Chapter 6

Bi-Matrix Games with Fuzzy Goals and Fuzzy Payoffs

6.1 Introduction

We discussed bi-matrix games with fuzzy payoffs in chapter 4. Fuzziness in bi-matrix games can arise in its goals also. In this chapter we are going to discuss bi-matrix games with fuzzy goals and fuzzy payoffs. In chapter 4 we established that a fuzzy bi-matrix game is equivalent to a crisp non linear programming problem. The same concept is applied here to obtain a solution for the bi-matrix games with fuzzy goals and fuzzy payoffs.
6.2 Preliminaries

Definition 6.1.

Let $S^m, S^n, \tilde{A}$ and $\tilde{B}$ as introduced earlier. Let $\tilde{v}, \tilde{w}$ be fuzzy numbers representing the aspiration levels of the player I and II respectively. Then a two person zero - sum matrix game with fuzzy goals and fuzzy payoffs, denoted by $BGFGFP$ is defined as

$$BGFGFP = (s^m, S^n, \tilde{A}, \tilde{B}, \tilde{v}, \tilde{p}, \tilde{p}', \tilde{w}, \tilde{q}, \tilde{q}', \preceq, \succeq);$$

where $\preceq$ and $\succeq$ have their meanings as explained earlier and $\tilde{p}, \tilde{p}'$ and $\tilde{q}, \tilde{q}'$ are fuzzy tolerance levels for player I and II respectively.

Definition 6.2.

A point $(x^*, y^*) \in S^m \times S^n$ is called an equilibrium solution to the fuzzy bi-matrix game $BGFGFP$ if

$$x^T \tilde{A} y^* \preceq_{\tilde{p}} \tilde{v}; \forall x \in S^m$$

$$(x^*)^T \tilde{B} y \succeq_{\tilde{q}} \tilde{w}; \forall y \in S^n$$

$$(x^*)^T \tilde{A} y^* \succeq_{\tilde{p}'} \tilde{v}; \forall x \in S^m$$

and

$$(x^*)^T \tilde{B} y^* \preceq_{\tilde{q}'} \tilde{w}; \forall y \in S^n$$

6.3 Ranking Function Approach

Using the above definitions for the game $BGFGFP$, we construct the following pair of fuzzy non linear programming problems $(FNP_1)$ for player
6.3 Preliminaries

I and II.

\((FNP_1)\)

Find \((x, y) \in \mathbb{R}^m \times \mathbb{R}^n\) such that

\[
\begin{align*}
x^T \tilde{A}y & \preceq_p \tilde{v}; \quad \forall \ x \in S^m \\
x^T \tilde{B}y & \preceq_{\tilde{q}} \tilde{w}; \quad \forall \ y \in S^n \\
x^T \tilde{A}y & \succeq_{\tilde{p}'} \tilde{v}; \quad \forall \ x \in S^m \\
\text{and} \\
x^T \tilde{B}y & \succeq_{\tilde{q}'} \tilde{w}; \quad \forall \ y \in S^n
\end{align*}
\]

Now using the resolution method of Yager (Yager 1981) for the double fuzzy constraints (discussed in earlier section) and following Zimmermann's approach (Zimmermann 1996) the above fuzzy non-linear programming problems \((FNP_1)\) reduces to

\((FNP_2)\)

\[
\begin{align*}
\max \ & \lambda \\
x^T \tilde{A}y(\leq) \tilde{v} + \tilde{p}(1 - \lambda), \ \forall \ x \in S^m \\
x^T \tilde{B}y(\leq) \tilde{w} + \tilde{q}(1 - \lambda), \ \forall \ y \in S^n \\
x^T \tilde{A}y(\geq) \tilde{v} - \tilde{p}'(1 - \lambda), \ \forall \ x \in S^m \\
x^T \tilde{B}y(\geq) \tilde{w} - \tilde{q}'(1 - \lambda) \ \forall \ y \in S^n \\
x \in S^m \\
y \in S^n
\end{align*}
\]
Next, utilize the defuzzification function $F : N(\mathbb{R}) \to \mathbb{R}$ for the constraints of $(FNP_2)$, to get $(NLP_1)$

$$\begin{align*}
& \text{max } \lambda \\
& \text{subject to} \\
& F(x^T \tilde{A}y) \leq F(\tilde{v}) + (1 - \lambda)F(\tilde{p}), \quad \forall x \in S^m \\
& F(x^T \tilde{B}y) \leq F(\tilde{w}) + (1 - \lambda)F(\tilde{q}), \quad \forall y \in S^n \\
& F(x^T \tilde{A}y) \geq F(\tilde{v}) - (1 - \lambda)F(\tilde{p}'), \quad \forall x \in S^m \\
& F(x^T \tilde{B}y) \geq F(\tilde{w}) - (1 - \lambda)F(\tilde{q}'), \quad \forall y \in S^n \\
& x \in S^m \\
& y \in S^n
\end{align*}$$

The defuzzification function $F$ preserves the ranking when fuzzy numbers are multiplied by non-negative scalars and therefore problem $(NLP_1)$ becomes $(NLP_2)$

$$\begin{align*}
& \text{max } \lambda \\
& \text{subject to} \\
& x^T F(\tilde{A})y \leq F(\tilde{v}) + (1 - \lambda)F(\tilde{p}), \quad \forall x \in S^m \\
& x^T F(\tilde{B})y \leq F(\tilde{w}) + (1 - \lambda)F(\tilde{q}), \quad \forall y \in S^n \\
& x^T F(\tilde{A})y \geq F(\tilde{v}) - (1 - \lambda)F(\tilde{p}'), \quad \forall x \in S^m \\
& x^T F(\tilde{B})y \geq F(\tilde{w}) - (1 - \lambda)F(\tilde{q}'), \quad \forall y \in S^n \\
& x \in S^m \\
& y \in S^n
\end{align*}$$
Since $S^m$ and $S^n$ are convex polytopes it is sufficient to consider only the extreme points (i.e., pure strategies) of $S^m$ and $S^n$ in the constraints of $(NLP_2)$. This leads to the following non-linear programming problem for player I and II.

\[(NLP_3)\]

\[
\begin{align*}
\max & \quad \lambda \\
\text{subject to} & \\
F(\tilde{A})_i y & \leq F(\tilde{v}) + (1 - \lambda)F(\tilde{p}), \quad i = 1, 2, \ldots, m \\
x^T F(\tilde{B})_j & \leq F(\tilde{w}) + (1 - \lambda)F(\tilde{q}), \quad j = 1, 2, \ldots, n \\
x^T F(\tilde{A}) y & \geq F(\tilde{v}) - (1 - \lambda)F(\tilde{p}') \\
x^T F(\tilde{B}) y & \geq F(\tilde{w}) - (1 - \lambda)F(\tilde{q}') \\
x & \in S^m \\
y & \in S^n \\
\lambda & \in [0, 1]
\end{align*}
\]

Here $F(\tilde{A})_i$ denotes the $i^{th}$ row of $F(\tilde{A})$ and $F(\tilde{B})_j$ denotes the $j^{th}$ column of $F(\tilde{B})$; where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

From the above section it can be seen that for solving the fuzzy bi-matrix game $BGFGFP$ we have to solve the non-linear programming problem $(NLP_3)$. Also, if $(x^*, y^*, \lambda^*)$ is an optimal solution of $(NLP_3)$, then $(x^*, y^*)$ is an equilibrium solution for the game $BGFGFP$ and $\lambda^*$ is
the degree to which the aspiration levels $F(\bar{v})$ and $F(\bar{w})$ of player I and II can be met.

All these findings can be presented in the form of a theorem given below.

**Theorem 6.1.**

The fuzzy bi-matrix game $BGFGFP$ described by

$$BGFGFP = (s^m, S^n, \tilde{A}, \tilde{B}, \tilde{v}, \tilde{p}, \tilde{p}', \tilde{w}, \tilde{q}, \tilde{q}', \preceq, \succeq)$$

is equivalent to the crisp non-linear programming problem ($NLP_3$).

**Remark 6.1.**

In this chapter we have studied fuzzy bi-matrix games having fuzzy goals and fuzzy payoffs. When the goals and payoffs are both fuzzy a ranking function approach is developed to solve such fuzzy bi-matrix games, whereas in the crisp scenario, it is easier to solve the problem. A summary of the findings is presented in the next chapter.