CHAPTER – IV

A -, D - AND E-OPTIMAL n-ARY BLOCK DESIGNS WITH UNEQUAL REPLICATIONS AND UNEQUAL BLOCK SIZES

4.1 INTRODUCTION

Let $D (V; B; G)$ denote the class of all connected n-ary block designs having $V$ treatments arranged in $B$ blocks and the total number of observations $G$. The number of replications of treatments and block sizes of the designs belonging to this class are not fixed. For some designs $d \in D (V; B; G)$, however $R_{d1}$, $R_{d2}$, … $R_{dV}$ denote the replications of treatments and $K_{d1}$, $K_{d2}$, …., $K_{dB}$ denote the block sizes, $\sum_i R_{di} = G = \sum_j K_{dj}$ for all $d \in D (V; B; G)$. Without any loss of generality let us assume that the $p$th treatment has the smallest number of replications; that is $R_{dp} = \min(R_{d1}, R_{d2}, \ldots R_{dV})$. A subclass of $D (V; B; G)$ is $D (V; K_t; B-1; G)$, which contains all connected designs having $V$ treatments, arranged in $B$ blocks with arbitrary sizes and smallest replication fixed; that is $R_{dp} = R_p$ for all $d$. The remaining $(V-1)$ treatments have arbitrary replications $R_{di}$ for all $i \neq p$, but $R_p + \sum_{i=p} R_{di} = G$.

Similarly let $K_{dt} = \max(K_{d1}, \ldots K_{dB})$. Another subclass of $D (V; B; G)$ is $D (V; K_t; B-1; G)$, which contains all connected designs having $V$ treatments arranged in $B$ blocks with maximum block size fixed, that is $K_{dt} = K_t$ and arbitrary replication of treatments $R_{d1}$, …, $R_{dV}$. The remaining $(B-1)$ block sizes are arbitrary, but $K_t + \sum_{j \neq t} K_{dj} = G$. Further, $D (R_p, V-1; K_t; B-1; G)$ denotes the subclass of all designs in $D (V; B; G)$ having $V$ treatments arranged in $B$ blocks and the smallest replication and the largest block size fixed at $R_p$ and $K_t$ respectively.

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2 This chapter forms a part of the paper “ON OPTIMAL n-ARY BLOCK DESIGNS” presented at the 99th sessions of INDIAN SCIENCE CONGRESS, 2012, Section of Statistics, Part III, No. 20.
Let \( d \) be a \( n \)-ary block design such as described above. The combinatorial structure of \( d \) is determined by its \( V \times B \) incidence matrix \( N_d \) whose entries \( n_{dij} \) gives the number of time treatment \( i \) occurs in block \( j \). The \( i \)-th row sum of \( N_d \) is denoted by \( R_{ji} \) and represents the number of times treatment \( i \) is replicated in the design. The \( j \)-th column sum of \( N_d \) is denoted by \( K_{dj} \) and represents the size of the \( j \)-th block of the design. The matrix \( N_d \) is referred to as the concurrence matrix of the design where \( N_d' \) is the transpose of \( N_d \) and its entries are denoted by \( \Lambda_{dij} \). Observations \( Y_{ij} \) obtained after applying \( i \)-th treatment to an experimental unit occurring in the \( j \)-th block are assumed to follow the usual additive two-way classification model, ie.

\[
Y_{ij} = \alpha_i + \beta_j + e_{ij}; \quad 1 \leq i \leq V-1; \quad 1 \leq j \leq B
\]

where \( \alpha_i \) = the effect of \( i \)-th treatment \( \beta_j \) = the effect of the \( j \)-th block and \( e_{ij} \) is a random error term with zero expectation. All observations are also assumed to be uncorrelated and have constant variance \( \sigma^2 \). The reduced normal equations for estimating the treatment effect in \( d \) are

\[
C_d = R_d - N_d K_d^{-1} N_d'
\]

where \( R_d = \text{diag} (R_{d1}, R_{d2}, \ldots R_{dV}) \) is a diagonal \((V \times V)\) matrix

\[
K = \text{diag} (K_{d1}, K_{d2}, \ldots K_{dB}) \text{ is a diagonal and}
\]

\[
(P_d) = N_d K_d^{-1} N_d' = (\delta_{dii})
\]

\( C_d \) is called the information matrix or \( C \)-matrix of \( d \) and is known to be positive semi-definite with zero rows sums and zero column sums. For \( d \in \mathbb{D} \) \((V, B, G)\), let \( Z_{d0} = 0 < Z_{d1} \leq Z_{d2} \leq \ldots Z_{dV-1} \) denote the eigen values of \( C_d \). The Optimality criterion considered here for selecting optimal designs in \( \mathbb{D} \) \((V, B, G)\) is the E-Optimality criterion. E-Optimality maximizes the smallest non-zero eigen values of the \( C \)-matrix. This criterion was introduced by Ehrenfeld (1955), Takeuchi (1961, 1963), Kiefer (1975), Cheng (1978) and Jacroux (1980, 1983) and proved that the E-Optimality of various balanced and
partially balanced binary block designs in the subclass of proper designs $D(v, b, k)$ where $bk/v$ is an integer and $k$ is the common value of the block sizes. The present authors Soundarapandian, et al. in a yet unpublished work have obtained results which imply the E-Optimality of certain types of balanced and partially balanced n-ary block designs. Jacroux (1980) obtained a sufficient condition for a block designs to be E-Optimal over $D(r_1, r_2, r_v; b; k)$ and $D(v; b; k)$, where $bk/v$ is an integer. Gupta, et al., (1989) characterized and constructed E-Optimal block designs within various subclasses of connected designs $D$. The present paper presents sufficient condition for a n-ary block designs to be E-Optimal over $D(R_1, R_2, ..., R_v; B; K)$ and $D(V; B; K)$ where $BK/V$ is not an integer. The purpose of this paper is to characterize and construct E-Optimal n-ary block designs within various subclasses of connected designs $D$.

4.2 PRELIMINARY RESULTS OF N-ARY BLOCK DESIGNS

Jacroux (1980) characterized the binary E-optimal block designs in the sub classes of proper designs. Now we extend those results to n-ary block designs with unequal replications as well as unequal block sizes. Let $d \in D(R_p, V-1; K_t, B-1; G)$ with incidence matrix $N_d$ and information matrix $C_d$. Consider,

$$T_{dx} = K_t C_d - xV (V-1)^{-1} (I_V - J_V J_V' / V)$$  (4.2.1)

where $x$ is any real number. $I_V$ is an identity matrix of order $V$, and $J_V$ is a $V$-component column vector with all elements unity. The eigen-values of $T_{dx}$ are

$$0 < K_t Z_d1 - xV / (V-1) \leq ... < K_t Z_d(V-1) - xV(V-1)$$  (4.2.2)

For a given value of $x$, $t_{x_{dij}}$ is used to denote the $(i,j)$th entry of $T_{xd}$. Here

$$t_{x_{dii}} = R_d K - \Lambda_{dii} - x, \quad t_{x_{dij}} = -\Lambda_{dij} + x/(V-1)$$  (4.2.3)

Now $T_{xd_{mn}}$ denote the sub matrix
of $T_{xd}$ and $|T_{xdmn}|$ to denote the determinant of this sub matrix.

Soundarapandian (1981a) established that $B \geq V$ for a balanced n-ary block (BNB) design with $V$ treatments and $B$ blocks. This inequality was shown to be true for a wider class of BNB designs and also for similar balanced n-ary block designs but with blocks of different sizes. The parameter relations given there may be recollected and modified for our present use.

The parameter relations (3.3.4) (3.3.5), (3.3.6) (3.3.7), (3.3.8) (3.3.9) and (3.3.10) may be recollected in this chapter also for proving theorems, corollaries, constructions and examples. Before that we present the following results for A- and D- Optimality

In this section we give some preliminary results, which are needed to derive our main results, of the end of this chapter. We begin by looking at majorization. The definitions and results given here concerning majorization can be found in Marshall and Olkin (1979). For any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (Euclidean n-space), we will let $x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(n)}$ denote the components of $x$ ordered in terms of magnitude.

4.2.1 Definition

For $x, y \in \mathbb{R}^n$, we say $y$ weakly majorizes $x$, written $x \prec^w y$, if and only if

$$\sum_{i=1}^{s} y(i) \leq \sum_{i=1}^{s} x(i), \quad \text{for } s = 1, \ldots, n$$

The relationship between weak majorization and the $\phi_j$ – optimality criteria defined in the previous chapter is given in the following lemma.

4.2.2 Lemma
For vectors \( x, y \in \mathbb{R}^n \), \( x \preceq^w y \) if and only if \( \sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i) \) holds for all continuous, convex non increasing functions \( f \).

**Proof:**


Our next preliminary results concern sums of eigenvalues. Let \( A \) denote a \( v \times v \) positive semi-definite matrix with zero row sums and let \( \{\sigma_i\} \) denote a collection of \( n \) permutations on the symbols \( 1, \ldots, V \). Now let \( \bar{A} = (1/n) \sum_{i=1}^n A \sigma_i = (1/n) \sum_{i=1}^n P_i A P_i' \) with \( P_i \) representing the \( v \times v \) matrix representation of \( \sigma_i \) and let \( 0 = m_0 \leq \ldots \leq m_{v-1} \) and \( 0 = m_0 \leq \ldots \leq m_{v-1} \) denote the eigenvalues of \( A \) and \( \bar{A} \), respectively. The following lemma is essentially proven in Constantine (1981), hence it is stated without proof.

**4.2.3 Lemma**

Let \( A \) and \( \bar{A} \) be as described in the previous paragraph. Then the eigenvalues of \( A \) are weakly majorized by the eigenvalues of \( \bar{A} \), i.e. for \( s=1, \ldots, v-1 \), \( \sum_{i=1}^v m_i \leq \sum_{i=1}^s m_i \).

The matrix \( A \) of the previous lemma is called an averaged version of \( A \).

For \( d \in \mathcal{D} (V; B; K) \), we shall assume for the remainder of this chapter that the treatments have been labeled so that \( R_{d1} \leq \ldots \leq R_{dj} \). We shall say that \( C_d \) has been averaged over treatments \( 1, \ldots, p_1 \) if in Lemma 4.2.3, we take \( A = C_d \) and \( P_i (i=1, \ldots, p_1!) \) to be the matrix representations for the symmetric group of permutations on the symbols \( \{1, \ldots, p_1\} \) extended by the identity to the rest of the treatments. If in addition to averaging over treatments \( \{1, \ldots, p_1\} \), we also average over the sets of treatments \( \{p_1+1, \ldots, p_2\}, \{p_2+1, \ldots, p_3\}, \ldots, \{p_{s-1}+1, \ldots, p_s\} \), then there are \( p_1!(p_2-p_1)! \ldots (p_s-p_{s-1})! \) permutations to use in Lemma 4.2.3.
4.2.4 Lemma

Suppose $d \in D (V; B; K)$ has C-matrix $C_d = (c_{dij})$. Now, for $i < V-1$, let

$$\alpha_d = k \sum_{i=1}^{V-s} C_{dii} / (V-s), \quad \alpha_d = -k \sum_{i=1}^{V-s} \sum_{j=i}^{V-s} C_{dij} / (V-s)(V-s-1).$$

$$\beta_d = k \sum_{i=v-s+1}^{v} C_{dii} / s, \quad \beta_d = k \sum_{i=v-s+1}^{v} \sum_{j=i}^{v} C_{dij} / s(s-1).$$

$$\gamma_d = k \sum_{i=1}^{v-s} \sum_{j=v-s+1}^{v} C_{dij} / s(v-1).$$

Then

(i) $k \sum_{d=1}^{V-s-1} z_{di} \leq (v-s-1) (\bar{a}_d + \bar{\alpha}_d),$

(ii) $k \sum_{d=1}^{V-s-1} z_{di} \leq (v-s-1) (\bar{a}_d + \bar{\alpha}_d), \forall \gamma_d.$

Proof: Consider the averaged version $\bar{C}_d$ of $C_d$ obtained by averaging separately over the sets of treatments $\{1,2, \ldots, v-s\}$ and $(v-s+1, \ldots, v}$. After averaging, we note that

$$k \bar{C}_d = \begin{bmatrix} \bar{a}_d + \bar{\alpha}_d & -\gamma_d \bar{J}_{v-s,v} & -\bar{\gamma}_d \bar{J}_{v-s,v} \\ -\gamma_d \bar{J}_{s,v} & \bar{\beta}_d + \bar{\beta}_d & \bar{\beta}_d \bar{J}_{ss} \end{bmatrix}$$

where $I_m$ denotes the m x m identity matrix and $J_{mn}$ denotes the m x n matrix of ones. It is now easily verified that the nonzero eigen values of $K \bar{C}_d$ are $\bar{a}_d + \bar{\alpha}_d$ occurring with multiplicity $v - s - 1$, $V \gamma_d$ occurring with multiplicity one and $\bar{B}_d$ and $\bar{\beta}_d$ occurring with multiplicity $s-1$. The result now follows directly from Lemma 4.2.3.
4.3 RESULTS ON E-OPTIMALITY OF N-ARY BLOCK DESIGNS HAVING UNEQUAL REPLICATIONS AND UNEQUAL BLOCK SIZES

To obtain E-Optimality of n-ary block designs having unequal replications and unequal block sizes, we follow the approach of Jacroux (1980) for binary designs and Soundarapandian et al. (1995) for n-ary block designs. Here we can assume that $v \geq k_t$ for binary designs and the incidence matrix $N$ of n-ary design consists of some zero elements in the $t$-th block of the n-ary block designs. If the incidence matrix $N$ does not contain zero elements in the $t$-th block $B_t$ or $V < K_t$, sharper bounds can be obtained by using the smallest possible value of $\delta_{d_{pp}}$ over all designs in the respective subclass. Let $p$ corresponds to the treatment with the smallest number of replications. Also let $R^*=[G/V]$ and $K^*=(G/B)$, where $(x)$ denotes the largest integer contained $x$.

In our varying replicate and unequal block sized n-ary block designs, the eigenvalues of $T_{dx}$ (from 3.3.1) are given as (3.3.2), they are

$$0 < K_t Z_{d1} - XV / (V-1) \leq K_t Z_{d2} - XV (V-1) \leq \ldots \leq K_t Z_{d(V-1)} - XV / (V-1) \quad (4.3.1)$$

Then our following Theorem 4.3.1 gives upper and lower bounds for the minimum nonzero eigenvalue of a C-matrix associated with n-ary block design in $\mathbb{D}$ ($R_p$, $V$-1, $K_t$, B-1, G).

4.3.1 Theorem

Let $d \in \mathbb{D}$ ($R_p$; $V$-1; $K_t$; B-1; G) have C-matrix $C_d$ and let $m$ equal the smallest off-diagonal entry occurring in $N_d N'_d = ((\Lambda_{dij}))$ or $m = \min_{i \neq j} (K_i \delta_{dij})$

Then $mV/K_t \leq Z_{d1} \leq (R_p K_t - \Delta_p) V / ((V-1) K_t)$

Further if $m = (R_p K_t - \Delta_p) V / ((V-1) K_t)$ then,

$Z_{d1} = (R_p K_t - \Delta_p) / ((V-1) K_t)$ and $d$ is E-Optimal in $\mathbb{D}$ ($R_p$, $V$-1; $K_t$; B-1; G).
Proof:

Take $T_{dx}$ as defined in (4.2.1) and consider $x = (R_p K_t - \Delta_p)$ in Theorem 4.3.1 of the paper by Soundarapandian, et.al (1995), or in Theorem 3.4.1 of chapter III, then the proof follows automatically and we get the above required results without explanation or proof.

4.3.2 Corollary

If some $d \in D (R_p; V-1; K_t; B-1; G)$ satisfies the conditions of Theorem 4.3.1. and $R^* = R_{dp}$, then $d$ is E-Optimal in a wider class $D (V; K_t; B-1; G)$.

4.3.3 Example

Let us consider the incidence matrix of the following ternary block design with parameters $(V=7; B=9; R_{dp} = R^* = K_t = 7; G=55)$

$$N_d = \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
2 & 1 & 0 & 0 & 2 & 0 & 2 & 0 \\
2 & 2 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 2 & 2 & 1 & 0 & 0 & 2 & 0 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 2 & 2 & 1 & 0 & 2 \\
0 & 0 & 2 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & 1 & 2 & 1
\end{bmatrix}$$

This ternary block design is E-Optimal over $D (R_p; V-1; K_t; B-1; G)$ as well as $D (V; K_t; B-1; G)$.

Here $m = 6 = (R_p K_t - \Delta_p) / (V-1)$ and $R_{dp} = R^*$. We can also find $Z_{dl} = 6 = V(R_p K_t - \Delta_p) / (V-1) K_t$

Now we will extend the above results further to the sub-class of n-ary block designs with arbitrary (ie unequal) block size. Consider the sub-class of n-ary block design $D (R_p, V-1; B; G)$. Let us define for some real number $x$ as before (4.2.1)
The eigenvalues are:

\[
0 < K^* Z_{d1} - XV / (V-1) \leq K^* Z_{d2} - XV / (V-1) \leq \ldots K^* Z_{d(V-1)} - XV / (V-1)
\]

(4.3.2)

Then we get the following theorem.

### 4.3.4 Theorem

Let \( d \in D (R_p, V-1; B; G) \) have C-matrix \( C_d \) and let \( m \) equal the smallest off diagonal entry occurring in

\[
N_d N'_d = ((\Delta_{dij})) \quad \text{or} \quad m = \min_{i \neq j} (K^* \delta_{dij}) \quad \text{and} \quad R_p \leq (K^* \delta_{dii}),
\]

then,

\[
mV/K^* \leq Z_{d1} \leq (R_p K^* - \Delta_p) V/K^* (V-1)
\]

Further if \( m = (R_p K^* - \Delta_p) / (V-1) \), then

\[
Z_{d1} = (R_p K^* - \Delta_p) V/K^* (V-1) \quad \text{and} \quad d \text{ is E-Optimal in } D (R_p; V-1; B; G).
\]

### 4.3.5 Corollary

If some \( d \in D (R_p, V-1; B; G) \) satisfies the conditions of Theorem 4.3.4 and \( R^* = R_{dp} \), then \( d \) is E-Optimal in \( D (V; B; G) \).

### 4.3.6 Example

Consider the incidence matrix of the following ternary block design with parameters \( (V = 7; B=8; R_{dp} = 7 = K^*) \).
The above ternary block design in E-Optimal over $D (R_p, V-1; B; G)$ for
$n = 6 = (R_p K^* - \Delta_p) / (V-1)$ and $Z_{d1} = 6 = V (R_p K^* - \Delta_p) / K^* (V-1)$. The
above ternary design is also E-Optimal over $D (V; B; G)$ for $R^* = R_{dp} = 7$.

4.4 CONSTRUCTION OF E-OPTIMAL N-ARY BLOCK DESIGN

The various Construction of E-Optimal n-ary block designs in different
subclasses considered in § 4.3 will be studied in detail in this section.

4.4.1 Construction

Let $N_1$ be the incidence matrix of a balanced n-ary block design with
parameters. $(V, B, R, K, A)$. Let $N_2$ (where the number of treatment to be the
same V) be the incidence matrix of a design with maximum block size K and
the number of blocks be B.

Then the design with the incidence matrix $N_{d\ast} = (N_1, N_2)$ is E-Optimal
in $D (R_p, V-1, K, B+B-1; G)$. To this series of designs, $Z_{d1} = \Lambda V/K$.

4.4.2 Example

Let $N_1 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $N_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Then $N_d = (N_1, N_2)$ is an E-Optimal design in $D (R_p, V-1; K, B+B-1; G)$
where $R_p=6$, $V=3$, $K=3$, $B+b-1=5$, $G=18$, $Z_{d1}=2$.

4.4.3 Construction
As per Theorem 4.3.1. we will have

\[ N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \]

Be the incidence matrix of an E-Optimal n-ary block design \( d \in D (R_p, V-1; K_t, B-1; G) \). Then the n-ary block design with incidence matrix.

\[ N_d^* = \begin{bmatrix} N_1 & O_{v-e \times O_s} \\ N_2 & qJ_e X_{J_e^{'}} \end{bmatrix} \quad (qe \leq K_i; \ s > 1) \] is an E-Optimal block design \( D (R_p, V-1; K_t, B_0-1; G_0) \) where \( B_0 = B + s, \ G_0 = G + qe \). \( N_1 \) contains least replicate \( R_p = \sum n_{dij} \) and the least \( \Lambda_{di}/K_{di} = \sum n_{dij} n_{dij}/k_{dij} \) of \( N_d^* \).

### 4.4.4 Example

Consider the E-Optimal n-ary block design of Example 4.3.3

\[
N_1 = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 2 \end{bmatrix} \quad N_2 = \begin{bmatrix} 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 2 \end{bmatrix}
\]

Then,

\[
N_d^* = \begin{bmatrix} N_1 & O_4 \\ N_2 & J_3 \end{bmatrix}
\]

is E-Optimal n-ary block design in \( D (R_p, V-1, K_t, B_0-1, G_0) \) with parameters \( V=7, B_0=10, G_0 = 58, R_p = 7 = K_t, \) \( m = 6 = (R_p K_t - \Delta_p) / (V-1) \) and \( Z_{d1} = 6 = (V/(V-1) (R_p K_t - \Delta_p)/K_t. \)
4.4.5 Construction

By theorem 4.3.1 let $d_1 \in D(R_{1p}, V-1; K_{1t}, B_{1}-1, G_1)$ and $d_2 \in D(R_{2p}, V-1; K_{2t}, B_{2}-1, G_2)$ be two n-ary designs in their respective classes and let $N_1$ and $N_2$ be their respective incidence matrices with $K_{1t} = K_{2t} = K_t$. Then $N = (N_1, N_2)$ is the incidence matrix of an E-Optimal n-ary block design $d \in D(R_p, V-1; K, B-1; G)$ where smallest replication $R_{1p}$ of $d_1$ and $R_{2p}$ of $d_2$ correspond to the same treatment $p$, and $B = B_1 + B_2$, $R_p = R_{1p} + R_{2p}$ and $G = G_1 + G_2$.

4.4.6 Example

Taking $N_1$ as E-Optimal n-ary block design in Example 4.3.3 and $N_2$ of that example 4.4.4, then the new design $d$ with incidence matrix

$$N_d = (N_1, N_2)$$

is an E-Optimal ternary block design over $D(R_p, V-1; K, B-1; G)$ with parameter $V=7$, $B=19$, $R_p=14$, $K_t=7$, $G=113$, $m=12=(R_p K_t - \Lambda_p)/(V-1)$ and $Z_{d_1} = 12 = \{V/(V-1)\} \{R_p K_t - \Delta_p\}/K_t$.

4.4.7 Construction

Let $N$ be the incidence matrix of a balanced n-ary block design with parameters $(V, B, R, K, A)$ and let $K \leq e \leq (V-1)$, $s \geq 1$, and as $\leq (B-1)$. Then there exists an E-optimal n-ary block design in $d \in D(R_p, V-1; B_0; G_0)$ with incidence matrix

$$N * d = \left[ \begin{array}{c} N^r_v \circ N^r_i \circ N^r_j \end{array} \right]$$

where $B_0 = B+s$ and $G_0 = G + se$.

4.4.8 Construction

An attempt is made to develop a single initial block to construct a general balanced n-ary block design and from which we obtain characterization
of an E-Optimal n-ary block design with varying replicates and unequal block sizes.

Let there exists two symmetrically balanced incomplete block designs with parameters \((v, b, r, k, \lambda)\) and \((v', b', r', k', \lambda')\) respectively, each obtained through a single initial block. Let these two initial blocks be respectively:

\((x, x_2, \ldots, x_k)\) and \((y_1, y_2, \ldots, y_k)\)

Then by utilizing the Theorem 7 of Soundarapandian (1980a), we get the following Theorem 4.4.9, where \(k=k'\) and which needs no proof.

**4.4.9 Theorem**

If \(D_1\) is a symmetric BIB design having parameters \((v, b, r, k, \lambda)\) and single initial block \((x_1, x_2, \ldots, x_k)\) and \(D_2\) is another BIB design with parameters \((v'=v, b'=b, r'=r, k'=k, \lambda=\lambda')\) and single initial block \((y_1, y_2, \ldots, y_k)\) then the new initial block:

\[[(x_1+y_1), (x_1+y_2), \ldots, (x_1+y_k); (x_2+y_1), (x_2+y_2), \ldots, (x_2+y_k); \ldots, (x_k+y_1), (x_k+y_2), \ldots, (x_k+y_k)]\]

when developed mod \(v\) yields a balanced n-ary block design with parameters

\[V' = v = B', R' = kk' = K', \quad \Lambda' = k(\lambda + \lambda') + (\lambda \lambda')(v-1)\]

This BNB design with \((V', B', R', K', \Lambda')\) developed by single initial block can be utilized in Theorem 4.3.1 to obtain the characterization of E-Optimal n-ary block design with unequal block sizes as below.

**4.4.10 Corollary**

For any \(d' \in D (R'_p, V'-1; K'_t, B'-1; G')\), \(m = \min_{i \neq j} (K'_t \delta_{ij}), \) then \(mV' / K'_t \leq Z_{d'1} \leq (R'_p V'K'_t - \Delta'_p V') / (V'K'_t - K'_t). \) Further if \(m' = (R'_p K'_t - \Delta'_p)\)
/ (V’-1), then Z_{d’1} = (R’_p V’ K’_t - \Delta’ p V’)/(V’ K’_t - K’_t) and d’ is E-Optimal in D (R’_p, V’-1, K’_t, B’-1; G’).

### 4.4.11 Example

Consider two BIB designs with same parameters (v=11=b, r=5=k, \lambda=2) where v=v’, b=b’, k=k’ and \lambda=\lambda’ with initial block (1,3,4,5,9)

Then by Theorem 4.4.9 the new initial block with 25 elements I=[1, 1, 2, 2, 2, 3, 4, 4, 4, 5, 5, 6, 6, 7, 7, 7, 8, 8, 8, 9, 9, 9, 10, 10, 10] when I is developed module V, yields balanced ternary design with parameters V’ = 11 = B’, R’ =25=K’, \Lambda’ = 56 and incidence matrix N_1.

Then by Corollary 4.4.10, consider the following incidence matrix of a 4-ary block diagram design with parameters V* = 11, B*12, R*_dp = R** = K_t=25, G* = 631:

\[
N_{*,d} = \begin{bmatrix}
0 & 3 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 3 & 2 & 0 \\
2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 3 & 0 \\
3 & 2 & 0 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 0 \\
2 & 3 & 2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 2 & 0 \\
2 & 2 & 3 & 2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 0 \\
2 & 2 & 2 & 3 & 2 & 0 & 3 & 2 & 3 & 3 & 3 & 1 \\
3 & 2 & 2 & 2 & 3 & 2 & 0 & 3 & 2 & 3 & 3 & 1 \\
3 & 3 & 2 & 2 & 2 & 3 & 2 & 0 & 3 & 2 & 3 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 2 & 0 & 3 & 2 & 1 \\
2 & 3 & 3 & 3 & 2 & 2 & 3 & 2 & 0 & 3 & 1 \\
3 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 0 & 1
\end{bmatrix}
\]

This 4-ary block design is E-Optimal over \( \mathcal{D} \) (R* _p, V* -1; K*_t, B*_1; G*) as well as \( \mathcal{D} \) (V*; K*_t; B*_1; G*) for m=55=\( R*_p k*_t - \Delta*_p / (V* - i) \) and R* dp = R**. Also we see \( Z_{d1} = 121/5 = \{V*/(V*-1)\} \times \{(R*_p K*_t - \Delta*_p)/K*_t\} \).
4.5  A-OPTIMAL N-ARY DESIGNS FOR K = V-1

In this section we characterize the A-optimal n-ary designs in certain classes \(\mathcal{D}(V,B,K)\) where \(K = V-1\) and \(V \geq 3\) by using definition 4.2.1, lemma 4.2.2, 4.2.3 and lemma 4.2.4. To begin, suppose we let \(d^* \in \mathcal{D}(V,B,K)\), where \(b_1 = mv, m \geq 1\), be that balanced block n-ary design (BBD) having \(n_{d^*ij} = 0,1,2,\ldots,n-1\) for \(i=1,\ldots,v, j=1\ldots B_1\), and \(\lambda_{d^*ij} = \lambda\) for \(i,j = 1,\ldots, v, i \neq j\). Now let \(d \in \mathcal{D}(V,B,K)\), where \(B = B_1 + B_2, 1 \leq B_2 \leq V-1\), be that design obtained by adding \(B_2\) blocks to \(d^*\) where the \(i\)-th block added has treatment \(i\) replicated \(0\) times and all other treatments replicated once for \(i=1,\ldots,B_2\). In the main result of this section, we show that certain n-ary designs of the type \(d\) just described are in fact A-Optimal in their corresponding classes \(\mathcal{D}(V,B,K)\). The following facts concerning \(KC_d\) are easy to verify:

\[
Kz_{d1} = \ldots = Kz_{d,B_2-1} = a + \alpha = A,
\]

\[
Kz_{d1B_2} = V \gamma = B,
\]

\[
Kz_{d1B_2+1} = \ldots = Kz_{d1v-1} = B + \beta = C,
\]

where

\[
a = K \sum_{i=1}^{B_2} C_{dii} / B_2 = (mV - m + B_2 - 1)(V - 1) - \{mV - m + B_2 - 1\},
\]

\[
\alpha = -K \sum_{i=1}^{B_2} \sum_{j=1}^{B_2} C_{dii} / B_2 (b_2 - 1) = m(V - 2) + B_2 - 2,
\]

\[
\gamma = -K \sum_{j=1}^{B_2} \sum_{i=1}^{B_2} C_{dii} / B_2 (V - B_2) = m(V - 2) + B_2 - 1,
\]

\[
B = K \sum_{i=B_2+1}^{v} C_{dii} / (V - B_2) = (mV - m + B_2)(V - 1) - (mV - m + B_2),
\]

\[
\beta = -K \sum_{i=B_2+1}^{v} \sum_{j=B_2+1}^{v} C_{dii} / (V - B_2)(V - B_2 - 1) = m(V - 2) + B_2,
\]

We also note that

\[V \gamma - (a + \alpha) = B_2.\]
We shall assume for the remainder of this section that if \( d \in D(V, B, K) \) is arbitrary and \( x_{di} = \sum_{i=1}^{B} n_{di}, \ j=1, \ldots, B, \) then the columns of \( N_d \) have been arranged so that \( x_{d1} \geq x_{d2} \geq \ldots \geq x_{dB}. \)

Also with this notation, we note that

\[
\sum_{i=1}^{B_2} \sum_{j=1}^{B} \lambda_{dij} = \sum_{j=1}^{B} x_{di}^2 - \sum_{j=1}^{B} n_{dj}^2, \quad (4.5.2a)
\]

\[
\sum_{i=1}^{B_2} \sum_{j=V_i}^{B} \lambda_{dij} = \sum_{j=1}^{B} x_{di}^2 (K - x_{di}) = n_{dj}k - \sum_{j=1}^{B} x_{dj}^2, \quad (4.5.2b)
\]

Where \( n_{d1} = \sum_{j=1}^{B} x_{dj}, \)

\[
\sum_{i=V_i}^{V} \sum_{j=V_i}^{V} \lambda_{dij} = \sum_{j=1}^{B} (K - x_{di})^2 - \sum_{j=1}^{B} \sum_{i=V_i}^{V} n_{dj}^2 = BK^2 - 2n_{d1}K + \sum_{j=1}^{B} x_{dj}^2 - \sum_{j=1}^{B} \sum_{i=V_i}^{V} n_{dj}^2
\]

\[
(4.5.2e)
\]

For \( d \in D(V, B, K) \) defined previously, we shall for shorthand let \( x_{dj} = x_j \) and \( n_{d1}=n_1 \) and note that

\[
X_j = B_2 \quad \text{for } j = 1, \ldots, m(V-1) + B_2 - 1,
\]

\[
x_j = B_2 - 1 \quad \text{for } j = m(V - 1) + B_2, \ldots, B
\]

Using the notation just introduced and (4.5.2), it is easy to verify that the eigen values for \( d \) given in (4.5.1) can also be expressed as

\[
Z_{d1} = \ldots = Z_{d,B_2-1} = a + \alpha
\]

\[
Z_{d1} = \ldots = Z_{d,B_2-1} = a + \alpha
\]

\[
= B_2n_1K + \sum_{j=1}^{B} x_j^2 - B_2 \sum_{i=1}^{B} \sum_{j=1}^{B} n_{dj}^2 / B_2(B_2 - 1)
\]

\[
Z_{d,B_2-1} = V\gamma = (V / B_2(V - B_2)) \left\{ n_1K - \sum_{j=1}^{B} x_j^2 \right\}
\]

\[
(4.5.3a)
\]

\[
(4.5.3b)
\]
We now give that main result of this section. Since the proof of the following theorem is rather long and algebraic, we defer it to the Appendix.

4.5.1 Theorem

Let \( d \) be a design such as described at the beginning of this section.

(a) If \( B_2 = 1 \), then \( d \) is \( \varphi_f \)– optimal in \( \mathcal{D}(V, B, K) \) with respect to any function \( f \) satisfying Definition 4.1.1

(b) If \( 2 \leq B_2 \leq V \), the \( d \) is A-optimal in \( \mathcal{D}(V, B, K) \).

Proof:

See the Appendix.

4.6 A- OPTIMAL N- ARY DESIGNS FOR \( K = V + 1 \)

In this section we characterize the A-optimal n-ary designs in certain classes \( \mathcal{D}(V, B, K) \) where \( K = V + 1 \). To this end, we shall let \( d^* \in \mathcal{D}(V, B, K) \), where \( B_1 = mV, m \geq 1 \), be that BBD having \( n_{d^*_{ij}} = 1 \) or 2 for \( i=1,\ldots,V, j=1,\ldots,B_1 \) and \( \lambda_{d^*_{ij}} = \lambda \) for \( i,j = 1,\ldots, V, i \neq j \). Now let \( d \in \mathcal{D}(V,B,K) \) where \( B=B_1+B_2, 1 \leq B_2 \leq V-1 \), be that design obtained by adding \( B_2 \) blocks to \( d^* \) where the i-th block added has treatment \( V+1-i \) replicated 2 times and all other treatments replicated once for \( i=1, \ldots,B_2 \). In the main result of this section, we show that certain designs \( d \) of the type just described are A-optimal in their corresponding classes \( \mathcal{D}(V, B, K) \). The following facts holds for \( K\mathcal{C}_d \).

\[
\begin{align*}
Kz_{d_{1}} &= \ldots = Z_{d_{V-B_2-1}} = a + \alpha = A, \\
Kz_{d_{V-B_2}} &= V\gamma = B, \\
Kz_{d_{V-B_2+1}} &= \ldots = Kz_{d_{V-1}} = B + \beta = C,
\end{align*}
\] (4.6.1)

where
a = \(K \sum_{i=1}^{V-B_2} C_{di} / (V - B_2) = m\{(V + 1)_2 - (V + 3)\} + B_2 (V + 1) - B_2\),

\[\alpha = -K \sum_{i=1}^{V-B_2} \sum_{j=1}^{V-B_2} C_{dij} / (V - B_2) (V - B_2 - 1) = m(V + 2) + B_2,\]

\[\beta = -K \sum_{i=V-B_2+1}^{V} \sum_{j=V-B_2+1}^{V} C_{dij} / B_2 (B_2 - 1) = m(V + 2) + B_2 + 2.\]

We also note that
\[V\gamma - (\alpha + \alpha) = V - B_2.\]

We shall assume, as in the previous section, that if \(d \in D \subseteq (V, B, K)\) and \(x_{d_i} = \sum_{i=1}^{V-B_2} n_{dij}, j=1,\ldots,B\), then the columns of \(N_d\) have been arranged so that \(x_{d1} \geq x_{d2} \geq x_{dB}\). With this notation we note that
\[\sum_{i=1}^{V-B_2} \sum_{j=1}^{V-B_2} \lambda_{dij} = \sum_{j=1}^{B} x_{dj}^2 - \sum_{j=1}^{B} \sum_{i=1}^{V-B_2} n_{dij}^2, \quad (4.6.2a)\]
\[\sum_{i=V-B_2+1}^{V} \sum_{j=V-B_2+1}^{V} \lambda_{dij} = \sum_{j=1}^{B} x_{dj}^2 (K - x_{di}) = n_d / K - \sum_{i=1}^{B} x_{dij}, \quad (4.6.2b)\]

Where \(n_{di} = \sum_{j=1}^{B} x_{dj}\),
\[\sum_{i=V-B_2+1}^{V} \sum_{j=V-B_2+1}^{V} \lambda_{dij} = \sum_{j=1}^{B} (K - x_{dj})^2 - \sum_{j=1}^{B} \sum_{i=1}^{V-B_2} n_{dij}^2 - \sum_{j=1}^{B} \sum_{i=1}^{V-B_2+1} n_{dij}^2 = BK^2 - 2n_{dij}K + \sum_{j=1}^{B} x_{dij}^2 - \sum_{j=1}^{B} \sum_{i=1}^{V-B_2+1} n_{dij}^2 \quad (4.6.2c)\]

For \(d \in D \subseteq (V, B, K)\) defined previously, we shall for shorthand let \(x_{d} = x_j\) and \(n_{d1}\) and \(n_{1}\) and note that
\[x_j = B_2 + 1 \quad \text{for } j = 1,\ldots, m(V - B_2)\]
and

\[ x_j = B_2 \quad \text{for} \quad j = m(V - B_2) + 1, B. \]

Using the notation just introduced, it is easy to verify that the eigenvalues for \(d\) given in (4.6.1) can also be expressed as

\[ Z_{d_l} = ... = Z_{d, V - B_2} = a + \alpha \]

\[ = \left( (V - B_2) n_i k + \sum_{j=1}^{B} x_j^2 - (V - B_2) \sum_{i=1}^{B} \sum_{j=1}^{B} n_{ij}^2 \right) / (V - B_2) (V - B_2 - 1) \] (4.6.3a)

\[ Z_{d, V - B_2} = V_{\gamma} = \left( V / B_2 (V - B_2) \right) \left\{ n_i K - \sum_{j=1}^{B} x_j^2 \right\}, \] (4.6.3b)

\[ Z_{d, V - B_2 + 1} = ... Z_{d, V - 1} = B + \beta \]

\[ = \left\{ (B_2 - 1) (BK - n_i^2) K + BK^2 - 2n_i K + \sum_{j=1}^{B} x_j^2 - B_2 \sum_{i=V - B_2 + 1}^{V} \sum_{j=1}^{V} n_{ij}^2 \right\} / (4.6.3c) \]

\[ B_2 (B_2 - 1) (K - \left( V - B_2 - 1 \right) (a + \alpha) - V_{\gamma}) / (B_2 - 1) \]

We now give the main result of this section. Again, the proof of the result stated below is referred to the Appendix.

4.6.1 Theorem

Let \(d\) be a design such as described at the beginning of this section.

a) If \(B_2 = V - 1\), then \(d\) is \(\phi_l\) – optimal in \(D (V, B, K)\) with respect to any function \(f\) satisfying Definition 1.1.

b) If \(\frac{1}{2} V \leq B_2 \leq V - 2\), then \(d\) is A-optimal in \(D (V, B, K)\).

Proof

See the Appendix.

Comment

With regard to the results given in Sections 3 and 4, the author believes that they can be generalized in several directions. In particular, results similar to those given in Theorem 3.1 are probably true for \(\frac{1}{2} V < B_2 \leq V - 1\) and \(K = tv - 1, t \geq 2\) and results analogous to those given in Theorem 4.1 probably
hold for $1 \leq B_2 < \frac{1}{2}V$ and $K = tv + 1$, $t \geq 2$. However, the proofs of these extensions have so far eluded the author.

**Appendix**

In this appendix, we provide proofs for the main results given in sections 4.5 and 4.6

**Lemma A.1**

Let $A$ and $B$ be as defined in (3.1) and suppose $V \leq 3$ and $B_2 \geq 2$. Then $g(x) = (B_2^{-1}) / ((A+2x / (B_2^{-1})) + 1 / (B-2Vx / B_2(V-B_2))$ is an increasing function of $x$, $1 \leq x \leq B_2(V-B_2)B / 2V$.

**Proof**

To begin with note that we can write

$$g(x) = (W_1x+W_2) / (W_3+W_4x - W_5x^2),$$

Where

$$W_1 = 2B_2\{V-B_2-V(B_2^{-1})^2\},$$
$$W_2 = B_2^2 (B_2^{-1})(V-B_2)\{(B_2^{-1})B+A\},$$
$$W_3 = (V-B_2)(B_2^{-1})B^2AB,$$
$$W_4 = 2(V-B_2)B_2B - 2VB_2(B_2^{-1})A,$$
$$W_5 = 4V.$$

Now, upon differentiating,

$$g'(W_3+W_4x-W_5x^2)W_1-(W_1x+W_2)(W_4-2W_5x) / (W_3+W_4x - W_5x^2)^2$$
$$=(W_1W_5x^2+2W_2W_5x+(W_1W_3-W_2W_4)) / (W_3+W_4x - W_5x^2)^2.$$

Thus $g'(x)$ has the same sign as

$$h(x) = W_1W_5x^2+2W_2W_5x+(W_1W_3-W_2W_4).$$

But $W_1W_5 < 0$, thus $h(x) > 0$ for $x \in (x_1, x_2)$ where
\[ X_1 = (-W_2 W_5 - (W_2^2 W_5^2 - W_1 W_5 (W_1 W_3 - W_2 W_4))^{1/2}) / W_1 W_5, \]
\[ X_2 = (-W_2 W_5 + (W_2^2 W_5^2 - W_1 W_5 (W_1 W_3 - W_2 W_4))^{1/2}) / W_1 W_5. \]

Now, it is easy to verify that \( W_1 W_3 - W_2 W_4 > 0 \) since \( B_2 \geq 2 \) and \( V \geq 3 \). Thus \( x_1 \geq 0 \)
and
\[ x_2 \geq -2W_2W_5/W_1W_5 = -2W_2/W_1 \]
\[ = 2B_2^2 (B_2-1)(V-b_2)\{(B_2-1)B+A\} / 2B_2 \{V(B_2-1)^2 - (V-B_2)\} \]
\[ = B_2(B_2-1)(V-B_2)\{(B_2-1)B+A\} / \{V(B_2-1)^2 - (V-B_2)\} \]

But
\[ B_2(B_2-1)(V-B_2)\{(B_2-1)B+A\} / \{V(B_2-1)^2 - (V-B_2)\} \]
\[ \geq B_2(B_2-1)(V-B_2)\{(B_2-1)B-A\}/V(B_2-1)^2 \]
\[ \geq (V-B_2)\{(B_2-1)B+A\} / V \geq B_2(V-B_2)B/2V, \]

where the last inequality follows because \( V \geq 3 \).

Proof of Theorem 4.5.1 (a) To begin, we note that when \( B_2=1 \), \( kC_d \) has eigenvalues
\[ K_{d_1} = V \gamma = Vm(V-2) \]
and
\[ K_{d_2} = \ldots = K_{d_{v-1}} = B + \beta = (mV - m+B_2)+m(V-2)+B_2. \]

So let \( d \in \mathcal{D} (V,B,K) \) be arbitrary. Then \( r_{d1} \leq r = (B-1) K / V \) and by the results of Jacroux (1985),
\[ Z_{d1} \leq \left( V / (V-1) \right) C_{d1} \leq \left( V / (V-1) \right) (r_{d1}K - \lambda_{d_2}) / K \]
\[ \leq \left( V / (V-1) \right) r (K-1) / K = Z_{d1}. \]

Furthermore, \( tr C_d tr C_d \) since \( d \) has \( n_{dij} = 0 \) or \( 1 \) for \( i=1, \ldots, V, j=1, \ldots, B \). Hence we see that
\[ KZ_{d_1} \leq KZ_{d_1} \quad \text{and} \quad K \sum_{i=1}^{V-1} z_{d_1} \leq K \sum_{i=1}^{V-1} Z_{d_1} = V \gamma + (V-2)(B+\beta) \]
Using these facts, it follows that the eigenvalues of $C_d$ weakly majorize the eigenvalues of $C_d$ and that

$$\phi_t(C_d) = \sum_{i=1}^{V-1} f(Z_{di}) \geq \phi_t(C_d) = \sum_{i=1}^{V-1} f(Z_{di})$$

(b) We note that the eigenvalues of $kC_d$ are as given in (3.1). So now let $d \in D (V,B,K)$ be arbitrary and consider the averaged version $C_d$ of $C_d$ obtained by averaging separately over the sets of treatments $\{1,...,B_2\}$ and $\{B_2+1,...,V\}$. Then, using (3.2) and Lemma 2.4, $kC_d$ has eigenvalues

$$a_d + \alpha_d = \left( B_1 n_d K + \sum_{j=1}^{B_1} x^2 d_j - B_2 \sum_{i=1}^{B_1} \sum_{j=1}^{B_2} n^2 d_{ij} \right) / B_2 (B_2 - 1)$$

Occurring with multiplicity $B_2 - 1$,

$$V_\gamma d = \left(V / B_2 (V - B_2)\right) \left( n_d K - \sum_{i=1}^{B_2} x^2 d_i \right)$$

Occurring with multiplicity 1,

$$B_d + \beta_d = (V - B_2 - 1)(K - n_{dl}) K + B K^2 - 2 n_{dl} K$$

$$+ \sum_{j=1}^{B_1} x^2 d_j - (V - B_2) \sum_{i=1}^{V} \sum_{j=1}^{B_2} n^2 d_{ij} / (V - B_2)$$

$$= (Ktr C_d - (B_2 - 1)(a_d + \alpha_d (- V_\gamma d)) / (V - B_2 - 1))$$

Occurring with multiplicity $V - B_2 - 1$,

We now consider separately several special cases.

**Case 1**

$n_{dl} = n_1$, $tr C_d = tr C_d$

For this case, it is easy to verify that $d$ must have

$$X_{dij} = X_j, \quad j=1,...,B,$$
Thus \( a_d + \alpha_d = a + \alpha, V_\gamma d = V_\gamma \) and \( B_d + \beta_d = B + \beta \) and we have from Lemma 4.2.2 and 4.2.3 that

\[
\frac{1}{K} \sum_{i=1}^{V-1} (1/Z_{di}) \geq \left( B_2 - 1 \right) (a_d + \alpha_d) + 1/V_\gamma d + \left( V - B_2 - 1 \right)/(B_d + \beta_d)
\]

\[
= \left( B_2 - 1 \right) (a + \alpha) + 1/V_\gamma + \left( V - B_2 - 1 \right)/(B + \beta) = \frac{1}{K} \sum_{i=1}^{V-1} (1/Z_{di})
\]

**Case 2**

\( n_{di} = n_1, \text{tr} C_d - 2w/k \) for some \( 1 \leq w \leq B_2 - 1 \).

Since \( n_{di} = n_1 \), the following facts are easily verified:

\[
\sum_{i=1}^{B} \sum_{j=1}^{B} n_{di} \geq \sum_{i=1}^{B} \sum_{j=1}^{B} n_{dij}, \quad (A2a)
\]

\[
\sum_{i=1}^{B} \sum_{j=1}^{B} n_{dij} \geq \sum_{i=1}^{B} \sum_{j=1}^{B} n_{dij}, \quad (A2b)
\]

\[
\sum_{i=1}^{B} x_{di}^2 = \sum_{i=1}^{B} x_{dij}^2 + 2t \quad \text{for some } t, 0 \leq t \leq 1/2w(w+1) \quad (A2c)
\]

From (4.5.3), (A1) and (A2), we see that

\[
a_d + \alpha_d \leq a + \alpha + 2t/B_2 (B_2 - 1),
\]

\[
V_\gamma d = V_\gamma - 2Vt/B_2 \left( V - B_2 \right),
\]

\[
B_d + \beta_d = \left\{ \text{Ktrace} C_d - \left( B_2 - 1 \right) (a_d + \alpha_d) - V_\gamma d \right\} / \left( V - B_2 - 1 \right)
\]

\[
\geq \left\{ \text{Ktrace} C_d - 2W - \left( B_2 - 1 \right) (a + \alpha) - 2t/B_2 - V_\gamma + 2Vt/B_2 \left( V - B_2 \right) \right\} / \left( V - B_2 - 1 \right)
\]

\[
\geq \left\{ \text{Ktrace} C_d - \left( B_2 - 1 \right) (a + \alpha) - V_\gamma - 2w + 2t/(V - B_2) \right\} / \left( V - B_2 - 1 \right)
\]

Hence by Lemmas 4.2.2 and 4.2.3.
\[
(1/K) \sum_{i=1}^{V-1} (1 - z_{a_i}) \geq (B_2 - 1)/(a_d + \alpha_d) + 1/V\gamma_d + (V - B_2 - 1)/(B_d + \beta_d) \\
\geq (B_2 - 1)/(a + \alpha_d + 2t / B_2)(B_2 - 1) + 1/(V\gamma - 2Vt / B_2(V - B_2)) \\
+ (V - B_2 - 1)/[\text{Ktr C}_d - (B_2 - 1)(a + \alpha) - V\gamma - 2W - 2t / (V - B_2)]/ \\
(V - B_2 - 1) = (B_2 - 1)/m_1(t) + 1/m_2(t) + (V - B_2 - 1)/m_3(W, t)
\]

Where
\[
m_1(t) = a + \alpha + 2t / B_2(B_2 - 1), \quad m_2(t) = V\gamma - 2Vt / B_2(V - B_2)
\]

and
\[
m_3(W, t) = \{\text{Ktr C}_d - (B_2 - 1)(a + \alpha) - V\gamma - 2W - 2t / (V - B_2)]/(V - B_2 - 1)
\]

In the above expressions, we note that \(m_1(t) \leq m_2(t)\) for \(t \leq B_2(B_2 - 1)\)

\((V - B_2)/4(V - 1)\).

Now let
\[
M(t) = \begin{cases} 
(B_2 - 1)/m_1(t) + 1/m_2(t) & \text{if } t \leq B_2(B_2 - 1)(V - B_2)/4(V - 1) \\
(B_2^2 / [(B_2 - 1)m_2(t) + m_2(t)]) & \text{if } t > B_2(B_2 - 1)(V - B_2)/4(V - 1)
\end{cases}
\]

and
\[
N(w, t) = \begin{cases} 
(V - B_2 - 1)^2 / (K\text{tr C}_d - (B_2 - 1)(a + \alpha) - V\gamma) & \text{if } t \leq w(V - B_2) \\
(V - B_2 - 1)/m_3(w, t) & \text{if } t \geq w(V - B_2)
\end{cases}
\]

Now, by Lemma A.1 and 2.2, we see that \(M(t)\) is an increasing function of \(t\),

\(M(t) \leq (B_2 - 1)/m_1(t) + 1/m_2(t)\),

and

\(N(w, t) \leq (V - B_2 - 1)/m_3(w, t)\)

Also, we clearly have that \(N(w, t)\) is a decreasing function of \(t\) and an increasing function of \(w\). Hence, if

\[
\sum_{j=1}^{B} x^2d_j = \sum_{j=1}^{B} x^2j + 2t, \quad 0 \leq t \leq 1/2w(w + 1)
\]

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It follows that
\[
\frac{1}{K} \sum_{i=1}^{V-1} \left( \frac{1}{z_{di}} \right) \geq (B_2 - 1)/m_1(t) + 1/m_2(t) + (V - B_2 - 1)/m_3(w, t)
\]
\[
\geq M(t) + N(w, t) \geq M(t) + N(w, 1/2w(w + 1))
\]
\[
\geq M(t) + (V - B_2 - 1)^2/(K \text{tr } C_d - (B_2 - 1)(a + \alpha) - V\gamma)
\]
(Since \( w \leq B_2 - 1 \leq V - B_2 - 1 \leq 2V - 2B_2 - 1 \))
\[
\geq M(0) + (V - B_2 - 1)^2/(K \text{tr } C_d - (B_2 - 1)(a + \alpha) - V\gamma)
\]
\[
= (B_2 - 1)/(a + \alpha) + 1/V\gamma + (V - B_2 - 1)^2/(K \text{tr } C_d - (B_2 - 1)(a + \alpha) - V\gamma)
\]
\[
= (1/K) \sum_{i=1}^{V-1} \left( \frac{1}{z_{di}} \right)
\]

Case 3

\( n_{di} = n_1, \quad \text{tr } C_d = \text{tr } C_d - 2w/K, \quad w > B_2 - 1. \)

For this case, since \( n_{di} = n_1 \) the conditions given in (A2) and (A3) still hold. So let \( m_1, m_2, m_3, M \) and \( N \) be as defined in case 2 above. If
\[
\sum_{j=1}^{B} x_j^2 = \sum_{j=1}^{B} x_j^2 + 2t, \quad 0 \leq t \leq \frac{1}{2}B_2(B_2 - 1)
\]

Then the arguments given in case 2 above hold since \( w > B_2 - 1 \).

If \( t > \frac{1}{2}B_2(B_2 - 1) \), then we see that \( M(t) = B_2^2/(B_2 - 1)m_1(t) + m_2(t) \)
\[
(1/K) \sum_{i=1}^{V-1} \left( \frac{1}{z_{di}} \right) \geq (B_2 - 1)/m_1(t) + 1/m_2(t) + (V - B_2 - 1)/m_3(w, t)
\]
\[
\geq M(t) + N(w, t)
\]

If \( \frac{1}{2}B_2(B_2 - 1) < t \leq w(V - B_2) \), then we clearly have

\[
M(t) + N(w, t) \geq M(0) + N(w, t) = (1/K) \sum_{i=1}^{V-1} \left( \frac{1}{z_{di}} \right)
\]
On the other hand, if \( t > \max \left\{ \frac{1}{2}B_2(B_2 - 1), 2(V - B_2) \right\} \), then

\[
m_3(w, w(V - B_2)) \quad \text{and since} \quad (B_2 - 1)m_1(t) + m_2(t) + (V - B_2 - 1)m_3(w, t) = (B_2 - 1)m_1(w(V - B_2) + m_2(V - B_2)) + (V - B_2 - 1)m_3(w, w(V - B_2))
\]

It follows from Lemma 2.2. that

\[
M(t) + N(w, t) \geq M(w(V - B_2)) + N(w, w(V - B_2)) = M(w(V - B_2)) + (V - B_2 - 1)^2/(K tr C_d - (B_2 - 1)(a + \alpha) - V_\gamma)
\]

\[
\geq M(0) + (V - B_2 - 1)^2/(K tr C_d - (B_2 - 1)(a + \alpha) - V_\gamma)
\]

\[
= (B_2 - 1)/(a + \alpha) + 1/V_\gamma + (V - B_2 - 1)^2/(k tr C_d - (B_2 - 1)(a + \alpha) - V_\gamma)
\]

\[
= \left( \frac{1}{k} \sum_{i=1}^{V-1} \left( \frac{1}{z_{di}} \right) \right)
\]

**Case 4**

\( n_{d_1} < n_1 \)

In this case, consider \( a_d + \alpha_d, V_\gamma_d \) and \( B_d + \beta_d \) as defined previously.

Since \( n_{d_1} < n_1 \), it follows that there must exist some \( f, 1 \leq f \leq B \), such that \( x_{df} < x_f \). For the value \( f \) that satisfies this condition, there must also exist \( n_{df}, 1 \leq S \leq B_2 \) such that \( n_{df} = 0 \) and \( n_{df}, B_2 + 1 \leq t \leq V \), such that

\[
\begin{align*}
  n_{df} & \geq \begin{cases} 
    2 & \text{if } x_f = B_2 - 1, \\
    1 & \text{if } x_f = B_2,
  \end{cases} \\
\end{align*}
\]

From the design \( d \), create the new design \( d \) having \( n_{df} = n_{df} + 1 \), \( n_{df} = n_{df} - 1 \) and \( n_{dij} = n_{dij} \) for all other \( i, j \). Then the following conditions are easily verified:

\[
\begin{align*}
  n_{d_1} &= n_{d_1} + 1, \\
  a_d + \alpha_d &\leq a_d + \alpha_d, \\
  V_\gamma_d &\leq V_\gamma_d \quad \text{(since } B_2 \leq \frac{1}{2}V), \\
  B_d + \beta_d &\geq B_d + \beta_d.
\end{align*}
\]
Using these latter conditions, it follows from Lemma 2.2 that
\[
\begin{align*}
(B_2 - 1)(a_d + \alpha_d) + 1 / V \gamma_d + (V-B_2-1) / (B_d + \beta_d) \\
\leq (B_2 - 1) / (a_d + \alpha_d) + 1 / V \gamma_d + (V-B_2-1) / (B_d + \beta_d)
\end{align*}
\] (A3)

Continuing in the same manner, we eventually arrive at a design \(d\) having \(n_d = n_1\) and such that the inequality given in (A3) is satisfied. But by our previous arguments concerning designs having \(n_d = n_1\), it follows that
\[
\begin{align*}
(B_2 - 1)/(a_d + \alpha_d)+1/V \gamma_d+(V-B_2-1)/(B_d+\beta_d) \\
\geq (B_2 - 1)/(a+a)+1/V \gamma+(V-B_2-1)/(B+\beta)
\end{align*}
\]
and we have the desired result.

**Lemma A.2**

Let \(A\) and \(B\) be as defined earlier and suppose \(B_2 \leq V-2\). Then
\[
g(x) = (V-B_2-1) / (A+2x / (V-B_2)(V-B_2-1)) + 1 / (B-2Vx / B_2(V-B_2))
\]
is an increasing function of \(x\), \(1 \leq x \leq B_2(V-B_2)B / 2V\).

**Proof**

The proof is similar to that of Lemma A.1 and is omitted.

Proof of Theorem 4.6.1 (a) To begin with, we note that when \(B_2 = V-1\), \(KC_d\) has eigenvalues
\[
KZ_{d1} = V \gamma = V \{m(V+2)+V\}
\]
and
\[
KZ_{d2} = \ldots = KZ_{d,V-1} = B+\beta = m(V+1)^2+V^2-3V+2m-3.
\]
So let \( D(V, B, K) \) be arbitrary. Then \( r_{d1} = (B_1K / V) + B_2 \) and by the results of Jacroux(1985),

\[
z_{d1} \leq \frac{V}{V-1} C_{d11} \leq \frac{V}{V-1} (r_{d1} K - \lambda d_{11}) / K \\
\leq \frac{V}{V-1} (rK - \lambda d_{11}) / K = z_{d1}.
\]

Furthermore, \( tr C_d \leq tr Cd \) since \( d \) has \( n_{dij} = 1 \) or 2 for \( i=1,\ldots, V, \)
\( j=1,\ldots, B, \) Hence we see that

\[
Kz_{d1} \leq Kz_{d1} \quad \text{and} \quad K \sum_{i=1}^{V-1} z_{di} \leq \sum_{i=1}^{V-1} z_{di} = V\gamma + (V-2)(B+\beta)
\]

Using these facts, it follows that the eigenvalues of \( C_d \) weakly memorize
the eigenvalues of \( Cd \) and that

\[
\phi_r(C_d) = \sum_{i=1}^{V-1} f(z_{di}) \geq \phi_r(C_d) = \sum_{i=1}^{V-1} f(z_{di})
\]

(b) Using Lemma A.2, the proof of this part of Theorem 4.6.1 is essentially the same as the proof of Theorem 4.5.1(b) with \( V-B_2 \) replacing \( B_2 \) everywhere.