CHAPTER IV

FLOW BETWEEN A ROTATING
AND A STATIONARY DISC*

* Part of this chapter is to appear in Fluid Dynamics Research (1992)
4.1 Introduction

In the last two chapters we have discussed the effect of squeezing and sliding of viscous incompressible fluid between two plates or discs using series analysis. In this chapter we show the applicability of series method in the study of flow between rotating discs. To be specific we consider the problem of the viscous flow between a rotating and a stationary disc.

The steady flow of a viscous incompressible fluid between two infinite coaxial rotating disks has been discussed by several authors. Batchelor [31], while giving the extension of von-Karman's solution, briefly explains about the general nature of such flows. His predictions are based on (after reducing Navier - Stokes equations into a set of ordinary differential equations using similarity transformation) the general properties of the ordinary differential equations and physics of the problem. Later Stewartson [32] investigated the nature of solution based on small perturbation expansion. Both Batchelor [31] and Stewartson [32], illustrating the nature of solution of general problem, were primarily interested in knowing the way two disks flow approaches one disk flow of von-Karman. Alternative views of these authors sparked healthy interest by many others in the field. The analytical studies on the problem are due to Stewartson [32], Hoffman [33], Phan - Thien and Bush [30] and others. Some careful
numerical studies on this problem are by Lance and Rogers [34], Pearson [35] and Mellor et al [36]. A clear qualitative picture of the solution of the problem emerged only after the numerical works of Lance and Rogers [34] and Mellor et al [36]. Mellor et al [36], in their investigation of the steady problem for the case of one fixed disk, predicted the non-uniqueness of the solution at high Reynolds numbers. They demonstrated that the Batchelor solution evolves from the zero-Reynolds number solution; the Stewartson solution does not appear until $R = 217$. Hoffman [33] extended the low Reynolds number perturbation series to moderately high Reynolds number using computer. He preferred employing adhoc recurrence relation in generating coefficients of the series solution. Phan-Thien and Bush [30] solved the problem using power series in conjugation with unconstrained optimization technique.

The recent reviews by Parter [37], Zandbergen et al [38] and Greenspan [38a] provide the history of the problem and the current status of investigations into the problem of Karman's flow between two impermeable disks. Brady and Durlofsky [39] give extensive analysis of the case of flow between finite disks of arbitrarily large aspect ratio (non-self similar flows) with different end conditions and find close agreement with similar solutions near the axis of rotation.
In the present case using similarity transformation the equations of motion reduce to coupled nonlinear ordinary differential equations. We seek the solution in the form of series with polynomial coefficients as in previous cases. The convergence of such series is limited by the appearance of non physical singularity. We give accurate analysis of locating the position and predicting the nature of this singularity. The extraction of this singularity yields analytic continuation of the series to high Reynolds number.

4.2 Formulation of the problem

We consider the flow of a viscous fluid between two rigid circular disks of infinite extent. The lower disk is stationary and the upper disk, which is separated by a distance 'd', is rotating with an angular velocity \( \Omega \) (Fig. 4.1). The governing equations for an incompressible viscous flow consist of the continuity equation and the equations of motion. Assuming axial symmetry, (1.1.1) and (1.1.2) take the forms (in cylindrical coordinates)

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0 \quad (4.2.1)
\]

\[
u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{\partial (u/r)}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) \quad (4.2.2)
\]
Fig. 4.1: Schematic diagram of the problem.
The velocity components satisfy no slip conditions. For seeking similarity solutions we define

\[ \xi = \frac{z}{d} \]

\[ u = r \Omega \ h'(\xi) \]

\[ v = r \Omega \ g(\xi) \]

\[ w = -2 \Omega \ d \ h(\xi). \]

The no-slip boundary conditions are

\[ h(0) = h'(0) = h(1) = h'(1) = 0 \]

\[ g(0) = 0, \ g(1) = 1. \]

With the transformation (4.2.5) the equations (4.2.2) and (4.2.3) reduce to the following pair of coupled non-linear ordinary differential equations

\[ g'' - 2 R \ ( h' \ g - h \ g') = 0 \]

\[ h'''' + 2 R \ ( g' \ g' + h \ h'') = 0 \]

where \( R = \frac{\Omega d^2}{\nu} \) is the rotation Reynolds number. Here dashes denote derivatives with respect to \( \xi \).
4.3 Method of solution

For small Reynolds numbers we seek a solution of equations (4.2.7) in power series of $R$ in the form

$$g = g_0 + \sum_{n=1}^{\infty} R^n g_n(\xi)$$

$$h = h_0 + \sum_{n=1}^{\infty} R^n h_n(\xi).$$  \hspace{1cm} (4.3.1)

Substituting (4.3.1) into (4.2.7) and equating like powers of $R$ on both sides, we get

$$g_n'' = 2 \sum_{r=1}^{n} (h_r' - h_{r-r} - h_r' g_{n-r})$$

$$h_n''' = -2 \sum_{n=1}^{n} (g_{r-1} g_{n-r} + h_{r-1} h_{n-r}''').$$ \hspace{1.5cm} (4.3.2)

The relevant boundary conditions are

$$h_0(0) = h_0'(0) = 0$$

$$h_0(1) = h_0'(1) = 0$$

$$h_n(0) = h_n'(0) = 0$$

$$h_n(1) = h_n'(1) = 0$$ \hspace{1cm} (4.3.3)

$$g_0(0) = 0, g_0'(1) = 1$$

$$g_n(0) = 0, g_n'(1) = 0.$$ \hspace{1cm} $n = 1, 2, 3,...$

The solutions of above equations up to $O(R^2)$ are

$$g_0 = \xi$$
It is essential to get higher approximations (large number of coefficients) in the series if it has to reveal the true nature of the function represented by it. But algebra involved is quite prohibitive the in direct evaluation of higher order coefficient functions. We propose an alternative scheme which serves this purpose and at the same time the whole process can be made automatic using computer. For achieving these aims we propose $g_n$ and $h_n$ to be of the forms

$$g_n = \sum_{k=1}^{3n} A_{nk} (1 - \zeta)^k \zeta^{k+1}$$

$$h_n = \sum_{k=1}^{3n-1} B_{nk} (1 - \zeta)^2 \zeta^{k+1}$$

(4.3.5)

$n = 1, 2, 3...$

in (4.3.1). These expressions yield exactly the earlier calculated terms of $g_i, h_i$ ($i = 0, 1, 2$). Besides this it enables us to find $g_i, h_i$ for $i \geq 2$ using computer. We substitute expressions (4.3.5) into (4.3.2) and equate various powers of $\zeta$ on both sides to
obtain a recurrence relation for unknowns \( A_{n(k)} \), \( B_{n(k)} \) in the forms

\[
A_{n(k)} = A_{n(k+1)} - \frac{1}{(k+1)k} \left[ \sum_{i=1}^{3} B_{(n-4)(k-1-i)} P_{(k-1-i)} \right.
\]

\[
+ \sum_{r=1}^{n-2} \left( \sum_{i=0}^{3} \left( \sum_{j=1}^{n(k+i)} A_{(m-4)(t-2)} P_{(4+i)} (l-nk+i+1) \right) \right)
\]

where

\( nk = 3n - 2 - k, \ m = n - r, \ l = 3r + j - 1, \ t = 3m - j, \ k = 1, 2, \ldots, 3n. \)

and

\[
B_{n(k)} = 2B_{n(k+1)} - B_{n(k+2)} - \frac{1}{(k+3)(k+2)(k+1)(k)} \left[ \sum_{i=1}^{2} A_{(n-4)(k-i)} Q_{(k+i)} \right.
\]

\[
+ \sum_{r=1}^{n-2} \left( \sum_{i=0}^{4} \left( \sum_{j=1}^{nk+i} B_{(m-1)(t-3)} Q_{(j+i)(t-3)} \right) \right)
\]

\[
+ \sum_{r=1}^{n-2} \left( \sum_{i=0}^{2} \left( \sum_{j=1}^{nk'+i} A_{(m-4)(t-3)} Q_{(j+i)(t-3)} \right) \right)
\]

\[
\quad (4.3.6)
\]

where

\( nk' = 3n - 4 - k, \ m = n - r, \ l = 3r + j - 1, \ t = 3m - j, \ k = 1, 2, \ldots, (3n-1). \)

\[
P_{1}(k) = 2k,
\]

\[
P_{2}(k) = -4(k+1),
\]

\[
P_{3}(k) = 2(k+2),
\]

\[
P_{4}(k, k_1) = 2(k-k_1 + 1),
\]

\[
P_{5}(k, k_1) = 3(3k_1 - 3k - 4),
\]
\[ P_0(k, k) = 2(3k-3k +5), \]
\[ P_1(k, k) = 2(k-1k-2), \]
\[ Q_1(k) = 2(k+1), \]
\[ Q_2(k) = -2(k+2), \]
\[ Q_3(k) = 2k, \]
\[ Q_4(k) = -2(k+1), \]
\[ Q_5(k) = 2(k+1), \]
\[ Q_6(k) = 2(k+1)(k-1), \]
\[ Q_7(k) = -4(k+1)(k-1) - 4(k+2)(k+1), \]
\[ Q_8(k) = 2(k+1)(k-1) + 8(k+2)(k+1) + 2(k+3)(k+2)(k+1), \]
\[ Q_9(k) = -4(k+2)(k+1)(k+1) - 4(k+3)(k+2)(k+1), \]
\[ Q_{10}(k) = 2(k+3)(k+2)(k+1), \]
\[ A_{i1} = 0, A_{i2} = 0, A_{i3} = 0, B_{i1} = -2/60, B_{i2} = -1/60. \]

The expressions for \( g(\xi) \), \( h(\xi) \) and \( h'(\xi) \) representing velocity components \( v, w \) and \( u \) are, respectively given by

\[ g(\xi) = \xi + \sum_{n=1}^{\infty} \sum_{k=1}^{9n} R_k \xi^k \left( 1-\xi \right) \]
\[ h(\xi) = \sum_{n=1}^{\infty} R^n \sum_{k=1}^{3n-1} B_{n(k)} \xi^{k+1} (1-\xi)^2 \] (4.3.8)

\[ h'(\xi) = \sum_{n=1}^{\infty} R^n \sum_{k=1}^{3n-1} B_{n(k)} ((k+1) \xi^k - 2 \xi^{k+2} \xi^{k+1} + (k+3) \xi^{k+2}) \]

Dimensionless pressure gradient per radius is

\[ p = h''''(0) - R^2 \left( h'^2(0) - g^2(0) - 2h(0) h''(0) \right) = h''''(0) \] (4.3.9)

4.3(a) Analysis and improvement of the series

We have from (4.3.9)

\[ p = h''''(0) = R \sum_{n=1}^{\infty} C_{2n-1} R^{2n-2} \]

\[ = R \sum_{n=1}^{\infty} b_n R^{2n-2} \]

\[ = (R/\epsilon)^{1/2} \sum_{n=1}^{\infty} \frac{R_0}{b_n} \epsilon^{n} \] (4.3.10)

where \( C_{2n-1} = 6B_{(2n-1)2} - 12B_{(2n-1)1} \), \( C_{2n-1} = b_n \), \( R_0 = 24.36819857 \), \( \epsilon = (R/R_0)^2 \). Using recurrence relations (4.3.6) and (4.3.7) we generate the universal coefficients \(((A_{nk}), k = 1,2,..,3n), n = 1,2,..,50 \) and \(((B_{nk}), k = 1,2,..,3n-1), n = 1,2,..,50 \) of the series (4.3.5). The generation of these universal coefficients is an elegant interactive process. Using one of the arithmetic programming languages this calculation is made possible. We run the program in double precision and calculate coefficients up to n
The coefficients so generated are used in obtaining pressure gradient and velocity profiles (series (4.3.9) and (4.3.8) respectively). The coefficients, $b_n$ of pressure gradient (which is obtained using universal coefficients $A_{nk}$ and $B_{nk}$) are listed in Table 4.1. These coefficients are decreasing in magnitude and alternate regularly in sign. This indicates that the nearest singularity restricting the convergence of the series (4.3.10) lies on the negative real axis of $R^2$. Fig.4.2 shows Domb-Sykes plot for the series (4.3.10). The intercept and slope of the curve indicate the presence of simple pole. It corresponds to conjugate pair of square root singularities lying on the imaginary axis of the similarity variable $R$. For extrapolation ($1/n \rightarrow 0$) we use rational approximation (Press et al [21]). The radius of convergence of the series (4.3.10) representing $h'''(0)$ is found to be $R_o = 24.36819857$.

4.3(ai) Euler transformation

The range of validity of the series is restricted by the appearance of a singularity on the negative real axis of $R^2$ which has no physical significance. To overcome this artificial restriction we use the bilinear Euler transformation, as in section 2.3(ai) of chapter 2.
\[ \omega = \frac{\varepsilon}{1+\varepsilon} \]

then, \[ \varepsilon = \frac{\omega}{1 - \omega} \]

these will recast the series (4.3.10) into the new form

\[ h''(0) = \left( R_0 \omega \right)^{1/2} \sum_{n=1}^{\infty} \frac{b_n}{1 - \omega} \]

\[ = \left( R_0 \omega / (1 - \omega) \right)^{1/2} \sum_{n=1}^{\infty} D_n \omega^{n-1} \] (4.3.11)

where

\[ D_1 = b_1 \]
\[ D_2 = R_0^2 b_2 \]
\[ D_3 = b_2 R_0^2 + b_3 R_0^4 \]
\[ D_4 = b_2 R_0^2 + 2 b_3 R_0^4 + b_4 R_0^6 \]
\[ D_n = (-1)^{n-1} \Delta^{n-2} e_2 \]

with \[ \Delta e_j = e_{j+1} - e_j \]
\[ e_j = (-R_0^2)^{j-1} b_j \]

For the summation of above series we use Pade'approximants which further increase the validity of the transformed series (4.3.10). A Pade' approximant is the ratio of two polynomials that, when expanded, agrees with a power series to as many terms as possible. Such rational fractions are known to have remarkable properties of analytic continuation (Baker [9]). A Pade' approximant can
simulate a singularity only by the poles that are zeros of its denominators.

4.4 Results and discussion

The flow of a viscous incompressible fluid between a stationary and a rotating disk is governed by non-linear system of ordinary differential equations (4.2.7) together with boundary conditions (4.2.6). The series expansion scheme (4.3.1, 4.3.5) proposed enables in obtaining recurrence relations (4.3.6) and (4.3.7). Using these interactive relations we generate large number \((n = 50)\) of universal coefficients \(((A_{nk}), k = 1,2,\ldots,3n), n = 1,2,\ldots,50\), \(((B_{nk}), k = 1,2,\ldots,3n-1), n = 1,2,\ldots,50\). A careful FORTRAN program consisting of number of DO loops makes it possible in performing complex algebra involved. The series representing the velocity profiles, (4.3.8), is analysed using Padé approximants. These are shown in Fig.4.3, Fig.4.4 and Fig.4.5. The velocity profiles are identical with those obtained by Phan-Thien and Bush [30] (Fig.4.3 and Fig.4.4). In Fig.4.5 the variation of radial velocity component is shown, we find that these calculations agree favourably with that of Brady and Durlofsky [39] near the axis of rotation. Using pure numerical techniques Lance and Rogers [34] and Mellor et al [36] in their studies have calculated torque and pressure gradient respectively. Phan - Thien and Bush [30] calculated pressure gradient using
power series in conjugation with unconstrained optimization. But they could not proceed beyond \( R = 18 \) in their analysis. Using the universal coefficients of the series (4.3.5) we obtain series expression for non-dimensional pressure gradient, (4.3.9). The non zero coefficients \( b_n \) of these series are listed in Table 4.1. They decrease in magnitude and alternate in sign. This indicates the presence of nearest singularity on negative real axis which is restricting the convergence of the series. Domb-Sykes plot (Fig.4.2) provides location and nature of nearest singularity. The rational extrapolation fixes the (location) radius of convergence of the series (4.3.10) with an error of the order of \( 10^{-5} \) to be \( R_Z = -593.8091 \). The slope of the Domb - Sykes plot shows that the singularity is a simple pole on the negative real axis of \( R^2 \) which in turn corresponds to a conjugate pair of square root singularities lying on the imaginary axis of similarity variable \( R \) itself. This singularity has no physical significance and it limits the range of applicability of the series. Applying Euler transformation and using Pade' approximants the summation of series (4.3.11) for different values of \( R \) is performed. The variation of \( 'p' \) with \( R \) is shown in Fig.4.6. The results of non-dimensional pressure gradient agree most favourably with numerical results of Mellor et al [36] up to \( R = 100 \). Also, it is of interest to note that [4/4] and [5/5] Pade' approximants for
non-dimensional pressure gradient are found to bracket the numerical values of Mellor et al [36]. Beyond $R = 100$ our results deviate slightly with numerical findings (Mellor et al [36]). Our results for non-dimensional pressure gradient agree with the numerical results of Brady and Durlofsky [39] near the axis of rotation which are shown in Fig.4.6. Phan-Thien and Bush [30] calculated pressure gradient only up to $R = 18$ and our results agree most accurately with that of Phan-Thien and Bush [30]. Moreover, the method proposed here is quite flexible and efficient in implementing it on computer compared with that of Phan-Thien and Bush [30] and other pure numerical methods. Once the universal coefficients are generated rest of the analysis can be done at a stretch requiring hardly any computer time and storage. Whereas other semi analytical and numerical methods require huge storage and long computing time.
Table 4.1: The coefficients of the series (4.3.10)

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<th>$b_n$</th>
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Fig. 4.2: Domb-Sykes Plot for finding singularity of the series representing pressure gradient (h''(0)).
Fig. 4.3: The transverse velocity profiles (gC2) for different Reynolds number.
Fig. 4.4: The longitudinal velocity profiles \( \text{h}(\xi) \) for different Reynolds number.
Fig. 4.5: The transverse velocity profiles \( h'(\xi) \) for different Reynolds number.
Fig. 4.6: Pressure gradient versus R.