6.1 INTRODUCTION

Many industrial, agricultural and biological experiments, the following problem is encountered frequently; \(p\) (new) varieties (or treatments) become available; an existing one is to eventually be replaced by one of these newer kinds. The purpose is to design an efficient experiment to compare the \(p\) new treatments (labeled 1, 2, ..., \(p\)) with the old one (hereafter called the control and labeled as 0) and to compare the new treatments among themselves. Important emphasis will be on the comparison of the \(p\) treatments with the control, hence higher precision is desired for these estimates.

If the block size is large enough to accommodate one replication of each test treatment and additional controls as well, then the design and analysis of the experiment can be carried out using the optimal allocations described in Bechhofer (1969) and Bechhofer and Nocturne (1970), perhaps with some slight modifications. Robson (1961) suggested the use of balanced incomplete block designs (BIBD) between all of the treatments, including the control. Cox (1958) noted that BIBD may not be very appropriate for the multiple comparisons with the control because of the special role played by the control. A brief exposition to the history of this problem and a list of references the reader will be referred to the article by Bechhofer and Tamhane (1981). Infact this chapter is basic to the later developments in our research for optimal designs for comparing test treatments with controls under 0-, 1- and 2-way elimination of heterogeneity models particularly for n-ary balanced treatments designs.
6.1.1 Notation, Definitions and Basic results Linear Model

The following usual additive linear model will be assumed:

\[
Y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} \quad (i = 0, 1 \ldots p, j = 1, 2, \ldots, b)
\]

Where \(Y_{ijh}\) the observation on the \(i\)-th treatment when it is assigned to the \(h\)-th unit in the \(j\)-th block.

\(\alpha_i\) is the \(i\)-th treatment effect with \(\sum_{i=0}^{p} \alpha_i = 0\)

\(\beta_j\) is the \(j\)-th block effect with \(\sum_{j=1}^{b} \beta_j = 0\)

\(e_{ijh}\) are assumed to be iid \(N(0, \sigma^2)\) random variables.

Desired to provide an exact joint confidence statement above the \(p\) differences \(\alpha_0 - \alpha_i\) based on their BLUE’s \(\hat{\alpha}_0 - \hat{\alpha}_i\) \(1 \leq i \leq p\)

Here a design is a \(k \times b\) array of integers from \(\{0, 1\ldots p\}\).

\(r_{ij}\) = number of replications of the \(i\)-th treatment in the \(j\)-th block.

\(\lambda_{ii'} = \sum_{j=1}^{b} r_{ij} r_{i'} = \) number of times the \(i\)th treatment appears with the \(i'\)-th treatment in the same block over the whole design.

6.1.2 Definition

In BT, a balanced treatment incomplete block design (BTIB) is defined to be a design that satisfies

\[\lambda_{01} = \lambda_{02} = \ldots \lambda_{0p} = \lambda_0\text{ (say)}\]

and

\[\lambda_{12} = \lambda_{13} = \ldots \lambda_{p-1,p} = \lambda_1\text{ (say).} \quad (6.1.1)\]
6.1.3 Definition

We define a BALANCED TREATMENT n-ARY BLOCK (BTNB) designs as an arrangement of \( V(=p+1) \) treatments in \( B \) blocks of size \( K \), such that the \( i \)-th treatment occurs in the \( j \)-th block \( n_{ij} \) times and altogether \( R \) times when \( n_{ij} \) can take the values \( 0,1,2\ldots (n-1) \). We say the design is balanced treatment n-ary block (BTNB) design if the following conditions are satisfied.

\[
\wedge_0 = \sum_{j=1}^{B} n_{ij} n_{kj} \text{ is a constant for } 0 \neq k; k=1,2\ldots p \text{ and }
\]

\[
\wedge_1 = \sum_{i=1}^{B} n_{ij} n_{kj} \text{ is another constant } i \neq k; k = 1,2\ldots p \quad (6.1.2)
\]

Hence a BTNB design is such that each test treatment appears in the same block with the control, the same number of time \( (=\wedge_0) \) over the design, and any pair of test treatment appears together (in the same block) the same total number of times \( (=\wedge_1) \) over the design.

6.1.4 Definition

Let \( C(b,k,p) \) denote the class of all possible incomplete block designs with \( b \) blocks of size \( k \) each, and \( p+1 \) treatments indexed \( 0,1\ldots p; 0 \) being the control.

In BT, it is shown that for a design \( d \in C(b,k,p) \) the information matrix for estimating all \( \alpha_0 - \alpha_i \) \( (1 \leq i \leq p) \) is the symmetric non-negative definite \( p \times p \) matrix \( M(d) \), whose \((i,L)\)-th entry is

\[
M_{il} (d) = \begin{cases} 
  r_i (d) - \frac{1}{k} \sum r_{ij}^2 (d) & (i = L) \\
  \frac{-1}{k} \gamma_{il} (d) & (i \neq L) 
\end{cases}
\]

Where \( r_i (d) = \sum_{j=1}^{b} r_{ij} (d) \) = number of times \( i \)-th treatment appears in the entire design.
For a BTIB design $M$ becomes completely symmetric, i.e., all diagonal elements are equal and all off-diagonal elements are equal. Since $M^{-1}$ is the variance-covariance matrix of the vector of estimates $(\hat{\alpha}_0 - \hat{\alpha}_1, \hat{\alpha}_0 - \hat{\alpha}_2, \ldots, \hat{\alpha}_0 - \hat{\alpha}_p)'$, a BTIB design provides BLUE’s $\hat{\alpha}_0 - \alpha (1 \leq i \leq p)$ with the property that

$$\text{Var}(\hat{\alpha}_0 - \hat{\alpha}_i) = \tau^2 \sigma^2 (1 \leq i \leq p) (\tau^2 \text{is a constant and}$$

$$\text{Cor}(\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_i) = \rho(l \leq i, i' \leq p) (\rho \text{is a constant}) \quad (6.1.3)$$

In BT, the basic theory underlying BTIB designs is provided. It is observed that for given $(p,k)$ a BTIB design for any $b$ can be built out of a set of elementary designs called generator designs. Fortunately, a small subset of these generator designs happens to be sufficient, in the sense that essentially all admissible BTIB designs can be constructed from this set, i.e., any admissible design either is a union of designs in this set, or is equivalent to a union of designs from this set. This set is called the "minimal complete class of generator designs", the phrases "generator design", "admissible design" and "equivalent design" are explained below.

First it is noted that the quantities $\tau^2$ and $\rho$ of (6.1.2) are given by

$$\tau^2 = \frac{k(\lambda_0 + \lambda_1)}{\lambda_0(\lambda_0 + p\lambda_1)} \quad \rho = \frac{\lambda_1}{\lambda_0 + \lambda_1} \quad (6.1.4)$$

Where $\lambda_0$ and $\lambda_1$ are as in (6.1.1) above.

**6.1.5 Definition**

For given $(p,k)$ a generator design is a BTIB design no proper subset of whose blocks forms a BTIB design, and no block of which contains only one of the $p + 1$ treatments.

Any BTIB design is either a generator design or a union of generator designs. In fact, if for given $(p,k)$ there are $n$ generator designs $D_i (1 \leq i \leq n)$
where $D_i$ has parameters $(b_i, \lambda_{0,i}, \lambda_{1,i})$ $(1 \leq i \leq n)$ then the BTIB design $D = \bigcup_{i=1}^{n} f_i D_i$ obtained by taking unions of $f_i$ replications of $D_i$ has the parameters $(b, \lambda_0, \lambda_1)$ given by

$$b = \sum_{i=1}^{n} f_i b_i, \lambda_0 = \sum_{i=1}^{n} f_i \lambda_{0,i}, \lambda_1 = \sum_{i=1}^{n} f_i \lambda_{1,i}.$$ 

### 6.1.6 Definition

Given $(p,k)$ consider two BTIB designs $D_1$ and $D_2$ with parameters $(b_1, \lambda_{0,1}, \lambda_{1,1})$ and $(b_2, \lambda_{0,2}, \lambda_{1,2})$. $D_2$ is inadmissible w.r.t $D_1$ iff $b_1 \leq b_2$, $\tau_1^2 \leq \tau_2^2$ and $\rho_1 \geq \rho_2$ with at least one inequality strict, where $\tau_1^2$ and $\rho_1$ are given by (6.1.4). If a design is not inadmissible then it is said to be admissible.

### 6.1.7 Definition

For given $(p,k)$ let $\mathcal{D} = \{D_i|1 \leq i \leq n\}$ be the smallest set of generator designs with the following property: Any admissible design $\mathcal{D}$ can either be constructed from $\mathcal{D}$ or it is equivalent to a design $D_1$ which is constructed from $\mathcal{D}$. Such a set $\mathcal{D}$ is called the minimal complete class of generator designs. In BT, the concept of strong inadmissibility defined as follows.

### 6.1.8 Definition

For given $(p,k)$ a design $D_2$ is called strongly inadmissible (s.i) w.r.t. $D_1$ if $D_2$ is inadmissible w.r.t $D_1$ if for any BTIB design $D_3$, $D_2 \cup D_3$ is inadmissible w.r.t $D_1 \cup D_3$.

It is shown in BT that a sufficient condition for the strong inadmissibility of $D_2$ w.r.t. $D_1$ is that $b_1 \leq b_2, \lambda_{0,1} = \lambda_{0,2}, \lambda_{1,1} \geq \lambda_{1,2}$ with at least one inequality being strict.

### 6.1.9 Historical Conclusions
We can observe in BT that in order to choose optimal designs for given (p,k) one needs to construct the minimal complete class of generator designs (MCCGD). Although some methods to construct BTIB designs are described, it is mentioned that they will not make it possible to construct MCCGD for given (p,k). This problem is left unanswered in BT. Later Notz and Tamhane (1981) used a method to construct MCCGD for k = 3, p= 3(1)10. But their method involves so extensive trial and error that, as they pointed out, it will not be practical for k>3. Also the inequalities they use to prove that the sets they have constructed are indeed MCCGD will not be adequate to obtain similar results for k > 3. In the next section some concepts are introduced by which it becomes possible to construct the MCCGd for k > 3, at least for moderate k (say k ≤ 10).

6.2 MINIMAL COMPLETE CLASS OF GENERATOR DESIGNS (MCCGD) AND THEIR CONSTRUCTIONS

6.2.1 Definitions

For convenience all the examples will be given for the case k = 4 and k = 5. A block that contains control treatments is called a "basic block" (BB). Otherwise it is called a "treatment block" (TB). For any k, there are several types of BBs. When k = 4, for example, there are six different types of BBs, namely

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & 0 & 0 & 0 \\
y & x & x & x & x & 0 \\
x & y & x & y & x & x \\
\end{array}
\]

where x, y, z represent distinct numbers from \{1,2,...,p\}. Also there are 4 different types of TBs, as follows:
six different types of TBs, as follows

\[ \begin{bmatrix} x & x & x & x \\ y & x & x & x \\ z & y & y & x \\ t & z & z & y \end{bmatrix} \]

TBs can be distinguished by the number of treatment pairs they contain, for example, a 6-TB refers to the first one above, a 3-TB refers to the last one above.

Another example for \( k=5 \), there are ten different types BBs, namely

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & x & x & x & x & 0 \end{bmatrix}
\]

where \( x, y, z, s \) represent distinct numbers from \((1,2, p)\). Also there are six different types of TBs, as follows

\[
\begin{bmatrix} x & x & x & x & x & x \\ y & x & x & x & x & x \\ z & y & y & x & x & x \\ s & z & z & y & y & x \\ t & s & z & z & y & y \end{bmatrix}
\]

TBs can be distinguished by the number of treatments pairs they contain, for example, a 10-TB refers to the first one above, a 7-TB refers to the 4-th one above and a 4-TB refers to the last one above respectively.

### 6.2.1 Truth

Given \((p,k)\) and A BTIB design with parameters \((b, \lambda_0, \lambda_1)\), the control appears a total of \( p \lambda_0 \) times with the test treatments. These \( p \lambda_0 \) control
treatment pairs can be allocated to different possible combinations of BBs, where each combination must have exactly $p \lambda_0$ such pairs.

### 6.2.1 Example

$p = 4$, $k = 4$. Consider a BTIB with parameters $(b, \lambda_0, = 8, \lambda_1)$ ($b$, $\lambda_1$ are not relevant here). Here, there are $p \lambda_0 = 32$ treatment control pairs (abbreviated as TC pairs) in this BTIB design. It is seen that a BB contains 3 TC pairs if it has 1 or 3 controls in it, contains 4 TC pairs if it has 2 controls in it. (In general a BB has $m(k-m)$ TC pairs, where $m$ is the number of controls in it). Hence there are 3 possible combinations of BBs with 32 TC pairs, namely

1. **Eight 4-BBs:**

   \[
   \begin{bmatrix}
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\
   2 & 2 & 3 & 4 & 3 & 4 & 4 & 4 \\
   \end{bmatrix}
   \]

2. **Five 4-BBs and four 3-BBs**

   \[
   \begin{bmatrix}
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
   1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
   2 & 3 & 4 & 3 & 4 & 4 & 4 & 4 \\
   \end{bmatrix}
   \]

3. **Two 4-BBs and eight 3-BBs**

   \[
   \begin{bmatrix}
   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\
   1 & 3 & 2 & 2 & 2 & 2 & 3 & 3 \\
   2 & 4 & 3 & 3 & 4 & 4 & 4 & 4 \\
   \end{bmatrix}
   \]

Treatment-treatment (TT) pairs are arbitrarily matched in these example. Note that different variations of (2) and (3) above can be obtained by replacing a 3-BB of the form

\[
\begin{bmatrix}
0 \\
x \\
y \\
z \\
\end{bmatrix}
\]

by a 3-BB of the form

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
x \\
\end{bmatrix}
\].
Some quantities will play a vital role in the construction of BTIB designs and in derivation of lower bounds for $b$. These are defined below:

### 6.2.2 Definition

Given $(p,k)$ and a BTIB design with parameters $(b, \lambda_0, \lambda_1)$

- $N =$ Total number of TT pairs in the design $= \binom{p}{2}\lambda_1$
- $K =$ Total number of times that a treatment must be paired with other treatments $= (p - 1)\lambda_1$
- $M =$ Total number of TT pairs in a particular combination of BBs
- $K_i =$ Total number of times $i$-th treatment is paired with other treatments in a particular combination of BBs. $i = 1,2,\ldots, p.$

### 6.2.2 Example

In Example (6.2.1) above, let $\lambda_i = 3$, Then

$$N = \binom{p}{2}3 = 18$$

$$= (4-1)3 = 9$$

For combination (1): $M = 8$ $K_i = 4$ $i = 1,2,3,4$
For combination (2): $M = 8$ $K_1 = K_2$ $K_3 = K_4 = 10$
For combination (2): $M = 26$ $K_i = 13$ $i = 1,2,3,4$.

For given $(p,k)$ and $(\lambda_0,\lambda_1)$ and a specific BB-combination two possibilities arise:

(a) $K_i > K$ for some $i, 1 \leq i \leq p$.

This means the $i$-th treatment appears in too many TT pairs. $K_i$ must be reduced so that $K_i = K$. This can be done either by choosing a BB combination with fewer blocks, or by replicating the $i$-th treatment within a block. These will be explained below.
(b) $K_i < K$ for some $i$, $1 \leq i \leq p$.

Then $K_i$ must be increased. This can be done either by choosing a BB with more blocks, or by adding some more TBs.

6.2.2 Truth

(i) If a treatment $i$ is replicated twice in a block in a BB combination, then $K_i$ is reduced by 2, assuming the number of times $i$ appears in this BB combination is fixed.

(ii) If a TB in which treatment $i$ appears $m$ times is added to the BB combination at hand then $K_i$ is increased by $m(k-m)$ ($m<k$).

Proof:

(i) Two typical possibilities will be considered.

(a) Suppose treatment $i$ appears in two blocks with $m$ controls in each; then it is paired $2(k-m-1)$ times with other treatments. Now take treatment $i$ from one of these two blocks and place it in the other. Of course another treatment will also change blocks necessarily. Then treatment $i$ appears twice in one of these two blocks and is paired $2(k-m-2)$ times with other treatments. Hence $K_i$ is reduced by 2. Here, treatment $i$ may appear in other blocks as well, but these are not considered, because their contribution to $K_i$ does not change.

6.2.3 Example

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(b) Suppose treatment $i$ appears in two blocks, one with $m-u$ controls and the other with $m+u$ controls ($1 \leq m-u \leq k-1$, $1 \leq m+u \leq k-1$). Suppose there exist some blocks with $m$ controls. Now remove $i$ from these two blocks and
place both in a block with m controls. Before the operation treatment i was
appearing in \((k - m + u - 1) + (k - m - u - 1) = 2k - 2m - 2\) TT pairs. After, it
appears in \(2(k - m - 2) = 2k - 2m - 4\) TT pairs. Again Ki is reduced by2.
Other possibilities can be examined similarly,

(ii) Follows trivially.

for \(k = 4\), assume \(K - K_i \neq 0\) for a certain BB combination. To get a BTIB
design \(K-K_i = 0\) must be true for all \(i = 1, \ldots, p\). Let \(K - K_i = m \mod(3)\)
consider two possibilities:

1. \(K-K_i = m \mod (3) > 0\) \(m = 1\) or 2.

Then \(K_i\) must be increased that \(K - K_i = 0\). There are four ways to
increase \(K_i\) (Not all of these may be possible for a specific case.)

(a) Choose another BB-combination with bigger \(K_i\), say \(K_i'\). We have \(K_i' - K_i = 0 \mod (3)\) (see Truth 4.2.5) then still \(K_i - K_i' = m \mod (3) \neq 0\) (<0 or > 0).

(b) Add a TB in which treatment i appears once or three times. Then \(K_i\)
becomes \(K_i' = K_i + 3\). Hence \(K - K_i' = m \mod (3) \neq 0\).

(c) Add a TB in which treatment i appears twice, then \(K_i\) becomes \(K_i' = K_i + 4\). Hence \(K - K_i' = (m - 1) \mod (3)\).

(d) Replicate i-th treatment twice in a block in that BB-combination, then
add a TB in which i-th treatment appears once or three times.
Then \(K_i \rightarrow K_i - 2\) (Truth 6.2.2) \(\rightarrow K_i + 1 = K_i'\). Hence \(K - K_i' = (m - 1) \mod (3)\).

2. \(K - K_i' = m \mod (3) < 0\).

Then \(K_i\) must be decreased. There are two ways to do that:

(a) Choose another BB-combination with smaller \(K_i\), say \(K_i'\). Since \(K - K_i' = 0 \mod (3)\); then \(K - K_i' = m \mod (3) \neq 0\).

(b) Replicate i-th treatment twice in a block in that BB-combination, then
\(K_i' = K_i - 2\). Hence \(K - K_j' = (m - 1) \mod (3)\).

The above considerations lead us to establish the following result.
6.2.3 Truth.

For K=4, if K-K_i = m mod (3) for m =0, 1, 2 for some treatment i, then the i-th treatment must be replicated twice in at least m blocks.

It was observed before that for given p and \( \lambda_0 \) there may be several different BB combinations each containing \( p \lambda_0 \) TC pairs. In fact, a general expression for all possible BB combination can be obtained for given (p,k). For (p=1, k=4) this is done below. Remember that for k=4, there are two different types of BBs. These are

\[
\begin{bmatrix}
0 \\
0 \\
x \\
y
\end{bmatrix}
\]

Type 1: referred as 4-BB, since it contains 4 TC pairs.

\[
\begin{bmatrix}
0 \\
x \\
y \\
z
\end{bmatrix}
or
\begin{bmatrix}
0 \\
0 \\
0 \\
x
\end{bmatrix}
\]

Type 2: a 3-BB, since it contains 3 Tc pairs.

From now on a BB combination is represented by an ordered pair, like (U,V), which means there are U of 4-BBs and V of 3-BBs in that combination.

6.2.4 Truth

For p=4, k=4 the BB combinations are given by

(i) \((\lambda_0 - 3j + 3, 4j - 4) \quad j = 1, 2, \ldots, \left\lfloor \frac{\lambda_0}{3} \right\rfloor + 1\) if \( \lambda_0 \) is even with

\[ M_j = \lambda_0 + 9j - 9 \quad \text{and} \quad b_j = \lambda_0 + j - 1. \]

(ii) \((\lambda_0 - 3j, 4j) \quad j = 1, 2, \ldots, \left\lfloor \frac{\lambda_0}{3} \right\rfloor + 1\) if \( \lambda_0 \) is odd > 1 with

\[ M_j = \lambda_0 + 9j \quad \text{and} \quad b_j = \lambda_0 + j. \]

Where \( M_j \) is the number of TT pairs (assuming the most efficient types of BBs are used and no replications are made) included in the j-th BB
combination and $b_j$ is the number of blocks it contains.

**6.2.5 Truth**

We also observe For $(p=k, k=l)$

$$K_i = \begin{cases} 
0 & \text{mod}(3) \text{ if } \lambda_0 = 0 \mod(3) \\
1 \mod(3) & \text{if } \lambda_0 = 2 \mod(3) \\
2 \mod(3) & \text{if } \lambda_0 = 1 \mod(3) 
\end{cases}$$

The method of construction can be described briefly as follows:

For given $(p,k)$ start with $\lambda_0 = 0$. Then use Truth (6.2.6) below to construct any BTIB designs available from $(p-l,k)$ immediately.

**6.2.6 Truth**

If for given $(p-l,k)$, a BTTB design with parameters $b, \lambda_0, \lambda_i = \lambda_0$ exists, then by replacing all zeros by integer $p$, a BTIB design with parameters $(b,0,\lambda_i)$ for $(p,k)$ is obtained.

Assuming that one has the MCCGD for $(p-l,k)$ one obtains all BTIB designs for $(p,k)$ and $\lambda_0 = 0$ through Truth (6.2.6.)

The following can also be used to construct BTIB designs with $\lambda_0 > 0$

**6.2.7 Truth**

If for given $(p, k-1)$ a BTIB design which is equireplicate in test treatments exists then by adding one control to each block of this design a BTIB design for $(p,k)$ is obtained. If the original design has parameters $(b, \lambda_0, \lambda_i)$ and "$r$" replications of any test treatment then the new design will have parameters $(b, \lambda_0 + r, \lambda_i)$
To obtain the BTIB designs that cannot be constructed through the above results the following algorithm is employed.

For given (p,k) start with $\lambda_0 = 1$. Take any BB combination for that case. (If no BB combination exists no BTIB design exists so increase $\lambda_0$.) Start with $\lambda_1 = 1$ (case $\lambda_1 = 0$ is trivial as will be seen in the example below) compute $K = (p-1)\lambda_1$ and $K - K_i, 1 \leq i \leq p$. Then try to make $K - K_i = 0$ using any possible ways as described before Truth (6.2.3.) Check if any within block replications are necessary. (This will be obvious through Truth (6.2.3). The minimum number of blocks needed for that BTIB design will be obtained via inequalities in the next section. After a BTIB design can't exist for this $(\lambda_0, \lambda_1)$ increase $\lambda_1$ by 1. Check if a BTIB design with this $(\lambda_0, \lambda_1)$ can be obtained as the union of the already constructed designs. If not, repeat the same procedure as above. It will be seen that from same $\lambda_1$ value on all BTIB designs can be obtained as unions of the previously constructed designs in a definite pattern. Once that $\lambda_1$ value is achieved, increase $\lambda_0$ by 1 and proceed, as before. Again, when is big enough all the BTIB designs thereon will be unions of the existing designs, so stop.

This is an “adhoc” procedure, i.e., some trial and error will be needed. But after having constructed hundreds of BTIB designs for several values of (p,k) it is the author’s experience that only in a very small percentage of cases an extensive amount of trial and error is needed. This is true at least for values of (p,k) in our range of interest. In fact, using the computerized method of construction that will be presented later, it becomes possible to construct all binary BTIB designs with little effort, hence the method described above can be used to construct the remaining (non-binary) designs only. (A binary BTIB design is such that no test treatment can appear more than once in any block.)
6.2.3 Example

Suppose it is desired to construct a BTIB design for \((p=5, \ k=4)\) and 
\((\lambda_0 = Z, \lambda_1 = 2)\). There are \(N = 20\) TT pairs and \(p\ \lambda_{10} = 10\) TC pairs. They only BB combination is \((1, 2)\) i.e.,

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & . & . & 0 & 3 & 3 \\
. & . & 1 & 4 & 4 & K_3 = K_4 = K_5 = 4 \\
. & . & 2 & 5 & 5 & \\
\end{array}
\]

\(K = (p-1)\lambda_1 = 8 = K - K_i = 1 \mod (3)\) each treatment must appear twice in a block at least once. From Lemma (6.3.3) the minimum number of blocks required is

\[
b = \left( \frac{N - M_{j_o} + m}{1/2k(k-1)} \right)^+ + \left( \frac{20 - 7 + 5}{6} \right)^+ + 3 = 6
\]

Hence at least three TBs must be used. Since there can be only one within block replication in each block (because \(\lambda_1 < 4\)) we must have the following situation:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 1 & 2 & 5 \\
0 & 3 & 4 & 1 & 2 & 5 \\
1 & 3 & 4 & \ldots & \\
2 & 5 & 5 & \ldots & \\
\end{array}
\]

Now, treatments 3 and 4 must meet twice with each other. Since they already appear twice with treatment 5 we are bound to have

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 5 \\
0 & 3 & 4 & 1 & 2 & 5 \\
1 & 3 & 4 & 3 & 3 & 1 \\
2 & 5 & 5 & 4 & 4 & 2 \\
\end{bmatrix} \equiv (6,2,2)
\]
6.3 BASIC INEQUALITIES

6.3.1 Lemma

Consider all possible BB combinations for a specified \((p, k, \lambda_0, \lambda_1)\). Then the following is true

\[
b \geq \min_j \left\{ \left( \frac{N - M_j}{2} \right)^+ + b_j \right\}
\]

where \((x)^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if otherwise} \end{cases}\)

and \(M_j\) is the M-value of the \(j\)-th BB combination which consists of \(b_j\) BBs.

Proof

Remember that \(N\) is the total number of TT pairs in the design with parameters \((b, \lambda_0, \lambda_1)\) for given \((p,k)\).

The lower bound is trivially obtained for \(N - M_j < 0\), so suppose \(N - M_j \geq 0\). Consider the \(j\)th BB combination which contains \(M_j\) TT pairs. The remaining \(N - M_j\) TT pairs must be allocated to the most efficient TBs, each of which has \(\frac{1}{2}k(k-1)\) TT pairs. Hence is \(\left( \frac{N - M_j}{2} \right)^+\) is the number of TBs that are needed. Since \(b_j\) BBs are already used up, the BTIB design for given \((p, k, \lambda_0, \lambda_1)\) cannot have less than \(\left( \frac{N - M_k}{2} \right)^+ + b_j\) blocks all together.

Taking minimum over all possible BB combinations the result is established. The following form of the inequality will be of frequent use.
6.3.2 Lemma

\[ b \geq \left( \frac{N - M_{j_0}}{\frac{1}{2} k(k - 1)} \right)^+ + b_{j_0} \]

where \( j_0 \) is the index of the BB combination with \( b_{j_0} \) BBs for which \(|N-M_{j_0}|\) is closest to zero. (If there is more than one BB combination with \( b_{j_0} \) BBs when \( M_{j_0} \) is the M-value of that particular one which makes \(|N-M_{j_0}|\) closest to zero.

To prove the Lemma (6.3.2) the following result is needed.

6.3.1 Truth

If there is a BB-combination with \( b_{j+1} \) BBs and \( M_{j+1} \) TT pairs, and a BB combination with \( b_j \) BBs and \( M_j \) TT pairs, then one can always have

\[ \frac{1}{2} k(k - 1) \leq M_{j+1} - M_j \leq k(k - 1) \text{ for } k \leq 12 \]

By choosing \( M_{j+1} \) and/or \( M_j \) properly. This may be true for all \( k \), but has been shown only for \( k \leq 12 \), because this covers all blocks sizes of interest for practical purposes. (Note: Both BB combinations are considered for the same \((p, \lambda_0)\)).

Proof

- \( k = 4 \quad M_{j+1} - M_j = 9 \) by Truth (6.2.4)
- \( k = 5 \quad M_{j+1} - M_j = 12 \)
- \( k = 6 \quad \) Here a BB combination is of the form \((t,u,v)\) where
  - \( t \) = number of 5-BBs that contain one control and 10 TT pairs.
  - \( u \) = number of 8-BBs that contain two controls and 6 TT pairs.
  - \( v \) = number of 9-BBs that contain three controls and 3 TT pairs.
  - \((t,u,v)\) has \( M_j = 10t + 6u + 3v \) with \( b_j + 1 \) blocks and equal number of TC pairs as \((t,u,v)\).
\[(t+i, u+9-4i, v-8+3i) i = I_1, I_1+1, \ldots, I_2\]

where \(I_1, I_2\) are such that \(u + 9 - 4I_2 \geq 0\) and \(v - 8 + 3I_1 \geq 0\).

Then \(M_{j+1}^i = M_j + 30 - 5i\). Hence, \(M_{j+1} - M_j \leq 30 = k(k - 1)\) no matter what \(i\) is. Now, \(M_{j+1}^i - M_j < \frac{1}{2} k(k - 1) = 15\) when \(i \geq 4\) for \(i = 4\) we have \((t+4, u-7, v+4)\) but whenever this is possible, i.e. \(u \geq 7\). We also have \((t+3, u-3, v+1)\) with \(M_{j+1} = M_j + 15\) so the result follows for \(k = 6\).

\[
\begin{align*}
\text{For } i & = 4 \text{ we have } (t+u, u-6, v+3). \text{ But whenever this is possible, i.e., } u \\
& \geq 6, \text{ we also can have } (t+3, u-3, v+1) \text{ with } M_{j+1} = M_j + 21. \text{ So the result follows for } k = 7. \text{ Similar analysis for } 8 \leq k \leq 12 \text{ leads to the same conclusion.}
\end{align*}
\]

### 6.3.2 Proof of Lemma

Suppose that we have (for fixed \(p, \lambda_0, \lambda_1\))

\[
\min_j \left\{ \frac{(N-M_j)^+}{k(k-1)} + b_j \right\} = \left\{ \frac{N-M_p^+}{k(k-1)} \right\} + b_p^+.
\]
and $|N - M_j|$ is not closest to zero. Consider two cases:

1. $N - M_j < 0$, i.e. $M_j > N$.

   That means the BB combination has more TT pairs than needed.
   Let $J_0 = j^* - 1$ then $b_{j_0} = b_{j^*} - 1$. Choose a BB combination with $b_{j_0}$ blocks that has $M_{j_0}$ TT pairs such that $|M_{j_0} - M_j| \leq k(k - 1)$.

   (a) If $N - M_j < 0$, then

   $$\left( \frac{N - M_j}{\frac{1}{2}k(k - 1)} \right)^+ + b_{j^*} = b_{j^*} > b_{j_0} = \left( \frac{N - M_j}{\frac{1}{2}k(k - 1)} \right)^+ + b_{j_0}$$

   Which is a contradiction.

   (b) If $N - M_j \geq 0$ then $N - M_j \geq \frac{1}{2}k(k - 1)$ since $M_{j_0}$ is closer to zero than $M_{j^*}$.

   Hence,

   $$\left( \frac{N - M_{j_0}}{\frac{1}{2}k(k - 1)} \right)^+ + b_{j_0} \leq 1 + b_{j_0} = b_{j^*}$$

   which is a contradiction.

2. $N - M_j \geq 0$ i.e. $N - M_j *$

   Let $j_0 = j^* + 1$ then $b_{j_0} = b_{j^*} + 1$ choose a BB combination with $b_{j_0}$ blocks that has $M_{j_0}$ TT pairs such that

   $$\frac{1}{2}k(k - 1) \leq M_{j_0} - M_{j^*} \leq k(k - 1)$$

   (a) If $N - M_{j_0} \geq 0$ then since $M_{j_0} - M_{j^*}$ one has

   $$\left( \frac{N - M_{j^*}}{\frac{1}{2}k(k - 1)} \right)^+ + b_{j^*} \geq \left( \frac{N - M_{j_0}}{\frac{1}{2}k(k - 1)} + \frac{1}{2}k(k - 1) \right)^+ + \left( \frac{N - M_{j_0}}{\frac{1}{2}k(k - 1)} + b_{j_0} \right)$$

   Which is a contradiction
(b) If \( N - M_{j_a} < 0 \) then \( N - M_{j_+} \geq \frac{1}{2} k(k - 1) \) since \( |N-M_{j_0}| \) is closer to zero than \( |N-M_{j_*}| \). Hence

\[
\left( N - \frac{M_{j_+}}{\frac{1}{2} k(k - 1)} \right)^+ + b_{j_+} \geq \left( \frac{\frac{1}{2} k(k - 1)}{\frac{1}{2} k(k - 1)} \right)^+ + b_{j_+} = b_{j_+} + 1 = b_{j_0} = \left( \frac{N - M_{j_0}}{\frac{1}{2} k(k - 1)} \right)^+ + b_{j_0}
\]

is a contradiction. That completes the proof.

To give some idea of the usefulness of the lower bound provided by Lemma (6.3.2) some comparisons are given for \( (p=6, k-4) \) below. Bounds are rounded to the next largest integer. Note that the inequality obtained by Notz and Tamhane (1981) is as follows:

\[
b \geq \frac{0.1^{p^{-1} \lambda_2}}{k(k - 1)}
\]

<table>
<thead>
<tr>
<th>((\lambda_0, \lambda_1))</th>
<th>Lower bound by Notz and Tamhane</th>
<th>Lower bound by Lemma (6.3.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,2)</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>(10,1)</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>(12,0)</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>(14,1)</td>
<td>17</td>
<td>21</td>
</tr>
</tbody>
</table>

**Comparison of lower bounds on b obtained by different inequalities**

The inequalities given above assume that the most efficient form of blocks are used in the BTIB design. But as observed in Truth (6.2.3) in many cases within block replications of treatments become necessary. As mentioned before whenever a treatment occurs twice in a block the number of TTs contained in this block decreases by one. In cases where within block replication is a must, the lower bound on b can further be improved. The following result is established easily.
6.3.3 Lemma

Suppose for given \((p,k)\) and \((\lambda_0, \lambda_1)\) a total of \(m\) within block replications are necessary (at minimum: note that \(m\) will be obtained through Truth (6.2.3). Then the following lower bound on \(b\) can be established:

\[
b \geq \left( \frac{N - M_{j_0} + m}{\frac{1}{2}k(k-1)} \right) + b_{j_0}
\]

Where \(j_0\) is defined as in Lemma (6.3.3)

Proof

a. Assume all “m” within block replications take place in the \(j_0\)th BB combination. Then the BB combination contains \(M_{j_0} + m\) TT pairs instead of \(m_{j_0}\) of \(m\). So by Lemma (6.3.2) we have the result.

b. Assume all “m” within block replications take place in \(m\) TBs. Then those \(m\) TBs will contain \(\frac{1}{2}k(k-1)-1\) TT pairs each. The remaining \(N - M_{j_0} - m\frac{1}{2}k(k-1)-1\) TT pairs will be allocated to other TBs. That means (by Lemma (6.3.2))

\[
b \geq \left( \frac{N - M_{j_0} - m\left(\frac{1}{2}k(k-1)-1\right)}{\frac{1}{2}k(k-1)} \right) + m + b_{j_0} = \left( \frac{N - M_{j_0} + m}{\frac{1}{2}k(k-1)} \right) + b_{j_0}
\]

Minimum number of TBS needed number of TBs that contain replications.

For remaining TT pairs

The case in which some within block replications are observed in BBs and some in TBs can be considered similarly.

Now, the minimal complete class of generator designs will be given for \(k = 4, p = 4\) \((1)\) 10, \(k = 5, p = (1)\) 10 and \(k = 6, p = 6\). The proofs are very
similar to those given by Notz and Tamhane for \( k = 3 \), namely, they involve a case by case enumeration each time. The method has also been employed for \( k = 6 \) and \( k = 7 \) satisfactorily. Hopefully, using this method together with the computer algorithm that will be described later, the minimal completer class of generator designs will be constructed for practical values of \((p,k)\) and they will be made available in the form of a monograph in the near future.

From now on a BTIB design will be represented by an ordered triple \((b, \lambda_0, \lambda_1)\) for fixed \((p,k)\).

### 6.3.1 Theorem

For \( p = 4, k = 4 \) the MCCGD is as given in Table (6.3.1)

#### 6.3.1 Table MCCGD For \( p = 4, k = 4 \)

<table>
<thead>
<tr>
<th>LABEL</th>
<th>DESIGNS</th>
<th>( b )</th>
<th>( \lambda_0 )</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>1 2 3 4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>0 0 1 1 1 1 2 2 0 0 1 1 2 2 3 3 0 3 3 4 2 2 3 4 0 4 3 4 3 4 4 4</td>
<td>8</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>( D_3 )</td>
<td>0 0 0 0 0 1 1 2 0 2 3 3 0 4 4 4</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( D_4 )</td>
<td>0 0 0 0 0 0 0 0 0 2 3 4</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
Proof

We will take an arbitrary BTIB design $D$ with parameters $(b, \lambda_0, \lambda_1)$ and show that it is either equivalent to some union of BTIB designs from the table or it is strongly inadmissible w.r.t. such a union.

For $(p=4, k=4)$ $N = 6\lambda_1$, $K = 3 \lambda_i$

Case: $1$ $\lambda_0 = 0 \mod(3)$

(a) $3 \lambda_i - 2\lambda_0 \geq 0$

Let $D^* = \frac{\lambda_0}{3} D_3 + \frac{3\lambda_1 - 2\lambda_0}{3} D_1$

$D^*$ requires the minimum number of blocks, which is $\frac{1}{3}(2\lambda_0 + 3\lambda_1)$, because $D_1$ and $D_3$ are completely binary, i.e., no within-block replications
occur for any treatments, including the control. (Hence $D^*$ consists of only the most efficient types of BBs and TBs).

(b) $3\lambda_1 - 2\lambda_0 < 0$

Note that $6\lambda_1 - \lambda_0 = L \mod 9$ $L = 0, 3$ or 6.

We consider these separately

i) $6\lambda_1 - \lambda_0 = 0 \mod 9 = 9m$ (say) $m = 0, 1, \ldots$

Let $D^* = \frac{2\lambda_0 - 3\lambda_1}{9}D_o + \frac{6\lambda_1 - \lambda_0}{9}D_3$

Take $j_0 = \begin{cases} m & \text{if } \lambda_0 : \text{odd} \\ m+2 & \text{if } \lambda_0 : \text{even} \end{cases}$

Then by Truth (6.2.4) $M_{j_o} = \lambda_0 + 9m = 6\lambda_1 + N$; i.e. $j_0$ is the value that minimizes $|N - M_j|$. Hence, by Lemma (6.3.2) $b \geq b_{j_o} = \lambda_0 + m$. But $D^*$ has $b^* = \lambda_0 + m$, hence it has minimum number of blocks (for these $\lambda_0$ and $\lambda_1$ values). Case $6\lambda_1 - \lambda_0 = 0 \mod 9 < 0$ will be considered later.

ii) $6\lambda_1 - \lambda_0 = 6 \mod 9 = 9m + 6$ $m = 0, 1, 2, \ldots$

Let $D^* = \frac{2\lambda_0 - 3\lambda_1 + 3}{9}D_o + \frac{6\lambda_1 - \lambda_0 - 6}{9}D_3 + D_1$

Take $j_0 = \begin{cases} m+1 & \text{if } \lambda_0 : \text{odd} \\ m+2 & \text{if } \lambda_0 : \text{even} \end{cases}$

Then by Truth (6.2.4) $M_j = \lambda_0 + 9m + 9 = 6\lambda_1 + 3 = N + 3$. Hence, by Lemma (6.3.1) $b \geq b_{j_o} = \lambda_0 + m + 1$. Since $D^*$ has $b^* = \lambda_0 + m + 1$ it is the best.

iii) $6\lambda_1 - \lambda_0 = -3$, here $\lambda_0$ is odd. Take $j_0 = 1$

$\Rightarrow M_{j} = \lambda_0 + 9 = N + 12 \Rightarrow b \geq b_{j_o} = \lambda_0 + 1$

i.e. any $D = (b, \lambda_0, \lambda_1)$ must have $b \geq \lambda_0 + 1$. Now consider $D^* = D_j + \lambda_i$
D_G = (b^*, \lambda_0^*, \lambda_1^* = \lambda_1 + 2) where b^* = 6\lambda_1 + 4 = \lambda_0 + 1 \leq b. Hence, D is s.i. with respect of D^*. Case 6\lambda_1 - \lambda_0 = 6 \mod (9) < -3 will be considered later.

iv) 6\lambda_1 - \lambda_0 = 3 \mod (9) = 9m - 6 \text{ m} = 0,1,2,\ldots

Take j_0 = \begin{cases} m & \text{if } \lambda_0 : \text{odd} \\
2m + 2 & \text{if } \lambda_0 : \text{even} \end{cases}

\Rightarrow M_{j_0} = N + 6 \Rightarrow b \geq b_{j_0} = \lambda_0 + m \ i.e., \ and \ D \ with \ (b, \lambda_0, \lambda_1) \ must \ have 
b \geq \lambda_0 + m. \ Now \ consider \ D^* \ with \ (b^*, \lambda_0^*, \lambda_1^*) \ where \ \lambda_0^* = \lambda_0, \lambda_1^* = \lambda_1 + 1. \ Then \ 6\lambda_1 - \lambda_0 ^* = 9m = 0 \ \mod (9). \ So \ D^* \ belongs \ to \ case \ (i) \ above \ \Rightarrow b^* = \lambda_0 + m.

Now \ we \ have \ D^* \ with \ (b^* = \lambda_0 + m, \lambda_0, \lambda_1 + 1) \ and \ D \ with 
(b \geq \lambda_0 + m, \lambda_0, \lambda_1). \ That \ means \ any \ BTIB \ design \ D \ with \ \lambda_0, \lambda_1 \ values \ satisfying 
(iv) \ is \ strongly \ admissible \ (s.i.) \ w.r.t. \ D^*. \ Case \ 6\lambda_1 - \lambda_0 = 3 \ \mod (9) < -6 \ will 
be considered later.

**Case: 2 \ \lambda_0 = 1 \ \mod (3)**

(a) \ 3\lambda_1 - 2\lambda_0 \geq -8

Let \ D^* = D_4 \frac{\lambda_0 - 4}{3} \ D_3 + \frac{3\lambda_1 \geq \lambda_0 + 8}{3} \ D_l

Note that 3\lambda_1 - 2\lambda_0 = 1 \ \mod (3) = 3m - 8 \ \text{ m} = 0, 1, 2\ldots

(i) \ m = 0 \ or \ 1 \ Take \ j_0 = \begin{cases} \lambda_0 - 4 + m & \text{if } \lambda_0 = \text{odd} \\
\lambda_0 - 1 + m & \text{if } \lambda_0 = \text{even} \end{cases}

\Rightarrow M_{j_0} = 4\lambda_0 - 12 + 9m = N + 3m + 4

\Rightarrow b \geq b_{j_0} = \frac{4\lambda_0 - 4}{3} + m = b^* \ \text{ Hence, } D^* \ is \ the \ best
(ii) \( m \geq 2 \) Take \( j_0 = \begin{cases} \frac{\lambda_0 - 1}{3} + m & \text{if } \lambda_0 = \text{odd} \\ \frac{\lambda_0 + 2}{3} + m & \text{if } \lambda_0 = \text{even} \end{cases} \)

Note that \( j_0 \) is the largest value \( j \) can take, Now,

\[
M_{j_0} = 4\lambda_0 - 3 = N - 6m + 13 \Rightarrow N - M_{j_0} = 6m - 13 \text{ and } b_{j_0} = \frac{1}{3}(4\lambda_0 - 1).
\]

From Truth (6.2.5) \( K - K_i = 1 \mod (3) \) since \( \lambda_0 = 1 \mod (3) \). Hence, at least 4 within block replications are necessary. Then Lemma (6.3.3) yields

\[
b \geq \left( \frac{6m - 13 + 4}{6} \right) + b_{j_0} = \frac{6m - 9 + 4\lambda_0 - 1}{3} = \frac{8\lambda_0 + 6m - 11}{6} = \frac{6\lambda_1 + 4\lambda_0 + 5}{6}
\]

but \( b^* = \frac{6\lambda_1 + 4\lambda_0 + 8}{6} \) = next largest integer to \( \frac{6\lambda_1 + 4\lambda_0 + 5}{6} \)

Hence, \( D^* \) is the best.

i. \( 3\lambda_1 - 2\lambda_0 < -8 \). Note that \( 6\lambda_1 - \lambda_0 = L \mod (9) \), \( L = 2, 5 \) or 8.

ii. \( 6\lambda_1 - \lambda_0 = 2 \mod (9) = 9m + 2 \quad m = 0, 1, 2, \ldots \)

Let \( D^* = D_1 + D_4 + \frac{2\lambda_0 - 3\lambda_1 - 5}{9}D_6 + \frac{6\lambda_1 - \lambda_0 - 2}{9}D_3 \)

Take \( j_0 = \begin{cases} m + 1 & \text{if } \lambda_0 : \text{odd} \\ m + 2 & \text{if } \lambda_0 : \text{even} \end{cases} \)

Then \( M_{j_0} = \lambda_0 + 9m + 9 = 6\lambda_1 + 7 = N + 7 \)

and \( b_{j_0} = \lambda_0 + m + 1 \)

\[
\Rightarrow \quad b \geq \lambda_0 + m + 1
\]

But \( b^* = \frac{6\lambda_1 + 8\lambda_0 + 7}{9} = \lambda_0 + m + 1 \)

\( D^* \) is the best.

(i) \( 6\lambda_1 - \lambda_0 = 5 \mod (9) = 9m - 4 \quad m = 0, 1, 2, \ldots \)
Let \( D^* = D_4 + \frac{2\lambda_0 - 3\lambda_1 - 8}{9} D_6 + \frac{6\lambda_1 - \lambda_0 + 4}{9} D_3 \)

Take \( j_0 = \begin{cases} 
  m & \text{if } \lambda_0 \text{ odd} \\
  m + 1 & \text{if } \lambda_0 \text{ even}
\end{cases} \)

Then \( M_{j_0} = 6\lambda_1 + 4 \)

\( b_{j_0} = \lambda_0 + m = b^* \)

\( * \) is the best

(ii) \( 6\lambda_1 - \lambda_0 = 8 \mod (9) = 9m - 1 \quad m = 0, 1, 2, \ldots \)

Take \( j_0 = 1 \quad \text{if } m = 0 \)

As above \( \text{if } m > 0 \)

\[ M_{j_0} = \begin{cases} 
  6\lambda_1 + 10 & \text{if } m = 0 \\
  6\lambda_1 + 1 & \text{if } m > 0
\end{cases} \quad \text{and} \quad b_{j_0} = \begin{cases} 
  \lambda_0 + 1 & \text{if } m = 0 \\
  \lambda_0 + m + \frac{1}{2} & \text{if } m > 0
\end{cases} \]

Then Lemma (6.3.3) yields (\( K - K_i = 1 \mod (3) \)) \( \Rightarrow m = 4 \)

\[ \left( -10 + 4 \right)^* \]

\[ b \geq \left( -1 + 4 \right)^* \]

Consider \( D^* = (b^*, \lambda_0^*, \lambda_1^* = \lambda_0 + \lambda_1 + 1) \).

Now, \( 6\lambda_1^* - \lambda_0^* = 9m + 5 = 9m' - 4 \quad (m' = m + 1) \)

Hence \( D^* \) belongs to case (ii) above, i.e., \( b^* = \lambda_0 + m' \), or \( b^* = \lambda_0 + m + 1 = \text{next largest integer to } \lambda_0 + m + \frac{1}{2} \).

\( \therefore \) Any \( D \) with \( (\lambda_0, \lambda_1) \) as in here will be s.i. w.r.t. \( D^* \) with \( \lambda_0^*, \lambda_1^* \) as in (ii)
Case: 3 $\lambda_0 = 2 \mod (3)$

(a) $\lambda_0 = 2$.

The only BB combination gives $M = 2$, $b = 2$, $K_i = 1$ all $i$. It is easy to see that no BTIB design can be constructed for $\lambda_i = 1, 2, 3$ and that $b \geq 8$ for $\lambda_i = 4$. Hence we start with $D_2 = (8, 2, 5)$.

Let $D^* = D_2 + (\lambda_1 - 5)D_1$ for $\lambda_1 \geq 5$. Note that $K - K_i = 3\lambda_i - 1 = 2 \mod (3)$. By Truth (6.2.1) each treatment must be replicated twice in at least two blocks. Hence, all together $m = 8$ within block replications necessary. Then by Lemma (6.3.3)

$$b \geq \left(\frac{N - 2}{6}\right)^+ + b_{\lambda_0} = \frac{6\lambda_i + 6}{6} + 2 = \lambda_i + 3$$

Since $D^*$ has $b^* = \lambda_i + 3$, it is the best.

(b) $\lambda_0 = 5$

Again it is easily seen it no BTIB design exists for $\lambda_i < 4$.

Let $D^* = D_5 + (\lambda_i - 4)D_1$ for $\lambda_i \geq 4$.

Here, too, $K - K_i = 2 \mod (3)$ and the fact that $D^*$ has the minimum possible number of blocks follows just as in (a).

(c) $\lambda_0 = 8$.

Let $D^* = \begin{cases} D_7 + (\lambda_i - 4)D_1 & \text{if } \lambda_i \geq 4 \\ ZD_4 + \lambda_i D_1 & \text{if } \lambda_i < 3 \end{cases}$

Here the result follows just as in (a). For $\lambda_0 = 3$ one gets s.i. BTIB designs only.

(d) $\lambda_0 \geq 11, 3\lambda_i - 2\lambda_0 \geq -7$.

Let $D^* = D_8 + \frac{\lambda_0 - 11}{3}D_3 + \frac{3\lambda_i - 2\lambda_0 + 7}{3}D_1$
Take $j_0 = \begin{cases} \frac{\lambda_0 - 2}{3} & \text{if } \lambda_0 \text{ odd} \\ \frac{\lambda_0 + 1}{3} & \text{if } \lambda_0 \text{ even} \end{cases}$

Then $M_{j_0} = 4\lambda_0 - 6$ and $b_{j_0} = \frac{4\lambda_0 - 2}{3}$ since $K - K_i = 2 \text{ mod } (3)$ Lemma (6.3.3) given (with $m = 8$).

$\text{Let } D^* = 2D_4 + \frac{6\lambda_1 + \lambda_0 + 8}{9} D_3 + \frac{2\lambda_0 - 3\lambda_1 - 16}{9} D_6$

$\text{Take } j_0 = \begin{cases} m & \text{if } \lambda_0 \text{ is odd} \\ m + 1 & \text{if } \lambda_0 \text{ is even} \end{cases}$

Then $M_{j_0} = N + 8$ and $b \geq b_{j_0} = \lambda_0 + m = b^*$ QED

(e) $\lambda_0 \geq 11, 3\lambda_1 - 2\lambda_0 \geq -7$

Observe that $6\lambda_1 - \lambda_0 - L \text{ mod } (3) L = 1, 4 \text{ or } 7$.

(i) $6\lambda_1 + \lambda_0 = 1 \text{ mod } (9) = 9m; \quad m = 0, 1, 2, \ldots$

Let $D^* = 2D_4 + \frac{6\lambda_1 + \lambda_0 + 8}{9} D_3 + \frac{2\lambda_0 - 3\lambda_1 - 16}{9} D_6$

$\text{Take } j_0 = \begin{cases} m & \text{if } \lambda_0 \text{ is odd} \\ m + 1 & \text{if } \lambda_0 \text{ is even} \end{cases}$

Then $M_{j_0} = N + 8$ and $b \geq b_{j_0} = \lambda_0 + m = b^*$ QED

(ii) $6\lambda_1 - \lambda_0 = 4 \text{ mod } (9) = 9m-5 \quad m = 0, 1, 2$

\[
\begin{cases} 1 & \text{if } m = 0 \\ \text{as above} & \text{if } m > 0 \end{cases}
\]

Then $M_{j_0} = \begin{cases} N + 14 & \text{if } m = 0 \\ N + 5 & \text{if } m > 0 \end{cases}$ and $b_{j_0} = \begin{cases} \lambda_0 + 1 & \text{if } m = 0 \\ \lambda_0 + m & \text{if } m > 0 \end{cases}$

Here $K - K_i = 2 \text{ mod } (3)$, hence Lemma (6.3.3) yields

$\left( -\frac{14 + 8}{6} \right)^+$

$\text{Let } j_0 = \begin{cases} 1 & \text{if } m = 0 \\ \text{as above} & \text{if } m > 0 \end{cases}$

Then $M_{j_0} = \begin{cases} N + 14 & \text{if } m = 0 \\ N + 5 & \text{if } m > 0 \end{cases}$ and $b_{j_0} = \begin{cases} \lambda_0 + 1 & \text{if } m = 0 \\ \lambda_0 + m + \frac{1}{2} & \text{if } m > 0 \end{cases}$

Here $K - K_i = 2 \text{ mod } (3)$, hence Lemma (6.3.3) yields

$\left( -\frac{14 + 8}{6} \right)^+$

$\text{Let } j_0 = \begin{cases} 1 & \text{if } m = 0 \\ \text{as above} & \text{if } m > 0 \end{cases}$

Then $M_{j_0} = \begin{cases} N + 14 & \text{if } m = 0 \\ N + 5 & \text{if } m > 0 \end{cases}$ and $b_{j_0} = \begin{cases} \lambda_0 + 1 & \text{if } m = 0 \\ \lambda_0 + m + \frac{1}{2} & \text{if } m > 0 \end{cases}$

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Consider $D^* = (b^*, \lambda_0^*, \lambda_1^* = \lambda_1 + 1)$ $D^*$ belongs to case (i) abc. Hence $b^* = \lambda_0 + m + 1 =$ next largest integer to $\lambda_0 + m + \frac{1}{2}$. So any $D$ with $(\lambda_0, \lambda_1)$ as in here will be s.i. w.r.t. a $D^*$ with $(\lambda_0, \lambda_1 + 1)$ as in (i).

(i) $6\lambda_0 - \lambda_1 = 7 \mod(9) = 9m - 2 \quad m = 0, 1, 2 \ldots$

Let $D^* = D_1 + 2D_1 + \frac{6\lambda_1 - \lambda_0 + 2}{9} D_3 + \frac{2\lambda_0 - 3\lambda_1 - 13}{9} D_6$

Take $j_0$ as in (i)

Then $M_{j_0} = N + 2$ and $b_{j_0} = \lambda_0 + m$

Since $K - K_i = 2 \mod(3)$ Lemma (6.3.3) yields

\[ b \geq \left( -2 + \frac{8}{6} \right)^* + b_{j_0} = \lambda_0 + m + 1 = b^* \]

Hence $D^*$ is the best.

It is noticed in above cases that $(6\lambda_1 - \lambda_0)$ was not allowed to take all possible values. Now it will be shown that for all such uncovered values of $(6\lambda_1 - \lambda_0)$ one gets s.i. BTIB designs.

Truth (6.2.4) together with Lemma (6.3.2) yields $b \geq \lambda_0$ no matter what $\lambda_1$ is.

(i) $\lambda_0 = 0 \mod(6)$

Consider $D^* = \lambda_1 D_6 = (b = \lambda_0, \lambda_0, \lambda_1 = \frac{\lambda_0}{6})$. Any design $D$ with $\lambda_1 < \frac{\lambda_0}{6}$ will be s.i. w.r.t. $D^*$ since it must have at least $b = \lambda_0$ blocks as $D^*$ but smaller $\lambda_1$. That means whenever $6\lambda_1 - \lambda_0 < 0$ and $\lambda_0 = 0 \mod(6)$ one can get only s.i. BTIB designs.
(ii) \( \lambda_0 = 2 \text{ mod } (6) \)

Let \( D^* = 2D_4 + \lambda_1 + D_6 = \left( b = \lambda_0, \lambda_0, \lambda_1 = \frac{\lambda_0 - 8}{6} \right) \).

Hence any \( D \) with \( \lambda_1 < \frac{\lambda_0 - 8}{6} \) or \( 6\lambda_1 - \lambda_0 < -8 \) will be s.i. w.r.t. \( D^* \).

(iii) \( \lambda_0 = 4 \text{ mod } (6) \)

Let \( D^* = D_4 + \lambda_1D_6 = \left( b = \lambda_0, \lambda_0, \lambda_1 = \frac{\lambda_0 - 4}{6} \right) \)

So any \( D \) with \( 6\lambda_1 - \lambda_0 < -4 \) is s.i. w.r.t. \( D^* \).

(b) \( \lambda_0 = \text{odd} \Rightarrow b \geq \lambda_0 + 1 \)

(i) \( \lambda_0 = 1 \text{ mod}(6) \geq 7 \) (\( \lambda_0 = 1 \) is trivial)

Let \( D^* = D_3 + D_4 + (\lambda_1 - 2)D_6 = \left( b = \lambda_0 + 1, \lambda_0, \lambda_1 = \frac{\lambda_0 + 5}{6} \right) \) so any \( D \) with \( 6\lambda_1 - \lambda_0 < 5 \) is s.i. w.r.t. \( D^* \).

(ii) \( \lambda_0 = 3 \text{ mod } (6) \)

Let \( D^* = D_3 + (\lambda_1 - 2) = \left( b = \lambda_0 + 1, \lambda_0, \lambda_1 = \frac{\lambda_0 + 9}{6} \right) \) so any \( D \) with \( 6\lambda_1 - \lambda_0 < 9 \) is w.r.t. \( D^* \).

(iii) \( \lambda_0 = 5 \text{ mod } (6) \)

Let \( D^* = D_3 + 2D_4 + (\lambda_1 - 2)D_6 = \left( b = \lambda_0 + 1, \lambda_0, \lambda_1 = \frac{\lambda_0 + 1}{6} \right) \) so any \( D \) with \( 6\lambda_1 - \lambda_0 < 1 \) is s.i. w.r.t. \( D^* \).

That makes the three cases considered before exhaustive, hence proof of the thermo is completed.
### 6.4 MINIMAL COMPLETE CLASS OF GENERATED DESIGNS (MCCGD) FOR K = 4

#### 6.4.1 TABLE MCCGD FOR p = 5, k = 4

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<tr>
<th>LABEL</th>
<th>DESIGNS</th>
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<th>( \lambda_0 )</th>
<th>( \lambda_1 )</th>
<th>n-ary</th>
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<td>0</td>
<td>3</td>
<td>2</td>
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<td>3</td>
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<td>3</td>
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<td>4</td>
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<td>3</td>
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<td>4</td>
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### 6.4.2 TABLE MCCGD FOR $p = 7$, $k = 4$

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<th>$\lambda_0$</th>
<th>$\lambda_1$</th>
<th>n</th>
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                   1 2 3 4 3 4 4  
                   2 5 5 6 6 5 5  
                   3 7 6 7 7 6 7  | 7 | 0 | 2 |
| D₂    | 0 1 1 1 1 1 2 2 2 2 2 2  | 11 | 0 | 3 4 |
| D₃    | 0 0 0 0 1 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3  
                   0 0 5 5 1 1 2 2 4 4 4 2 3 4 5 3 3 4  
                   1 3 6 6 3 5 2 3 5 6 7 6 5 4 6 7 5 6 4  
                   2 4 7 7 7 6 3 4 5 6 7 7 5 5 6 7 6 7 7  | 19 | 2 | 4 3 |
| D₄    | 0 0 0 0 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3  
                   3 4  
                   0 0 5 5 1 1 1 2 3 3 4 4 4 2 2 3 3 3 3 4 6 3  
                   4 5  
                   1 3 6 1 3 5 3 4 5 5 6 7 4 5 3 5 7 4 6 5  
                   4 6  
                   2 4 7 7 2 7 6 4 6 7 5 6 7 6 5 7 5 7 6 7 7  | 23 | 2 | 5 3 |
| D₅    | 0 0 0 0 0 0 0  
                   1 1 1 2 2 3 3  
                   2 4 5 4 5 4 6  
                   3 7 6 6 7 5 7  | 7 | 3 | 1 2 |
| D₆    | 0 0 0 0 0 0 0 1 1 1 1  
                   0 0 2 2 2 3 4 5 2 2 3  
                   0 3 3 4 4 6 4 5 4 5  
                   1 6 7 5 5 7 7 6 7 7 6  | 11 | 3 | 2 4 |
| D₇    | 0 0 0 0 0 0 0  
                   0 0 0 0 0 0 0  
                   1 2 3 4 5 6 7  
                   2 2 3 4 5 6 7  | 7 | 4 | 0 3 |
| D₈    | 0 0 0 0 0 0 0 0 1 1 1 1  
                   0 2 2 2 2 3 3 4 2 3 4 5  
                   1 4 5 6 7 3 4 5 5 2 4 6 6  
                   1 4 5 6 7 6 7 6 3 5 7 7  | 13 | 4 | 2 3 |
| D₉    | 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 2 2  
                   0 2 2 2 2 3 3 4 5 1 3 3 3 4 5 2 4  
                   1 3 4 5 6 3 4 7 6 1 4 4 5 6 6 3 5  
                   2 7 4 5 6 5 5 7 2 5 6 7 7 7 6  | 17 | 4 | 3 3 |
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<td>1 2 3 4 6 7 7 5 6 7 4 5 6 4 7 7</td>
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<td>0 1 2 3 3 3 4 4 4 6 1 2 3 4 5 2 3 5</td>
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<td>1 2 3 6 3 4 5 4 5 5 6 2 5 3 7 6 3 4 7</td>
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<tr>
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<td>1 2 7 7 5 7 7 6 5 6 7 4 5 6 7 6 6 4 7</td>
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<td>$D_{12}$</td>
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<tr>
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<td>0 0 0 0 0 0 0 1 1 1 2 3</td>
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<tr>
<td></td>
<td>0 2 2 3 3 4 5 2 4 6 4 5</td>
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<td>1 5 7 4 6 7 6 3 5 7 6 7</td>
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### 6.4.3 Table MCCGD FOR p = 8, k= 4

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<td>2</td>
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<td>4</td>
<td>3</td>
</tr>
<tr>
<td>D₃</td>
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<td>3</td>
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6.5 BALANCED TREATMENTS N-ARY BLOCK DESIGNS (BTNDB)

6.5.1 Linear Model and Definition

The usual additive linear model for n-ary block design will be assumed as

$$ y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} $$

(6.5.1)

Where \( y_{ijh}, \alpha_i, \beta_j \) and \( e_{ijh} \) are assumed as usual way with

$$ \sum_{i=1}^{p} \alpha_i = 0, \sum_{j=1}^{B} B_j = 0 \text{ and } e_{ijh} \text{ are assumed to be i.i.d. N}(0, \sigma^2). $$

6.5.2 Definition

We define a BALANCED TREATMENT n-ARY BLOCK (BTN) design as an arrangement of \( V(=p+1) \) treatments in \( B \) blocks of size \( K \), such that the \( i \)-th treatment occurs in the \( j \)-th block \( n_{ij} \) times and altogether \( R \) times when \( n_{ij} \) can take the values 0,1,2…… (n-1). We say the design is balanced treatment n-ary block (BTN) design if the following conditions are satisfied.

$$ \wedge_0 = \sum_{j=1}^{B} n_{ij} n_{kj} \text{ is a constant for } 0 \neq k; k = 1,2,\ldots,p \text{ and} $$

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\( \wedge_1 = \sum_{j=1}^{B} n_{ij} n_{kj} \) is another constant \( i \neq k; k = 1,2 \ldots p \) \quad (6.5.2)

Hence a BTNB design is such that each test treatment appears in the same block with the control, the same number of times (= \( \wedge_0 \)) over the design, and any pair of test treatment appears together (in the same block) the same total number of times (= \( \wedge_1 \)) over the design.

Let \( C(B,K,p) \) denote the class of all possible \( n \)-ary block designs with \( B \) blocks of size \( K \) each and \( p+1 \) treatment indexed \( 0,1,2 \ldots p; 0 \) being the control.

As usual in BTNB design, for a design \( d \in C(B,K,p) \) the information matrix \( M(d) \) of order \((n \times p)\), where \((i,L)\)th entry is

\[
M_{iL}(d) = \begin{cases} 
    n_i(d) - \frac{1}{K} \sum_{j=1}^{B} n^2_{ij}(d) & (i = L) \\
    -\frac{1}{K} \wedge_{iL}(d) & (i \neq L)
\end{cases}
\]

(6.5.3)

where \( R_i(d) = \sum_{j=1}^{B} n_{ij}(d) \) = number of times the \( i \)-th treatment appears in the entire design. This matrix has all diagonal elements equal \((= \frac{1}{K} \wedge_0 + \frac{1}{K} (p-1) \wedge_1)\) and off-diagonal elements equal \((= -\frac{1}{K} \wedge_1)\).

Since \( M^{-1} \) is the variance–covariance matrix of the vector of estimates \((\hat{\alpha}_0 - \hat{\alpha}_1, \hat{\alpha}_0 - \hat{\alpha}_2, \ldots, \hat{\alpha}_0 - \hat{\alpha}_p)\), BTNB design provides BLUEs \( \hat{\alpha}_i - \hat{\alpha}_1 \) \((1 \leq i \leq p)\) with the property that

\[
\text{Var} (\hat{\alpha}_0 - \hat{\alpha}_i) = \tau^2 \sigma^2 \quad (1 \leq i \leq p) \quad \text{Where } \tau^2 \text{ is a constant, and}
\]

\[
\text{Cov} (\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_{i'}) = \rho \quad (1 \leq i, i' \leq p) \quad \rho \text{ is a constant} \quad (6.5.4)
\]
The values of \( \tau^2 \) and \( \rho \) are given as

\[
\tau^2 = \frac{K(\Lambda_0 + \Lambda_1)}{\Lambda_0 (\Lambda_0 + \rho \Lambda_1)} \quad \text{and} \quad \rho = \frac{\Lambda_1}{\Lambda_0 + \Lambda_1}
\]  

(6.5.5)

All other definitions and conditions for binary designs are extended to n-ary designs with the modification of notation with capital letters from small letters. Notz (1981) obtained extensive results on d- and E-optimality of BTIB designs for binary cases. He showed that a BIBD (which is also a BTIB design) retains its D-optimality for the problem of multiple comparisons with the control. We recall that a D-optimal design minimizes the determinant of the variance-covariance matrix \( M^{-1} \), and E-optimal design minimizes the maximum eigenvalue of \( M^{-1} \), and an A-optimal design minimizes the trace of \( M^{-1} \), i.e., minimizes \( \sum_{i=1}^{p} \text{Var}(\hat{\alpha}_i - \alpha_i) \). He also provided E-optimality of a certain subset of BTIB designs. The results he obtained on A-optimality of BTIB designs are restrictive. Majumdar (1981), at the same time, obtained similar results on D- and E-optimality of BTIB designs, but he was able to get more extensive results on A-optimality. His work will be referred later.

The following example shows how reasonable an E-optimal design can be (From now on, without loss of generality \( \sigma^2 \) is taken to be equal to one).

### 6.5.1 Example:

Let \( p=5 \), \( K=4 \), \( B=10 \). From table (4.4.1) above all implementable (i.e. with \( \Lambda_0 > 0 \)) BTIB designs can be obtained. These are given in Table 4.5.1 (given below), together with the variances they provide.

\[
\text{(Note that } \text{Var}(\hat{\alpha}_i - \alpha_i) = \frac{2K\sigma^2}{\Lambda_0 + \rho \Lambda_1} \text{)} \quad i,j = 1,2,\ldots,p, (i \neq j).
\]
### 6.5.1 Table All Implementable BTIB Designs for (B=10, p=5,K=4)

<table>
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<th>$\wedge_1$</th>
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<th>$\text{Var}(\hat{\alpha}_0, \hat{\alpha}_i)$</th>
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<td>4</td>
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<td>0.3478</td>
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<tr>
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<td>3</td>
<td>0.2857</td>
<td>0.3810</td>
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</table>

If one is restricted to BTNB designs only, then $D_{16}$ is $E-$optimal because it minimizes $\wedge_0^{-1} (\text{maximum eigenvalue of } M^{-1} \wedge_0^{-1})$. But $D_{16}$ is certainly a bad choice because $D_{13}$ yields smaller variances for estimating both $(\alpha_0 - \alpha_i)$ and $(\alpha_0 - \alpha_j)$. It can be explained that D-optimality, either is not proper choice in this setting.

Since A-optimality has a natural statistical meaning and it picks up really desirable designs for the problem of multiple comparisons with the control, we will restrict our attention to this criterion.

In the next section some useful truths and lemmas required for our subsequent works are stated and proved utilizing the previous chapter concepts, definitions and conditions. Let $R_0$ be the number of units allocated to control in a design with parameters $(B, K, p)$. How should $R_0$ be chosen and how should the allocation of $RQ$ controls to $B$ blocks be performed so that an A-optimal design can be obtained will be discussed in detail in the subsequent section. The final section treats that this problem and a suitable table is framed to determine the optimal choice of $R_0$ exactly. The guidelines for the construction of the best design (in the sense of A-optimality) for this optimal value of $R_0$ will be explained in detail.
### 6.6 CONSTRUCTED TABLES FOR N-ARY DESIGNS

#### 6.6.1 Table MCCGD FOR p=9, K=4

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155
| Table 6.6.1 (Continued) |  |
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| 7 4 8 5 9 6 6 9 9 9 9 8 8 7 7 9 8 7 | | | | |
| **D_10**                 | 16 | 5 | 2 | 3 |
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| 1 3 5 2 4 6 8 8 8 5 5 6 6 6 5 6 6 6 5 6 6 7 7 | | | | |
| 2 4 5 7 8 9 9 9 9 7 8 9 7 8 9 9 9 7 8 8 8 9 7 9 8 | | | | |
| **D_11**                 | 9  | 4  | 0 | 3 |
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| 1 2 3 4 5 6 7 8 9 | | | | |
| **D_12**                 | 12 | 4 | 1 | 2 |
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| 2 3 4 5 3 4 5 4 7 6 8 7 | | | | |
| 9 9 7 6 6 6 8 7 5 9 9 9 8 | | | | |
| **D_13**                 | 21 | 4 | 2 | 3 |
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| 1 1 1 2 2 3 4 4 4 4 5 5 6 6 1 4 7 2 4 7 3 5 8 | | | | |
| 1 3 4 2 5 6 4 8 5 9 6 7 5 6 9 6 4 7 4 5 8 | | | | |
| 2 3 7 3 8 9 9 8 7 9 8 7 8 6 9 9 5 8 7 6 9 | | | | |
| **D_14**                 | 20 | 5 | 2 | 3 |
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| 9 7 5 6 8 8 9 8 5 7 6 8 9 7 9 9 9 9 9 9 8 | | | | |
| **D_15**                 | 29 | 5 | 3 | 3 |
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| 0 0 1 2 3 4 4 4 4 4 5 5 6 7 1 2 2 2 5 5 6 2 4 6 6 3 4 4 5 | | | | |
| 1 2 3 8 7 7 4 5 6 7 6 8 7 8 4 3 3 3 7 8 6 5 4 8 9 8 6 7 9 5 | | | | |
| 1 2 3 9 9 8 6 5 8 8 6 9 9 9 9 4 5 6 7 8 7 8 8 9 8 7 9 9 | | | | |
| **D_16**                 | 23 | 6 | 2 | 3 |
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| 0 0 1 1 1 2 2 3 3 4 4 5 6 7 7 8 2 3 5 3 4 4 | | | | |
| 1 2 3 1 2 4 2 5 3 6 5 5 5 6 7 9 8 6 5 6 4 6 5 | | | | |
| 4 5 6 7 3 8 8 9 9 7 9 9 9 8 8 9 9 9 8 8 7 7 8 | | | | |
| **D_17**                 | 18 | 7 | 1 | 3 |
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| 1 1 2 2 3 3 4 4 5 2 4 6 4 6 5 4 6 8 | | | | |
| 8 9 5 7 6 8 6 7 9 3 5 7 8 9 7 9 8 | | | | |
| **D_18**                 | 27 | 7 | 2 | 3 |
| 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 2 2 | | | | |
| 0 0 1 1 2 2 3 3 4 4 4 4 5 5 6 6 7 7 8 2 2 3 3 3 3 | | | | |
| 1 2 3 1 9 2 8 3 7 4 4 5 5 5 5 6 6 7 9 8 4 4 5 5 6 6 | | | | |
| 1 2 3 6 9 5 8 4 7 8 3 6 6 7 9 7 8 8 9 9 7 8 8 9 9 | | | | |
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</tr>
</tbody>
</table>

| D_{23} | All combinations of two digits from \{1,2,3,\ldots,9\} bordered by two rows of zeros. | 36 | 16 | 1 |
### Table MCCGD FOR $p = 10$, $K = 4$

<table>
<thead>
<tr>
<th>LABEL</th>
<th>DESIGNS</th>
<th>B</th>
<th>$\lambda_0$</th>
<th>$\lambda_1$</th>
<th>n</th>
</tr>
</thead>
</table>
| $D_1$ | $1111111222233346$  
223445436754557  
356767458969888  
8998xx76xx7xx99 | 15 | 0 | 2 | 2 |
| $D_2$ | $1111111111222222233446$  
133344567333445656557  
14585789845678697978678  
26796xx98979xx8xx989 | 23 | 0 | 3 | 4 |
| $D_3$ | $000001111111122222222333444445$  
000001122256782256783456445666  
1357934379x7834x97878956785679  
2468x56489x9x65x99xx789878x | 30 | 2 | 3 | 3 |
| $D_4$ | $00000001111111222222233344$  
45566677  
005566611222334444223334433444  
45566688  
1377885639957859x5659x57856775  
6787899  
2499x8874xx67869x8769x678x988x9  
9x9xx | 37 | 2 | 4 | 3 |
| $D_5$ | $00000000000$  
0000000000  
0000000000  
123456789x | 10 | 3 | 0 | 4 |
| $D_6$ | $000000000001122$  
0001234563434  
1357987875665  
246x8xx99x789x | 13 | 3 | 1 | 3 |
| $D_7$ | $0000000000011112223456$  
1112223342235344777  
44855866953656465889  
7x979x78x67x8998x9x | 20 | 3 | 2 | 2 |
| $D_8$ | $00000000000$  
0000000000  
123456789x  
123456789x | 10 | 4 | 0 | 3 |
| $D_9$ | $0000000000011111222235$  
00003456778922334434446  
12353456789x785566655587  
1246798xx899x9x78987xx | 24 | 4 | 2 | 3 |
| D | 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 2 3 3 3 4 4 |
| D | 0 2 2 3 3 4 5 5 6 6 8 8 9 9 2 2 2 3 3 3 4 4 4 4 3 4 5 6 4 5 7 5 6 |
| D | 1 2 4 5 3 6 7 5 6 7 8 x 9 5 6 5 7 6 5 6 7 7 6 9 7 5 6 8 7 8 |
| D | 1 3 4 8 x 9 6 7 7 9 x x 8 9 9 x x x 8 8 9 x x x 8 |
| D | 0 2 2 2 3 3 4 4 5 5 6 6 8 8 9 9 2 2 2 3 3 3 4 4 4 3 4 5 6 4 5 7 5 6 |
| D | 1 2 4 5 3 6 7 5 6 7 8 x 9 5 6 5 7 6 5 6 7 7 6 9 7 5 6 8 7 8 |
| D | 1 3 4 8 x 9 6 7 7 9 x x 8 9 9 x x x 8 8 9 x x x 8 |
| D | 0 2 2 2 3 3 4 4 5 5 6 6 8 8 9 9 2 2 2 3 3 3 4 4 4 3 4 5 6 4 5 7 5 6 |
| D | 1 2 4 5 3 6 7 5 6 7 8 x 9 5 6 5 7 6 5 6 7 7 6 9 7 5 6 8 7 8 |
| D | 1 3 4 8 x 9 6 7 7 9 x x 8 9 9 x x x 8 8 9 x x x 8 |
| D | 0 2 2 2 3 3 4 4 5 5 6 6 8 8 9 9 2 2 2 3 3 3 4 4 4 3 4 5 6 4 5 7 5 6 |
| D | 1 2 4 5 3 6 7 5 6 7 8 x 9 5 6 5 7 6 5 6 7 7 6 9 7 5 6 8 7 8 |
| D | 1 3 4 8 x 9 6 7 7 9 x x 8 9 9 x x x 8 8 9 x x x 8 |
| D | 0 2 2 2 3 3 4 4 5 5 6 6 8 8 9 9 2 2 2 3 3 3 4 4 4 3 4 5 6 4 5 7 5 6 |
| D | 1 2 4 5 3 6 7 5 6 7 8 x 9 5 6 5 7 6 5 6 7 7 6 9 7 5 6 8 7 8 |
| D | 1 3 4 8 x 9 6 7 7 9 x x 8 9 9 x x x 8 8 9 x x x 8 |

Table 6.6.2 (Continued)
<table>
<thead>
<tr>
<th>Table 6.6.2 (Continued)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>D22</strong> 0000000000000000000000000000000000000000000000000001</td>
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<tr>
<td>01111111222222222333333444444555566667782</td>
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<tr>
<td>1311345672234567335584455988x779l9983</td>
</tr>
<tr>
<td>2489xx56789xx567796686877999xx889xx4</td>
</tr>
<tr>
<td><strong>D23</strong> 0000000000000000000000000000000000001112345</td>
</tr>
<tr>
<td>011122222333333444455556667792466788</td>
</tr>
<tr>
<td>189x4578x4568967967xx889xx579xx9</td>
</tr>
<tr>
<td><strong>D24</strong> 0000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>0000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>02233444556622334455633447766445589988</td>
</tr>
<tr>
<td>18899xx67799xx88675566xx887799xxx9</td>
</tr>
<tr>
<td><strong>D25</strong> 000000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
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</tr>
<tr>
<td>123456789x33555666777555566677775555666777669369</td>
</tr>
<tr>
<td>123456789x44888899xxx999xx8888xx88899977xx47x</td>
</tr>
<tr>
<td><strong>D26</strong> 0000000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>0000000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>001111122223333344444445555666677883579</td>
</tr>
<tr>
<td>122789x345656789x56789x789x789x9x9x468x</td>
</tr>
<tr>
<td><strong>D27</strong> 0000000000000000000000000000000000000000000000000000000000</td>
</tr>
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</tr>
<tr>
<td>1133567145556778923555666789</td>
</tr>
<tr>
<td>224469x34669x889xx4377x8889x99x</td>
</tr>
<tr>
<td>0000000000000000000000000000000000000000000000000000000000</td>
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<td>0000000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>3333333334444445555566782</td>
</tr>
<tr>
<td>55556678855566789xx56793</td>
</tr>
<tr>
<td>9xx77899x6797xx889x99xx4</td>
</tr>
<tr>
<td><strong>D28</strong> 0000000000000000000000000000000000000000000000000000000000</td>
</tr>
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</tr>
<tr>
<td>224469x34669x889xx4377x8889x99x</td>
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<tr>
<td>0000000000000000000000000000000000000000000000000000000000</td>
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<tr>
<td>0000000000000000000000000000000000000000000000000000000000</td>
</tr>
<tr>
<td>3333333334444445555566782</td>
</tr>
<tr>
<td>55556678855566789xx56793</td>
</tr>
<tr>
<td>9xx77899x6797xx889x99xx4</td>
</tr>
</tbody>
</table>
| **D29** All Combinations of two digits from \([1,2,\ldots,9,x]\) bordered by a row of zeros. | 45 | 16 | 1 | 1