldots, m \]

\[ \sum_{i=1}^{m} F(\tilde{b}_{ij})x_i \leq W + (1 - \eta)F(\tilde{q}); \quad j = 1, 2, \ldots, n \]

\[ F(x^T \tilde{A}y) \geq V - (1 - \lambda)F(\tilde{p}') \]

\[ F(x^T \tilde{B}y) \geq W - (1 - \eta)F(\tilde{q}') \]

\[ x \in S^m; \]

\[ y \in S^n \]

\[ \lambda, \eta \in [0, 1]; \]

\[ \bar{V}, \bar{W} \in \mathbb{R} \]

Thus, although we know that "values" for player I and II are fuzzy with appropriate membership function, we shall only get numerical values \( V^* \) and \( W^* \) for player I and II respectively and the actual fuzzy value for player I and II will be "close to" \( V^* \) and \( W^* \) respectively. Thus, it can be concluded that we shall not get exact membership function for the fuzzy values of player I and II.

In particular, when \( F \) is Yager's first index (Yager 1981), the numerical values \( V^* \) (respectively \( W^* \)) will represent the "centroid" or "the average" value for player I (respectively player II).