CHAPTER – VII
A-OPTIMAL n-ARY DESIGNS FOR 1-, 2-, 3- WAY ELIMINATION OF HETEROGENEITY

This Chapter outlines the existing knowledge on binary optimal designs to n-ary designs for comparing test treatments with controls under 0-, 1- and 2-way elimination of heterogeneity models.

7.1 HISTORICAL REVIEW

Earlier Cox (1958) advocated augmenting a BIB design in test treatments with one or more replications of controls in each block as a means of getting good designs. He neither formally mathematized the problem nor gave any justification for his suggestion. However, based on what has been developed during the past several years, we know that this is an excellent method of getting efficient designs in many cases. Fieller (1947) gave a solution for A-optimal designs for the zero-way elimination of heterogeneity model which is applicable when $v$ is a square. Pearce (1960) proposed a class of designs for comparing test treatments with a control and gave their analysis for the one-way elimination of heterogeneity model; Freeman (1975) studied some designs for comparing two sets of treatments for the two-way elimination of heterogeneity model. Pesek (1974) compared a BIB design with an augmented BIB design, as suggested by Cox (1958) in estimating control-test treatments contrasts and noticed that the latter was more efficient. Das (1958) has also looked at augmented BIB designs.

Bechofer and Tamhane (1981) were the first to study the problem of obtaining optimal block designs. However their optimality consideration was neither A-nor MV-optimality, but for the problem of obtaining optimal simultaneous confidence intervals under a one-way elimination of heterogeneity.
heterogeneity model. Their discoveries led to the concept of BTIB designs; Notz and Tamhane (1983) studied their construction. Constantine (1983) showed that a BIB design in test treatments augmented by a replication of the control in each block is A-optimal in the class of designs with exactly one replication of the control in each block. Jacroux (1984) showed that Constantine’s conclusion remains valid even when the BIB designs are replaced by some group, divisible designs.

optimality and construction of designs in this area have been investigated by Pigeon (1984), Pigeon and Raghavarao (1987) and Majumdar (1988). Giovagnoli and Wynn (1985a) studied A-optimality of designs for one-way elimination of heterogeneity models set in the context of approximate theory i.e., with an infinite number of observations. Christof (1987) made some further investigations along these lines. Suprrier and Edwards (1986) did a similar study for optimal designs for finding simultaneous confidence intervals. Later Bayes optimal designs have been studied in the context of approximate theory. Owen (1970) studied Bayes A-optimal designs, Giovagnoli and Verdinelli (1983) studied Bayes $\phi_p$ criteria, including D- and E- optimality. Verdinelli (1983) gave methods for computing Bayes D- and A-optimal block designs, and Giovagnoli and Verdinelli (1985) investigated the Bayesian approach under a hierarchical linear model. This area of research continues to grow in several directions. Among some recent technical reports are: Toman and Notz (1987) on Bayes A-optimal designs for two-way elimination of heterogeneity models. Ting and Notz (1987a) on optimal designs for two-way elimination of heterogeneity models; Ting and Notz (1987b, 1988) and Jacroux and Majumdar (1987) on optimal designs for one-way elimination of heterogeneity models regarding n-ary block designs and their concept sufficient historical outlook and review were worked out and presented in a neat form in the previous chapter I and II.

The labelling of the treatments, experimental units under a zero-way elimination of heterogeneity model, blocks under a one-way elimination of heterogeneity model and rows and columns under a two-way elimination of heterogeneity model can be randomized.

7.2 PRELIMINARY OUTLOOK

In this section we consider the problem of comparing a set of test treatments with a controls or standard treatment. Such a problem arises, for example, in screening experiments or in the beginning of a long term
experimental investigation where it is initially desired to determine the relative performance of the new test treatments with respect to the control or controls or standard treatment. For specificity, suppose four new methods for performing a certain task become available and we wish to conduct an experiment to compare the new methods to the standard procedure currently being used to perform the given task. Any particular allocation of treatments to experimental units is called a design and is denoted by \( d \). The question of how to compare the test treatments with the control cannot be answered unless it is asked in a more precise manner. To begin with we need to postulate a model for the response observed upon application of a treatment, test treatment or control, to an experimental unit. In this section we shall consider three possible models; 0 way elimination of heterogeneity model in which all experimental units are homogeneous before application of treatments:

\[
y_{ij} = \mu + t_i + \varepsilon_{ij}; \quad (7.2.1)
\]

1-way elimination of heterogeneity model in which experimental units can be divided into several homogeneous blocks:

\[
y_{ij} = \mu + \tau_j + \beta_i + \varepsilon_{ij}; \quad (7.2.2)
\]

2-way elimination of heterogeneity model in which the experimental units can be conceptually arranged according to rows and columns:

\[
y_{ij} = \mu + \tau_j + \beta_i + \varepsilon_{ij}; \quad (7.2.3)
\]

In models (7.2.1), (7.2.2) and (7.2.3) the \( y \)'s denote observations obtained after applying treatment \( i \) once or more than once in times to an experimental unit occurring block \( j \) or column \( j \) and row 1, \( t_i \) represents the effect of treatment \( i \), \( \beta_j \) the effect of block or column \( j \), \( \rho_k \) the effect of row 1, and the \( \varepsilon \)'s are independent random error terms having expectation zero and constant variance \( \sigma^2 \).
We can be more precise about what we mean by comparing test treatments with a control. In particular, because our primary goal is to determine which among the test treatments might be better than the control, we would like to estimate the magnitude of each \( t_i - t_0 \) with as much precision as possible. More precise comparisons among test treatments found to perform better than the control at this initial stage is generally left to later experimentation. Under the assumptions made above, the method of least squares yields the best linear unbiased estimators \( \hat{\xi}_{di} - \xi_{d0} \) for the contrasts \( t_i - t_0 \) under a given design \( d \). In assigning treatments to experimental units, we have to make sure that the contrasts \( t_i - t_0 \) are estimable. A design satisfying this latter condition is said to be treatment connected and we shall restrict our attention to such designs. Clearly there are a number of designs available for the situation being considered here and we want to choose one which is best in some sense. For example, we might choose a design that given the minimal value among all available designs of

\[
\sum_{i=1}^{d} \text{var}(\hat{t}_{di} - \hat{t}_{d0}) \tag{7.2.4}
\]

or

\[
\max_{1 \leq i \leq 4} \text{var}(\hat{t}_{di} - \hat{t}_{d0}) \tag{7.2.5}
\]

where \( \text{var}(\{\hat{\xi}_{di} - \xi_{d0}\}) \) denotes the variance of \( \hat{\xi}_{di} - \xi_{d0} \). A design which gives the minimum in (7.2.4) is called an A-optimal design and one which gives the minimum in (7.2.5) is called an MV-optimal design. Further we give n-ary designs which are A- and MV-optimal under each of the three models.

Take each column of the following array as a block:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 1 & 1 & 2 & 2 & 3 & \\
2 & 3 & 4 & 2 & 4 & 4 & \\
\end{array}
\]
A- and MV-optimal design under model (7.2.3), where there are three rows and six columns:

Assign the treatments according to the following array:

\[
\begin{array}{cccccc}
3 & 1 & 0 & 0 & 2 & 4 \\
1 & 0 & 3 & 4 & 2 & 0 \\
0 & 3 & 4 & 2 & 0 & 1 \\
4 & 2 & 0 & 0 & 1 & 3 \\
\end{array}
\]

We note that even in a small experiment such as the example given above, the determination of an optimal design is usually not easy. During the past several years there has been a concentrated effort to identify and construct optimal designs for the general problem of comparing V test treatments with a control. The A- and MV-optimality criteria defined in (7.2.4) and (7.2.5) have been the most widely studied criteria with regard to the construction of such designs. Finding an A-optimal n-ary designs corresponds to minimizing mean square error in inference and finding an MV-optimal n-ary design is analogous to finding a minimax procedure. We point out that D-optimality is not a natural criterion for this type of problem; Even though other optimality criteria have also been considered for comparing test treatments with a control, it is our conclusion that the published literature on these other criteria has not reached a level of generality for summarization. In this section we shall attempt to summarize the known results on (A- and in the subsequent chapter MV-optimal n-ary designs which we hope will be useful to both the theoretician and the practitioner.

In this chapter we give general results for A-and in the subsequent chapter MV-optimal n-ary designs for comparing V test treatments with a control in each of the three models (7.2.1), (7.2.2) and (7.2.3). Under n-ary setup we give model robust A-optimal n-ary designs, we further suggest various approaches for finding efficient designs in those cases where A-optimal n-ary designs are unknown. We also give A-optimal n-ary designs for
comparing test treatments with two or more controls. We also outline Bayes A-optimal n-ary designs. Thus we give an overview of the literature of optimal n-ary designs for comparing test treatments with controls.

### 7.3 A-OPTIMAL N-ARY DESIGNS

Now we shall give A-optimal n-ary designs for comparing V number of test treatments with a control separately for the zero-way, one-way and two-way elimination of heterogeneity. Throughout this section the control will be denoted by the symbol 0 and the test treatments by 1, 2... V (treatments).

#### 7.3.1 A-optimal n-ary designs for the Zero-way Elimination of Heterogeneity

Our statistical n-ary set-up consists of N experimental units, and our model of response under n-ary design d is

\[ y_{d_{ij}} = \mu + t_i + e_{ij} \quad (7.3.1) \]

where \( j = 1...R_{di}, i = 0, 1...V \). Here and throughout the sequel \( R_{di} \) is the number of experimental units receiving treatment \( i \) under a particular design \( d \). We assume the model to be homoscedastic. The symbols in equation (7.3.1) have the same meaning as described earlier. The A-optimal n-ary design minimizes

\[ \sum_{i=0}^{V} \left( \frac{1}{R_{do}} + \frac{1}{R_{di}} \right) \quad (7.3.2) \]

subject to the restriction that \( (R_{do} + R_{d1} + ... + R_{dv} = n) \). It is easily seen that for a fixed value of \( R_{do} \) (7.3.2) is minimized when \( R_{di} = p (R_{do}) \) or \( p (R_{do}) +1 \) for \( i=1...V \) where \( p (R_{do}) = [(N-R_{do})/V] \) and (x) denotes the integral part of the decimal expansion for \( \ x > 0 \). Thus the problem of finding an A-optimal n-ary design in this case reduces to that of finding the value of \( R_{do} \) that minimizes
The minimization of (7.3.3) can easily be done using a calculator. In the case $V$ is a square and $n = m (V + \sqrt{V})$ for an integer $m$, the A-optimal n-ary design $d^*$

$$R_{d^*1} = \ldots = V_{d^*v} = m, R_{d^*0} = m V^2$$

### 7.3.2 A-optimal n-ary Designs for One-way Elimination of Heterogeneity

Our n-ary setup consists of $B$ blocks on size $K$ each $K \leq V$. The model of response under a design $d$ is

$$y_{dtfp} = \mu + \ell_d + \beta_j + \varepsilon_{tfp},$$

where $i = 0, 1, \ldots, V$; $j = 1, \ldots, B$ and $p = 0, 1, \ldots, n_{dij}$. Here $n_{dij}$ is the number of times treatment $i$ is used in block $j$ and the matrix $N_d = (n_{dij})$ is called the incidence matrix of the design. We note that

$$R_{di} = \sum_{j=1}^{n} n_{dij}$$

We shall also let

$$C_d = \text{diag}(R_{d0}, R_{d1}, \ldots, R_{dv}) - K^{-1} N_d' N_d'$$

and

$$P = \begin{pmatrix}
1 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & -1
\end{pmatrix}$$

$P$ being a $V \times (V +1)$ matrix. The matrix $C_d$ is called the information matrix of $d$ and $P C_d^{-1} P'$ is the covariance matrix of the vector of least squares estimators of the contrasts $t_1-t_0, \ldots, t_v-t_0$; here $C_d^{-1}$ is a generalized inverse of
We note that $PC_dP'$ is proportional to the inverse of the matrix obtained by eliminating the first row and first column of $Cd$ (Refer Bechhofer and Tamhane, 1981; Constantine, 1983 Soundarapandiyan, 1980-81, 1995). Then an A-optimal n-ary design minimizes trace

$$PC_dP'$$  \hspace{1cm} (7.3.4)

over all possible n-ary designs with parameters V, B and K.

Experience has shown that this minimization is usually not easy. As in other cases of exact design theory, it is highly unlikely that we can obtain one method which is capable of producing A-optimal designs and now n-ary for arbitrary values of V, B and K. Recently several examples of A-optimal n-ary designs have been discovered.

If there is no control and if we are interested in comparing V test treatments among themselves, then a balanced n-ary block (BNB) design (definition follows) in the V test treatments would be A-optimal (Kiefer, 1958; Kshirsagar, 1958; Roy, 1958; Kiefer, 1975 and Soundarapandiyan 1980-81, 1995). Here the A criterion is defined by expression (7.3.4) with $P$ denoting any $(V-l) \times V$ matrix having normalized rows orthogonal to each other and to the vector $(I \ldots I)'$, and $C_d$ denoting the $V \times V$ information matrix of the test treatments. In fact, it has been proved by Kiefer (1975) and Soundarapandiyan (1995) that BNB designs are "universally optimal" in the sense that they are optimal under a large family of criteria, which includes the A- and MV-optimality criteria.

7.3.1 Definition

A BNB design with parameters $V$, $B$, $R$, $K$, $\Lambda$ is a block n-ary design with $B$ blocks each containing $K<V$ distinct treatments such that each treatment is replicated $R$ times and each pair of treatments appears in $\Lambda$ blocks.
Unlike the case of BNB designs, the structure of optimal n-ary designs for treatment-control comparisons seems to depend heavily on the criterion used. Although A- and MV-optimal n-ary designs are often the same other criteria different n-ary designs, usually requiring either fewer or more replications of the control, but otherwise balanced with respect to test treatments. For example, the D-optimality criterion selects a block n-ary design from within a given class of n-ary designs which minimizes the determinant of the matrix $\text{PC}_d'O'$ defined in (7.3.4) and it can be shown that a BNB design is always D-optimal when such n-ary design exists. But for the problem of comparing test treatments with a control, the (D-optimality criterion does not seem to be either an intuitively or statistically suitable criterion because the n-ary designs it selects as being optimal generally do not provide any more information) about treatment-control comparisons that they do about comparisons among the test treatments. On the other hand, A-, 1995 and MV-optimality criteria each have a natural and statistically meaningful interpretation as given above.

Unfortunately, with the presence of a control and for the set of contrasts of interest a BNB design is almost never an A- or MV-optimal n-ary design. However, we can sometimes utilize BNB designs in the test treatments to construct an A- or MV-optimal n-ary design for our problem. We shall shortly give some sufficient conditions that can be used to establish the A-and MV-optimality of some example of such n-ary designs. For convenience, we introduce the notation $\text{ABNB (V,B,K- t;t)}$ to denote a BNB design in the V test treatments in B blocks of size K- t each augmented by t replications of the control in each block. The following three examples of ABNB designs are A-optimal for comparing a set of V test treatments to a control.
7.3.1 Example

An ABNB \((V, B, K-1;1)\) is A-optimal whenever \((K-2)^2 + 1 \leq V \leq (K-1)^2\). An example of an A-optimal n-ary design when \(V = 7\), \(B = 7\), \(K = 7\) is given below, where the columns are the blocks:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\
4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
4 & 5 & 6 & 7 & 1 & 2 & 3 & 4 \\
\end{array}
\]

For each \((V, K)\) satisfying \((K-2)^2 + 1 \leq V \leq (K-1)^2\), there are an infinite number of values of \(B\) for which A-optimal ABNB \((V, B, K-1; 1)\) designs exist. This can be seen as follows. For each \((V, K)\), form all \(B = \binom{V}{K-1}\) subsets of size \(K-1\) out of the \(V\) test treatments. Augment each subset with a copy of the control. Then these \(B\) augmented subsets form an ABNB \((V,B,K-1;1)\) design. Further, we note that the \(n\)-ary design which consists of the \(B_1\) blocks of an ABNB \((V,B,K-1;1)\) design and the \(B\) blocks of an ABNB \((V, B_2, K-1; 1)\) design is an ABNB \((V,B,K-1;1)\) design with \(B = B_1 + B_2\). For more details concerning example (7.3.3) of \(n\)-ary designs, the reader is referred to Hedayat and Majumdar (1985) and Soundarapandiyan (1980-81, 1995).

7.3.3 Example

An ABNB\((V,B,K-t;t)\) is A-optimal whenever \((k-t-1)^2 + 1 \leq t^2 \leq (k-t)^2\)

An A-optimal \(n\)-ary design when \(V = 8\), \(D = 28\), \(K = 14\) is given below:

\[
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Sometimes we can use two BNB designs to construct an A-optimal n-ary designs for our problem. We give below one such example.

**7.3.3 Example**

For \( V = \alpha^2 - 1, B = \gamma(\alpha + 2)(\alpha^2 - 1) \) and \( K = \alpha \), the union of an ABNB \((V, \gamma (\alpha + 1)(\alpha^2 - 2), \alpha - 1 ; 1)\) and a BNB design in all the \( V+1 \) treatments, test treatments and control, in \( \gamma \alpha (\alpha + 1) \) blocks of size \( K \) each is A-optimal whenever \( \alpha \) is a prime power, and \( \gamma \) is any integer.

A numerical example when \( \alpha = 3, \quad \gamma = 1 \quad V=8, \quad B = 40, \quad K = 5 \) is:

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 7 1 2 3 1 2 3 1 2 3
1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 4 4 5 5 5 6 6 7 2 5 8 4 5 6 5 6 4 6 4 5
2 3 4 5 6 7 8 3 4 5 6 7 8 4 5 6 7 8 5 6 7 8 6 7 8 7 8 8 3 6 0 7 8 0 0 7 8 8 0 7
1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 4 4 5 5 5 6 6 7 2 5 8 4 5 6 5 6 4 6 4 5
2 3 4 5 6 7 8 3 4 5 6 7 8 4 5 6 7 8 5 6 7 8 6 7 8 7 8 8 3 6 0 7 8 0 0 7 8 8 0 7
```

To establish the optimality of these examples starting point is a result due to Majumdar and Notz (1983) and later works of Soundarapandian (1995) which we shall shortly state. Anyhow we need some more definitions.
### 7.3.2 Definition

d is a balanced treatment incomplete n-ary block (BTNB) design if

\[
\land_{d_{01}} = \ldots = \land_{d_{0V}},
\]

\[
\land_{d_{12}} = \ldots = \land_{d_{V-1,V}},
\]

where \(\land_{d_{ij}} = \sum_{p=1}^{B} n_{d_{jp}} n_{d_{jp}}\). This definition is due to Bechhofer and Tamhand (1981) and Soundarapandiyan 1995.

### 7.3.3 Definition

for integers \(t \in \{0, 1 \ldots K-1\}\) and \(s \in \{0, 1 \ldots B-1\}\), d is a BTNB \((V, B, K; t, s)\) if it is a BTNB design with the additional property that

\(n_{d_{ij}} \in \{0, 1\}, \quad i = 1 \ldots V, \quad j = 1 \ldots B,\)

\(n_{d_{01}} = \ldots = n_{d_{0s}} = t + 1,\)

\(n_{d_{0s+1}} = \ldots = n_{d_{0B}} = t.\)

A BTNB \((V, B, K; t, s)\) is called a rectangular (R-) type design when \(s = 0\), and a step S-type design when \(s > 0\). The layout of these designs can be pictured as follows, with columns as blocks, in each of the two cases R-type and S-type.

(i) R-type.

(ii) S-type.

\[
\begin{array}{c}
  1 \\
  \vdots \\
  t \\
  t+1 \\
  \vdots \\
  K \\
\end{array}
\]

\[
\begin{array}{c}
  \cdot \\
  \cdot \\
  \cdot \\
  \cdot \\
  \cdot \\
  \cdot \\
\end{array}
\]

control

\[
\begin{array}{c}
  d_0 \\
\end{array}
\]

d_0 is a BNB design in the test treatments.
d1 and d2 are components of the n-ary design which involve the test treatments only.

Now we are ready to state the result of Majumdar and Notz (1983) and Soundarapandian (1995).

7.3.1 Theorem

Let V, B, K be integers with $K \leq V$. A BTNB $(V, B, K; t, s)$ is A-optimal in the class of all designs with the same values of V, B, and K if

$$g(t, s) = \min \{ g(x, z) : (x, z) \in ^\wedge,* \},$$

where

$$^\wedge,* = \{(x, z) : x = 0, \ldots, \lfloor K/2 \rfloor - 1; z = 0, \ldots, B \text{ with } z > 0 \text{ when } x = 0\},$$

$$g(x, z) = \frac{a}{A^*(x, z)} + \frac{1}{B^*(x, z)},$$

$$a = (V-l)^2, \quad c = BVK(K-l), \quad p = V(K-1) + K,$$

$$A^*(x, z) = \frac{c - p(Bx + z) + Bx^2 + 2xz + z}{VK},$$

$$B^*(x, z) = \frac{K(Bx + z) - (Bx^2 + 2xz + z)}{VK}.$$

It note that there are many parameter combinations $(V, B, K)$ which do not belong to any of the three examples of A-optimal BTNB designs given previously for which the result of Majumdar and Notz (1983), Soundarapandian can still be used to get an optimal n-ary design. Hedayat and
Majumdar (1984) have devised an algorithm for obtaining A-optimal designs based on Theorem (7.3.1) and gave a list of all designs available by this result when \( 2 \leq K \leq 8, \ K \leq V \leq 30, \ V \leq B \leq 50 \) and for n-ary designs refer Soundarapandiyan (1995). In particular, the algorithm given by Jacrous (1988b) and following Soundarapandiyan (1995) work produces A-optimal group divisible treatment n-ary designs (GDTND's). for binary and n-ary designs.

### 7.3.4 Definition

\( d \) is GDTND with parameters \( m, n, \wedge_0, \wedge_1 \) and \( \wedge_2 \) if the treatments 1...\( V \) can be divided into \( m \) groups \( V_1^* \ldots V_m \) of size \( n \) such that

- (i) \( \wedge_{d0i} = \wedge_0 \) for \( i = 1 \ldots V \) and for some constant \( \wedge_0' \),
- (ii) if \( i, j \in V^*, i \neq j, \wedge_{dij} = \wedge_1 \) for some constant \( \wedge_1 \),
- (iii) If \( i \in V_p^*, j \in V_q^*, p \neq q, \wedge_{dij} = \wedge_2 \) for some constant \( \wedge_2 \).

### 7.3.5 Definition

For integers \( t \in \{0,1\ldots K-1\} \) and \( s \in \{0,1\ldots B-1\} \), \( d \) is a GDTND \((V, B, K; t,s)\) if it is a GDTND with the additional property that

\[
\begin{align*}
n_{dij} &\in \{0,1,2\ldots n-1\} \, , \, i = 1 \ldots V, j = 1 \ldots B \\
n_{d01} &= \ldots = n_{do} = t + 1. \\
n_{d0,s+1} &= \ldots = n_{doB} = t.
\end{align*}
\]

A GDTND \((V, B, K; t,s)\) is called an R-type design when \( s = 0 \) and an S type design when \( s > 0 \).

The generalization of Theorem (7.3.1) can be stated as follows utilizing n-ary concept of Soundarapandiyan 1980.
7.3.2 Theorem

Let \( V, B, K \) be integers with \( K \leq V \), a BTNB \((V, B, K; \tilde{t}, s)\) or a GDTB \((V, B, K; \tilde{t}, \tilde{s})\) having \( m = 2 \), \( n = V/2 \) and \( \wedge_2 = \wedge_1 + f \) is A-optimal in the class of all \( n \)-ary designs if

\[
n(\tilde{t}, \tilde{s}) = \min \{n(x, z): (x, z) \in \wedge^*\},
\]

where \( a, c, p, A^*(x, z), B^*(x, z) \) and \( g(x, z) \) are as defined in (7.3.5) and where

\[
\wedge^* = \{(x, z): x = 0 \ldots K - 2; z = 0 \ldots B \text{ with } z > 0 \text{ when } x = 0\}
\]

and

\[
n(x, z) = \min \\{h(x, z), m(x, z)\}, \quad (7.3.6)
\]

with

\[
h(x, z) = a/(A^*(x, z) - 2/K) + 1/B^*(x, z),
\]

\[
g(x, z), \quad \text{if } B^*(x, z) > \{A^*(x, z) - (V - 1)/(V - 2) P^*(x, z)\}/(V - 1),
\]

\[
m(x, z) = 1/B^*(x, z) + (V - 2)/(V - 1)\{A^*(x, z) - ((V - 1)/(V - 2)^{1/2} P^*(x, z))\}
\]

\[
+ (V - 1)/\{A^*(x, z) + (V - 1)/(V - 2)^{1/2} P^*(x, z)\}, \text{ otherwise.}
\]

The quantities, \( a, A^*(x, z) \) and \( B^*(x, z) \) are as defined in (7.3.5) and

\[
P^*(x, z) = \{C^*(x, z) - B^{-2}(x, z) - A^{-2}(x, z)/(V - 1)\}^{1/2}
\]

\[
C^*(x, z) = \{BK - Bx - z - VR^*(x, z)\} x \{R^*(x, z) + 1\} (K^* - 1)^2
\]

\[
+ \{V-BK + Bx + z + VR^*(x, z)\} x \{R^*(x, z) (K - 1)\}^2/K^2
\]

\[
+ \{A^*(x, z) - V(V - 1) \wedge (x, z)\} \{\wedge (x, z) + 1\}^2/K^2
\]

\[
+ \{V(V - 1) - A^*(x, z) + V(V - 1) \wedge (x, z)\} \wedge (x, z)/K^2,
\]

\[
R^*(x, z) = [(BK-Bx-z)/V],
\]

\[
\wedge (x, z) = [(BK - Bx - z) (K - 1) - VKB^*(x, z)]/ V(V - 1).
\]

A note that many BTNB \((V, B, K; \tilde{t}, \tilde{s})\) designs not satisfying the conditions of Theorem (7.3.1) can be shown to satisfy the conditions of Theorem (7.3.1). In addition, Theorem (7.3.2) can be used to establish the A-
optimality of GDTND \((V, B, K; \tilde{t}, \tilde{s})\)'s having \(m = 2\), \(n = V/2\) and \(\wedge_2 = \wedge_1 + f\). Using some more elaborate computational techniques, Jacroux (1988b) and Soundarapandiyam (1995) have also developed some sufficient conditions for GDTND \((V, B, K; \tilde{t}, \tilde{s})\)'s having \(\wedge_2 = \wedge_1 + f\) or \(m = V/2\), \(n = 2\) \(\wedge_2 = \wedge_1 - f\) to be A-optimal among all n-ary designs with parameters \(V, B\) and \(K\). One example of an A-optimal GDTND is that GDTND \((9, 9, 7; 2, 1, 0)\) having \(m = 3\), \(n = 3\), \(\wedge_1 = 0\), \(\wedge_2 = f\) and treatment groups \((1, 2, 3), (4, 5, 6)\) and \((7, 8, 9)\) given below:

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 & 6 \\
7 & 8 & 9 & 8 & 9 & 7 & 9 & 7 & 8 & 8 \\
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 & 6 \\
7 & 8 & 9 & 8 & 9 & 7 & 9 & 7 & 8 & 8 \\
\end{array}
\]

### 7.3.3 A-optimal n-ary Designs for Two-way Elimination of Heterogeneity

The statistical n-ary set-up consists of \(BK\) experimental units arranged in a \(K \times B\) array, and the model of response under design \(d\) is

\[
y_{dij} = \mu + t_i + \beta_j + \rho_l + \epsilon_{ij}
\]

\((7.3.7)\)

\(i = 0, 1 \ldots V; j = 1 \ldots B; l = 1 \ldots K\), if treatment \(i\) is applied to the experimental unit in cell \((I, j)\).

Let

- \(n_{dij}\) = number of times treatment \(i\) occurs in column \(j\),
- \(m_{di1}\) = number of times treatment \(i\) occurs in row \(l\),
- \(R_{di} = \sum_{j=1}^{n} n_{dij}\),
- \(N_d = (n_{dij}), a(V + 1)xB\) matrix,
- \(M_d = (M_{dij}), a(V + 1)xK\) matrix,
P is the V x (B+1) matrix defined in (7.3.1)

\[ R_d = (R_{d0}, R_{d1}, \ldots, R_{dV})' \]

\[ C_{d(2)} = \text{diag} (R_{d0}, R_{d1}, \ldots, R_{dV}) - K^{-1}N_d N_d' - B^{-1}M_d M_d' + (BK)^{-1}R_d R_d'. \]

Then an A-optimal n-ary design minimizes trace \( PC_{d(2)}^{-1}P' \)

### 7.3.4 Example

Let \( p \) be an integer and \( V = p^2 \), \( B = K = p^2 + p \). Given B x B array in which each test treatment appears once in each row and in each column and the control appears \( p \) times in each row and in each columns is A-optimal.

One easy to construct members of this example is to start with a Latin square of order \( p^2 + p \) and change symbols \( p^2 + 1 \ldots p^2 + p \) to 0 (control). We illustrate this in the following example with \( V = 4, B = K = 6 \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 \\
6 & 1 & 2 & 3 & 4 & 5 & 0 & 1 \\
5 & 6 & 1 & 2 & 3 & 4 & 0 & 1 \\
4 & 5 & 6 & 1 & 2 & 3 & 0 & 0 \\
3 & 4 & 5 & 6 & 1 & 2 & 3 & 1 \\
2 & 3 & 4 & 5 & 6 & 1 & 2 & 0 \\
\end{array}
\]

This and some more general results are available for binary and n-ary designs.

Majumdar (1986), Soundarapandiyan have generalized the preceding example of A-optimal n-ary designs.

### 7.3.5 Example

Let \( p, \alpha \), and \( \gamma \) be integers, \( V = p^2 \), \( K = \alpha (p^2 + p) \) and \( B = \gamma (p^2 + p) \). Given KxB array in which each test treatment appears \( \alpha \) times in each column
and $\gamma$ times in each row, and the control appears $\alpha p$ times in each column and $\gamma p$ times in each row is A-optimal n-ary designs.

One way to construct members of this example is to form the array

$$(L_{ij}) = i = 1 \ldots \alpha; j = 1 \ldots \gamma$$

where each $L_{ij}$ is a member of example (7.3.4)

7.3.5 Example

Given KxB array is A-optimal n-ary designs if

(i) it is an A-optimal n-ary block design for 1-way elimination of heterogeneity with columns (7.3.8) as blocks, and

(ii) the total number of replications for each treatment, test treatment or control, is divided equally among the K rows.

This has been given by Jacroux (1986) Soundarapandiyam (1995). The following is an example when $V = 9, B = 24, K = 6$.

We note that when considering this last array as n-ary block designs with columns acting as blocks, it is a BTNB (9, 24, 6, 0, 78) design which satisfies the conditions of Theorem (7.3.1) (hence it is A-optimal) and because the total number of replications for each treatment are divided equally among the rows, condition (ii) of (7.3.8) is satisfied.

In general, the method used to construct most of the examples of row-column designs given in this section is to find an optimal block n-ary design.
having parameters $V$, $B$ and $K$ and having the numbers of replications assigned to the test treatments and control divisible by $K$, then arrange treatments within blocks so that condition (ii) of (7.3.8) is satisfied. The fact that an arrangement within blocks satisfying (7.3.8) (ii) can always be found when the number of replications assigned to the test treatments and control are divisible by $K$ essentially follows from Hall's (1935) "marriage lemma" and Soundarapandiyan n-ary works in 1980-81, 1995.