CHAPTER II
LOCAL FINITENESS AND SOME COVERING PROPERTIES IN FTS

1. INTRODUCTION

The concept of point - finite, locally finite and discrete collections of sets in general topology play a very important role, particularly in the study of paracompact spaces and their weaker and stronger forms such as Weakly paracompact, Strongly paracompact, Nearly paracompact, Nearly weakly paracompact, nearly strongly paracompact spaces etc.

The study of local finiteness, paracompactness and related concepts in fuzzy topology was initiated by S. R. Malghan and S. S. Benchalli [33] in 1981. In this paper, among other results, it was shown that a locally finite family of fuzzy sets is closure preserving. Further, paracompactness in fts was also introduced and studied.

Local finiteness in fts was also studied by J. G. Jiang [23] in 1981, S. L. Pu [41] in 1983. Mao- Kang Luo [35] in 1988 also introduced and studied local finiteness in fts which was based on the idea of quasi coincident introduced in [42]. The study of local finiteness in fts was also carried out by Bulbul and M. W. Warren [6] in 1993. All these concepts have certain limitations.
In this chapter the concept of $\alpha$-local finiteness of a family of fuzzy sets has been introduced as a natural generalization of local finiteness in general topology. The concepts of $\alpha$-point finite and $\alpha$-discrete families of fuzzy sets have also been introduced and studied. It is proved that every $\alpha$-discrete family of fuzzy sets is $\alpha$-locally finite and that every $\alpha$-locally finite family is $\alpha$-point finite. Further it is proved that every $\alpha$-locally finite family of fuzzy sets is closure preserving and that arbitrary union of a $\alpha$-locally finite family of closed fuzzy sets is a closed fuzzy set.

The class of perfect maps - continuous, closed surjections with compact point inverses - in general topology play an important role, particularly in the study of covering spaces. The concept of perfect maps in fts was first introduced and studied by F. T. Christoph [10] in 1977. This concept includes the compactness notion due to C. L. Chang [9] and the concept of fuzzy point due to Wong [53]. Since these two concepts have serious limitations there was a need for improvement of the concept of perfect maps.

The study of perfect maps was also carried out by K. K. Azad [3] in 1981, S. R. Malghan and S. S. Benchalli [34] in 1984 and Louis Friedler [32] in 1987 and others. The concept of perfect maps introduced by S. R. Malghan and S. S. Benchalli [34] is free from the concept of fuzzy point and includes the more general notion of $\alpha$-compactness due to Gantner et.al [16].
In the present chapter some properties of $\alpha$-perfect maps [34] have been explored. It is proved that the inverse image under a $\alpha$-perfect map, of a $\alpha$-compact crisp subset is $\alpha$-compact, the composition of $\alpha$-perfect maps is again a $\alpha$-perfect map and that the concept of $\alpha$-local finiteness is invariant under $\alpha$-perfect maps.

A. S. Mashhour and M. H. Ghanim [36] and A. Haydar E. S. [20] initiated the study of near compactness in fuzzy topological spaces. Near compactness is a generalization of compactness. A. Hayder E. S. [20] used the concept of a cover due to C. L. Chang [9] to introduce near compactness in fts. In this paper, it was proved among other results, that fuzzy almost continuous, fuzzy almost open image of a nearly compact fts is nearly compact fts, and that a fuzzy almost continuous image of a compact fts is a nearly compact fts.

In the present chapter, the concept of near $\alpha$-compactness in fts has been introduced and studied, using the concept of an $\alpha$-shading due to Gantner et.al [16] and fuzzy regular open sets due to K. K. Azad [4]. It is proved that near $\alpha$-compactness is a generalization of $\alpha$-compactness due to Gantner et.al [16]. The concept of near $\alpha$-compactness is characterized in terms of semiregular fts. It is also proved that a closed crisp subspace of a nearly $\alpha$-compact fts is nearly $\alpha$-compact fts. Further, it is also proved that near $\alpha$-compactness is invariant under fuzzy almost continuous, fuzzy almost open surjections.
The investigation of Lindelof property in fts was carried out by C. K. Wong [52], Pu Pao Ming and Liu Ying Ming [42], S. R. Malghan and S. S. Benchalli [33], Mao - Kang Luo [35] and many others.

In this chapter, nearly $\alpha$ - Lindelof property for fts has been introduced and studied. Nearly $\alpha$ - Lindelof property is a generalization of $\alpha$ - Lindelof property due to S. R. Malghan and S. S. Benchalli [33], $\alpha$ - compactness due to Gantner et.al [16] and nearly $\alpha$ - compactness [Definition 4.4]. Nearly $\alpha$ - Lindelof property is characterized in terms of semiregular fts. Further it has also been proved that a closed crisp subspace of a nearly $\alpha$ - Lindelof fts is nearly $\alpha$ - Lindelof. The invariance under maps, of nearly $\alpha$ - Lindelof property has also been obtained.

Most of the ideas in this chapter are motivated by the work of Gantner et.al [16].

Throughout this chapter, for any $\alpha$ - concept, $\alpha \in [0, 1)$ and for any $\alpha^*$ - concept, $\alpha \in (0, 1]$, unless otherwise mentioned.

2. LOCAL FINITENESS IN FTS

Locally finite families of sets and related concepts play a very important role in general topological spaces, particularly in the study of paracompact spaces.

The study of local finiteness and paracompactness in fuzzy topological spaces, was initiated by S. R. Malghan and S. S. Benchalli [33] in 1981. Such a

In this section, $\alpha$ - locally finite, $\alpha$ - point finite and $\alpha$ - discrete families of fuzzy subsets have been newly introduced and studied. Here the ideas are motivated by the concept of $\alpha$ - shading family introduced and studied by Gantner, Steinlage and R. H. Warren [16]. In the present section, interrelationships among various concepts and the closure preserving property of $\alpha$ - locally finite family of fuzzy sets have been established, among other results.

The concept of $\alpha$ - point finite family of fuzzy sets is introduced in the following.

2.1 Definition: Let $\alpha \in [0,1)$ (resp. $\alpha \in (0,1]$). A family $\{A_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in a fuzzy topological space $(X, T)$ is said to be $\alpha$ - point finite (resp. $\alpha^*$ - point finite) if for each $x \in X, A_\lambda(x) > \alpha$ (resp. $A_\lambda(x) \geq \alpha$) for at most finitely many $\lambda \in \Lambda$.

The following new concepts of "an empty fuzzy set of order $\alpha$" and "nonempty fuzzy set of order $\alpha$" are going to be useful in the sequel.

2.2 Definition: Let $\alpha \in [0,1)$ (resp. $\alpha \in (0,1]$). A fuzzy set $A$ in a fts $(X, T)$ is said to be an empty fuzzy set of order $\alpha$ (resp. $\alpha^*$) if $A(x) \leq \alpha$ (resp. $A(x) < \alpha$) for each $x \in X$. 
A fuzzy set \( A \) is said to be nonempty of order \( \alpha \) (resp. \( \alpha^* \)) if there exists \( x_0 \in X \) such that \( A(x_0) > \alpha \) (resp. \( A(x_0) \geq \alpha \)).

The concept of locally finite family of fuzzy sets is introduced in the following.

**2.3 Definition**: A family \( \{ A_\lambda : \lambda \in \Lambda \} \) of fuzzy subsets in a fts \( (X, T) \) is said to be \( \alpha \)-locally finite (resp. \( \alpha^* \)-locally finite) in \( X \) if for each \( x \in X \) there exists an open fuzzy set \( U \) in \( X \) such that \( U(x) = 1 \) and \( U \cap A_\lambda \) is nonempty of order \( \alpha \) (resp. nonempty of order \( \alpha^* \)) for atmost finitely many \( \lambda \in \Lambda \).

The following observations are clearly evident.

**2.4 Remark**: 1) Every finite family of fuzzy sets in a fts is \( \alpha \)-locally finite (resp. \( \alpha^* \)-locally finite) and also \( \alpha \)-point finite (resp. \( \alpha^* \)-point finite) family.

2) Every subfamily of an \( \alpha \)-locally finite (resp. \( \alpha^* \)-locally finite) family is \( \alpha \)-locally finite (resp. \( \alpha^* \)-locally finite).

3) Every subfamily of an \( \alpha \)-point finite (resp. \( \alpha^* \)-point finite) family is \( \alpha \)-point finite (resp. \( \alpha^* \)-point finite).

As in the case of general topology, every locally finite family is point finite in case of fuzzy sets, which is contained in the following.

**2.5 Theorem**: Every \( \alpha \)-locally finite (resp. \( \alpha^* \)-locally finite) family is \( \alpha \)-point finite (resp. \( \alpha^* \)-point finite).

21
**Proof**: Let \((X, T)\) be a fts and \(\{A_\lambda : \lambda \in \Lambda\}\) be a \(\alpha\) - locally finite family of fuzzy sets in \(X\). Let \(x \in X\) By definition 2.3 , there exists an open fuzzy set \(U\) in \(X\) such that \(U(x) = 1\) and \(U \land A_\lambda\) is nonempty of order \(\alpha\) for atmost finitely many \(\lambda \in \Lambda\). Then it follows that \(A_\lambda(x) > \alpha\) for atmost finitely many \(\lambda \in \Lambda\). For , if \(A_\lambda(x) > \alpha\) is true for infinitely many \(\lambda \in \Lambda\), then \(U(x) \land A_\lambda(x) = (U \land A_\lambda)(x) > \alpha\) for infinitely many \(\lambda \in \Lambda\), so that \(U \land A_\lambda\) is nonempty of order \(\alpha\), by definition 2.2, for infinitely many \(\lambda \in \Lambda\), which is a contradiction to the fact that \(U \land A_\lambda\) is nonempty of order \(\alpha\) for atmost finitely many \(\lambda \in \Lambda\). But \(x \in X\) is arbitrary . Therefore , for each \(x \in X\), \(A_\lambda(x) > \alpha\) for atmost finitely many \(\lambda \in \Lambda\). So, by definition 2.1 , it follows that \(\{A_\lambda : \lambda \in \Lambda\}\) is \(\alpha\) - point finite.

The proof for \(\alpha^*\) - case is similar.

Locally finite families have the following property.

**2.6 Theorem**: Let \(\alpha \in [0, 1)\) ( resp. \(\alpha \in (0, 1]\) ). If \(\{A_\lambda : \lambda \in \Lambda\}\) and \(\{B_\gamma : \gamma \in \Gamma\}\) are any two \(\alpha\) - locally finite ( resp. \(\alpha^*\) - locally finite ) families of fuzzy sets in a fts \((X, T)\) then the family \(\{A_\lambda \land B_\gamma : (\lambda, \gamma) \in \Lambda \times \Gamma\}\) is also \(\alpha\) - locally finite ( resp. \(\alpha^*\) - locally finite ) in \(X\).

**Proof**: Let \(x \in X\). By definition 2.3 , there exist open fuzzy sets \(G\) and \(H\) in \(X\) such that \(G(x) = 1\), \(H(x) = 1\), \(G \land A_\lambda\) is nonempty of order \(\alpha\), for atmost finitely many \(\lambda \in \Lambda\) and \(H \land B_\gamma\) is nonempty of order \(\alpha\), for atmost finitely many \(\gamma \in \Gamma\).

Then by definition 2.2, there exists \(x_0 \in X\) such that \((G \land A_\lambda)(x_0) > \alpha\) for atmost
finitely many $\lambda \in \Lambda$ and there exists $y_0 \in X$ such that $(H \wedge B_\gamma)(y_0) > \alpha$ for at most
finitely many $\gamma \in \Gamma$. Now it is required to prove that $(G \wedge H) \wedge (A_\lambda \wedge B_\gamma)$ is
nonempty of order $\alpha$ for at most finitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. In other words, to
prove, there exists $x_0 \in X$ such that $[(G \wedge H) \wedge (A_\lambda \wedge B_\gamma)](x_0) > \alpha$ for at most
finitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$.

Suppose that, for each $z \in X$, $[\{(G \wedge H) \wedge (A_\lambda \wedge B_\gamma)\}(z) > \alpha$ is true for
infinitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. Then it follows that
$[\{(G \wedge A_\lambda) \wedge (H \wedge B_\gamma)\}(z) > \alpha$ for infinitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. That is,
$(G \wedge A_\lambda)(z) > \alpha$ for infinitely many $\lambda \in \Lambda$ and $(H \wedge B_\gamma)(z) > \alpha$ for infinitely
$\gamma \in \Gamma$. Therefore in particular, it follows that $(G \wedge A_\lambda)(x_0) > \alpha$ for infinitely
many $\lambda \in \Lambda$ and $(H \wedge B_\gamma)(y_0) > \alpha$ for infinitely many $y \in \Gamma$, which is a
contradiction.

Thus for every $x \in X$, there exists an open fuzzy set $G \wedge H$ in $X$ such that
$(G \wedge H)(x) = 1$ and $(G \wedge H) \wedge (A_\lambda \wedge B_\gamma)$ is nonempty of order $\alpha$, for at most
finitely many $(\lambda, \gamma) \in \Lambda \times \Gamma$. Thus $\{A_\lambda \wedge B_\gamma : (\lambda, \gamma) \in \Lambda \times \Gamma\}$ is $\alpha$-locally
finite.

The proof for $\alpha^*$-case is similar.

The concept of discrete family of fuzzy sets is introduced in the following.

2.7 Definition : Let $\alpha \in [0, 1)$ (resp. $\alpha \in (0, 1]$). A family $\{A_\lambda : \lambda \in \Lambda\}$ of
fuzzy sets in a fts $(X, T)$ is said to be $\alpha$-discrete (resp. $\alpha^*$-discrete), if for
each x ∈ X there exists an open fuzzy set U in X such that U(x) = 1 and U ∩ A_λ is nonempty of order α (resp. α*) for atmost one member λ ∈ Λ.

**2.8 Theorem**: Every α - discrete (resp. α* - discrete) family of fuzzy sets in a fts is α - locally finite (resp. α* - locally finite).

**Proof**: The proof follows immediately from definitions 2.3 and 2.7.

It is well known in general topology that a locally finite family of sets is closure preserving and that arbitrary union of closed sets is a closed set if the family is locally finite. These results are extended to families of fuzzy sets in the following result and its corollaries.

**2.9 Theorem**: If a family \{ A_λ : λ ∈ Λ \} of fuzzy sets in a fts (X, T) is a α - locally finite (resp. α* - locally finite), then the following results hold good.

1) \{ A_λ : λ ∈ Λ \} is also α - locally finite family.

2) For each subset Λ' of Λ, v{ A_λ : λ ∈ Λ'} is a closed fuzzy set.

**Proof**: 1) Let \{ A_λ : λ ∈ Λ \} be a α - locally finite family of fuzzy sets in a fts (X, T). Let x ∈ X, since \{ A_λ : λ ∈ Λ \} is α - locally finite, there is an open fuzzy set U in X such that U(x) = 1 and U ∩ A_λ is nonempty of order α for atmost finitely many λ ∈ Λ. Therefore by definition 2.2, there exists x_0 ∈ X such that (U ∩ A_λ)(x_0) > α for atmost finitely many λ ∈ Λ. That is, Min{U(x_0), A_λ(x_0)} > α, for atmost finitely many λ ∈ Λ. Since A_λ ≥ A_λ, it follows that,
Min \{ U(x_0), \bar{A}_\lambda(x_0) \} > \alpha, \text{ for atmost finitely many } \lambda \in \Lambda. \text{ Therefore } U \wedge \bar{A}_\lambda \text{ is nonempty of order } \alpha \text{ for atmost finitely many } \lambda \in \Lambda. \text{ Thus } \{ \bar{A}_\lambda : \lambda \in \Lambda \} \text{ is } \alpha \text{- locally finite family in } (X, T).

2) Let \Lambda' \subset \Lambda. \text{ Let } B = \bigvee \{ \bar{A}_\lambda : \lambda \in \Lambda' \}. \text{ To prove that } B \text{ is a closed fuzzy set, it is required to prove that } 1 - B \text{ is an open fuzzy set. From Theorem 2.7 [49], it is known that a fuzzy set } A \text{ in } X \text{ is open iff for each } x \in X \text{ satisfying } A(x) > 0 \text{ there exists an open fuzzy set } N_x \text{ such that } N_x \leq A \text{ and } N_x(x) = A(x). \text{ Let } x \in X \text{ such that } (1 - B)(x) > 0. \text{ Then } (1 - \bigvee_{\lambda \in \Lambda'} \bar{A}_\lambda)(x) = \bigwedge_{\lambda \in \Lambda'} (1 - \bar{A}_\lambda)(x) = \inf_{\lambda \in \Lambda'} \{ 1 - \bar{A}_\lambda(x) \} > 0. \text{ Therefore } 1 - \bar{A}_\lambda(x) > 0 \text{ for each } \lambda \in \Lambda'. \text{ Now } x \in X \text{ and from (1) of this theorem, it follows that, } \{ \bar{A}_\lambda : \lambda \in \Lambda' \} \text{ is } \alpha \text{- locally finite. Therefore there exits an open fuzzy set } U \text{ with } U(x) = 1 \text{ such that } U \wedge \bar{A}_\lambda \text{ is nonempty of order } \alpha \text{ for atmost finitely many } \lambda \in \Lambda', \text{ say } \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \Lambda'. \text{ Therefore, there exists } x_0 \in X \text{ such that }

( U \wedge \bar{A}_\lambda)(x_0) > \alpha \text{ for } \lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \text{ and }

( U \wedge \bar{A}_\lambda)(x) \leq \alpha, \text{ for each } x \in X \text{ and for } \lambda \neq \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \ldots (I).
Define $N = \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}})$. Clearly $N$ is an open fuzzy set and

$N(x) > 0$. For, $U(x) = 1 > 0$, and $(1 - \widetilde{A}_{\lambda_{i}})(x) > 0$ for every $\lambda \in \Lambda'$ which implies $(1 - \widetilde{A}_{\lambda_{i}})(x) > 0$ for $i = 1, 2, 3, \ldots, n$. Therefore,

\[
\min\{U(x), \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}})(x)\} > 0. \text{ That is } N(x) > 0.
\]

Let $\Lambda'' = \Lambda' \setminus \{\lambda_{1}, \lambda_{2}, \lambda_{3}, \ldots, \lambda_{n}\}$. Then $N(x) = U(x) \wedge \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}})(x)$

\[
= 1 \wedge \left[ \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}})(x) \right] = \bigwedge_{i=1}^{\aleph} \left[ (1 - \widetilde{A}_{\lambda_{i}})(x) \right] \geq \bigwedge_{i=1}^{\aleph} (1 - \widetilde{A}_{\lambda_{i}})(x) \wedge \bigwedge_{\lambda \in \Lambda'} (1 - \widetilde{A}_{\lambda})(x)
\]

\[
= \bigwedge_{\lambda \in \Lambda'} \left(1 - \widetilde{A}_{\lambda}\right)(x) = (1-B)(x). \text{ Therefore } N(x) \geq (1-B)(x) \quad \text{-- (II).}
\]

Now we prove that $N \leq 1-B$. Suppose $N \leq 1-B$ is false. Then there exists $z \in X$ such that $N(z) > (1-B)(z)$.

That is $[U \wedge \left( \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}}) \right)](z) > (1 - \bigvee_{\lambda \in \Lambda} \widetilde{A}_{\lambda})(z) = \bigwedge_{\lambda \in \Lambda'} (1 - \widetilde{A}_{\lambda})(z)$.

Suppose $\bigwedge_{\lambda \in \Lambda'} (1 - \widetilde{A}_{\lambda})(z) = \inf_{\lambda \in \Lambda'} (1 - \widetilde{A}_{\lambda})(z) = (1 - \widetilde{A}_{\lambda_{0}})(z)$ for some $\lambda_{0} \in \Lambda'$. Then $U(z) \wedge \left( \bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}}) \right)(z) > (1 - \widetilde{A}_{\lambda_{0}})(z)$, which implies

\[
\bigwedge_{i=1}^{n} (1 - \widetilde{A}_{\lambda_{i}})(z) > (1 - \widetilde{A}_{\lambda_{0}})(z) \quad \text{-- (III).}
\]

If $\lambda_{0} = \lambda_{i}$ for some $i = 1, 2, 3, \ldots, n$, then from (III) it follows that

$(1 - \widetilde{A}_{\lambda_{i}})(z) > (1 - \widetilde{A}_{\lambda_{i}})(z)$, for each $i = 1, 2, 3, \ldots, n$, which implies

26
(1- $\bar{A}_{\lambda_i}(z)$) > (1- $\bar{A}_{\lambda_i}(z)$), which is impossible, and therefore there is a contradiction.

If $\lambda_0 \neq \lambda_i$ for $i = 1, 2, 3, \ldots, n$, then we have

$$\bigwedge_{i=1}^{n} (1- \bar{A}_{\lambda_i})(z) \geq \bigwedge_{x \in X} (1- \bar{A}_{\lambda_i})(z) \text{ and } \bigwedge_{x \in X} (1- \bar{A}_{\lambda_i})(z) = (1- \bar{A}_{\lambda_i})(z).$$

Therefore

$$\bigwedge_{i=1}^{n} (1- \bar{A}_{\lambda_i})(z) \geq (1- \bar{A}_{\lambda_i})(z) \text{ for } \lambda_0 \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n.$$ (IV)

Since, from (I), $(U \wedge \bar{A}_{\lambda})(x) \leq \alpha$ for every $x \in X$ and for $\lambda \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$, it follows that, $\operatorname{Min} \{ U(x), \bar{A}_{\lambda}(x) \} \leq \alpha$, for each $x \in X$ and for $\lambda \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. But $U(x) = 1$. Therefore $\operatorname{Min} \{ U(x), \bar{A}_{\lambda}(x) \} = \bar{A}_{\lambda}(x) \leq \alpha$, for each $x \in X$ and for $\lambda \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. That is, $\bar{A}_{\lambda_i}(x) \leq \alpha$ for each $x \in X$ and for $\lambda \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. Which implies $\bar{A}_{\lambda_i}(z) \leq \alpha$ for $z \in X$ and for $\lambda_0 \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. That is, $1- \bar{A}_{\lambda_i}(z) \geq 1-\alpha$ for $\lambda_0 \neq \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$.

Now from (IV), $\bigwedge_{i=1}^{n} (1- \bar{A}_{\lambda_i})(z) \geq 1-\alpha$ which implies $(1- \bar{A}_{\lambda_i})(z) \geq 1-\alpha$ for $i = 1, 2, 3, \ldots, n$. That is, $(1- \bar{A}_{\lambda})(z) \geq 1-\alpha$ for $\lambda = \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. That is,

$$1- \bar{A}_{\lambda}(z) \geq 1-\alpha \text{ for } \lambda = \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n \text{ which implies } \bar{A}_{\lambda}(z) \leq \alpha \text{ for } \lambda = \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n.$$ (V)

Again from (I), we have $(U \wedge \bar{A}_{\lambda})(z) \geq \alpha$ for $\lambda = \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$. That is

$$[U(z) \wedge \bar{A}_{\lambda}(z)] \geq \alpha \text{ for } \lambda = \lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n.$$ Which implies $\bar{A}_{\lambda}(z) > \alpha$ for
\[ \lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \] Which is a contradiction to (V). Therefore \( N > 1-B \) is not true. Hence \( N \leq 1-B \) \(-\) (VI).

Therefore from (II) & (VI), \( N(x) = (1-B)(x) \). Thus it follows that \( N(x) = (1-B)(x) \) and \( N \leq 1-B \). By theorem (2.7) \(-\) \( [4^\dagger] \), (1-B) is an open fuzzy set so that B is a closed fuzzy set. This completes the proof of the theorem.

**2.10 Corollary**: If \( \{ A_\lambda : \lambda \in \Lambda \} \) is a \( \alpha \)-locally finite family of fuzzy sets in a fts \((X, T)\) then it is closure preserving.

**Proof**: Let \( \{ A_\lambda : \lambda \in \Lambda \} \) be a \( \alpha \)-locally finite family of fuzzy sets in a fuzzy topological space \((X, T)\). Then for each \( \lambda, A_\lambda \leq \bigvee_{\lambda \in \Lambda} A_\lambda \). Therefore

\[ \bigwedge_{\lambda \in \Lambda} A_\lambda \leq \bigvee_{\lambda \in \Lambda} A_\lambda, \] for each \( \lambda \in \Lambda \), which implies that \( \bigvee_{\lambda \in \Lambda} \bigwedge_{\lambda \in \Lambda} A_\lambda \leq \bigvee_{\lambda \in \Lambda} A_\lambda \).

On the other hand, it is known that \( A_\lambda \leq \bigwedge_{\lambda \in \Lambda} A_\lambda \), for each \( \lambda \in \Lambda \), which implies that \( \bigvee_{\lambda \in \Lambda} A_\lambda \leq \bigwedge_{\lambda \in \Lambda} A_\lambda \). From Theorem 2.9 it follows that \( \bigvee_{\lambda \in \Lambda} \bigwedge_{\lambda \in \Lambda} A_\lambda \) is a closed fuzzy set which contains \( \bigvee_{\lambda \in \Lambda} A_\lambda \). But \( \bigvee_{\lambda \in \Lambda} A_\lambda \) is the smallest closed fuzzy set containing \( \bigvee_{\lambda \in \Lambda} A_\lambda \). Therefore \( \bigvee_{\lambda \in \Lambda} \bigwedge_{\lambda \in \Lambda} A_\lambda \) is a closed fuzzy set in \( X \).

**2.11 Corollary**: If \( \{ A_\lambda : \lambda \in \Lambda \} \) is an \( \alpha \)-locally finite family of closed fuzzy sets in a fts \( X \) then \( \bigvee_{\lambda \in \Lambda} A_\lambda \) is a closed fuzzy set in \( X \).
The following concept of an $\alpha$-shading was introduced and studied by Gantner, Steinlage and R. H. Warren [16] which is a natural generalization of a cover.

**2.12 Definition [16]:** Let $0 \leq \alpha < 1$ (resp. $0 < \alpha \leq 1$). Let $(X, T)$ be a fts. A family $\mathcal{U}$ of fuzzy sets in $X$ is called an $\alpha$-shading (resp. $\alpha^*$-shading) of $X$ if for each $x$ in $X$ there exists a $U \in \mathcal{U}$ such that $U(x) > \alpha$ (resp. $U(x) \geq \alpha$).

An $\alpha$-shading (resp. $\alpha^*$-shading) $\mathcal{U}$ is said to be an open $\alpha$-shading (resp. open $\alpha^*$-shading) if each member of $\mathcal{U}$ is an open fuzzy set in $X$.

A subcollection of an $\alpha$-shading (resp. $\alpha^*$-shading) of $X$ which is also an $\alpha$-shading (resp. $\alpha^*$-shading) of $X$ is called an $\alpha$-subshading (resp. $\alpha^*$-subshading) of $\mathcal{U}$ for $X$.

The refinement of an $\alpha$-shading family was defined as follows, in [33].

**2.13 Definition:** Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ and $\mathcal{V} = \{ V_\gamma : \gamma \in \Gamma \}$ be two $\alpha$-shadings (resp. $\alpha^*$-shadings) of a fts $(X, T)$. Then $\mathcal{U}$ is said to be a refinement of $\mathcal{V}$ written $\mathcal{U} \leq \mathcal{V}$ if for each $\lambda \in \Lambda$ there is some $\gamma \in \Gamma$ such that $U_\lambda \leq V_\gamma$.

**2.14 Remark:** If $\mathcal{U}$ is an $\alpha$-shading (resp $\alpha^*$-shading) of $X$ and $\mathcal{V}$ is an $\alpha$-subshading (resp. $\alpha^*$-subshading) of $\mathcal{U}$ for $X$ then $\mathcal{V}$ is a refinement of $\mathcal{U}$.

Now, the following result is proved.
2.15 **Theorem**: If a $\alpha$-shading $\{U_\lambda : \lambda \in \Lambda\}$ of a fts $X$ has a $\alpha$-locally finite open refinement, then there exists a $\alpha$-locally finite open $\alpha$-shading $\{V_\lambda : \lambda \in \Lambda\}$ such that $V_\lambda \leq U_\lambda$ for every $\lambda \in \Lambda$.

**Proof**: Let $\{U_\lambda : \lambda \in \Lambda\}$ be a $\alpha$-shading family in a fts $X$. Let $\{W_\gamma : \gamma \in \Gamma\}$ be a $\alpha$-locally finite open refinement of $\{U_\lambda : \lambda \in \Lambda\}$. Then for each $\gamma$ there exists some $U_\lambda$ such that $W_\gamma \leq U_\lambda$. For each $\lambda \in \Lambda$, let $V_\lambda = \{W_\gamma : W_\gamma \leq U_\lambda\}$. Now clearly $V_\lambda \leq U_\lambda$, as $V_\lambda = \{W_\gamma : W_\gamma \leq U_\lambda\} \leq U_\lambda$. Also for each $\lambda$, $V_\lambda$ is an open fuzzy set in $X$.

Further $\{V_\lambda : \lambda \in \Lambda\}$ is a $\alpha$-shading of $X$. Let $x \in X$. Since $\{W_\gamma : \gamma \in \Gamma\}$ is an $\alpha$-shading of $X$, there exists $W_{r_\gamma}$ such that $W_{r_\gamma}(x) > \alpha$. Since $\{W_\gamma : \gamma \in \Gamma\}$ is refinement of $\{U_\lambda : \lambda \in \Lambda\}$, for $\gamma_0 \in \Gamma$, there exists $\lambda_0 \in \Lambda$ such that $W_{r_\gamma} \leq U_{\lambda_0}$.

Therefore $V_{\lambda_0}(x) = \{W_\gamma : W_\gamma \leq U_{\lambda_0}\}(x) > \alpha$, as $W_{r_\gamma} \leq U_{\lambda_0}$ and $W_{r_\gamma}(x) > \alpha$.

Thus for each $x \in X$, there exists $V_{\lambda_0} \in \{V_\lambda : \lambda \in \Lambda\}$ such that $V_{\lambda_0}(x) > \alpha$.

Therefore $\{V_\lambda : \lambda \in \Lambda\}$ is an $\alpha$-shading of $X$.

Finally $\{V_\lambda : \lambda \in \Lambda\}$ is $\alpha$-locally finite; Let $x \in X$. Since $\{W_\gamma : \gamma \in \Gamma\}$ is $\alpha$-locally finite, there exists an open fuzzy set $G$ in $X$ such that $G(x) = 1$ and $G \land W_\gamma$ is nonempty of order $\alpha$ for atmost finitely many $\gamma \in \Gamma$. That is $\{\gamma \in \Gamma : G \land W_\gamma \text{ is nonempty of order } \alpha\}$ is atmost finite. Now $G \land W_\gamma$ is nonempty of order $\alpha$ iff $G \land V_\lambda$ is nonempty of order $\alpha$; Suppose $G \land W_\gamma$ is...
nonempty of order $\alpha$. That is $G \land (\forall \{ W_\gamma : W_\gamma \leq U_\lambda \})$ is nonempty of order $\alpha$. That is $G \land V_\lambda$ is nonempty of order $\alpha$. Conversely, suppose $G \land V_\lambda$ is nonempty of order $\alpha$. Then $G \land (\forall \{ W_\gamma : W_\gamma \leq U_\lambda \})$ is nonempty of order $\alpha$. That is $\forall \{ G \land W_\gamma : W_\gamma \leq U_\lambda \}$ is nonempty of order $\alpha$. That is $G \land W_\gamma$ is nonempty of order $\alpha$, for some $\gamma \in \Gamma$. Therefore, since $\{ \gamma \in \Gamma : G \land W_\gamma$ is nonempty of order $\alpha \}$ is atmost finite, it follows that $\{ \lambda \in \Lambda : G \land V_\lambda$ is nonempty of order $\alpha \}$ is atmost finite. Therefore $\{ V_\lambda : \lambda \in \Lambda \}$ is $\alpha$ - locally finite. This completes the proof of the theorem.

3. $\alpha$ - PERFECT MAPS AND $\alpha$ - REGULARITY IN FTS


In this section, some properties of perfect maps introduced and studied by S. R. Malghan and S. S. Benchalli [34] have been explored. Further the concept of $\alpha$ - regularity has been introduced and characterized. It is also proved that $\alpha$ - regularity is invariant under $F$ - continuous, $F$ - open, $F$ - closed surjections.

The following concept of $\alpha$ - compactness is due to Gantner, Steinlage and Warren [16].
3.1 Definition [16]: Let $0 < \alpha < 1$ (resp. $0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-compact (resp. $\alpha^*$-compact) if each open $\alpha$-shading (resp. $\alpha^*$-shading) of $X$ has a finite $\alpha$-subshading (resp. $\alpha^*$-subshading).

Note that $1^*$-compactness and compactness in the sense of Chang [9] are equivalent notions.

The following concept of $\alpha$-perfect maps is due to S.R. Malghan and S.S. Benchalli [34].

3.2 Definition [34]: Let $0 < \alpha < 1$ (resp. $0 < \alpha < 1$). An $F$-closed, $F$-continuous function $f: X \to Y$ from a fts $X$ onto a fts $Y$ is said to be $\alpha$-Perfect (resp. $\alpha^*$-Perfect) if $f^{-1}(y)$ is $\alpha$-compact (resp. $\alpha^*$-compact) for each $y \in Y$.

Now, the following property is obtained.

3.3 Theorem: If $f: X \to Y$ is a $\alpha$-Perfect (resp. $\alpha^*$-Perfect) map from a fts $X$ onto a fts $Y$ and $K$ is $\alpha$-compact (resp. $\alpha^*$-compact) crisp subset of $Y$ then $f^{-1}(K)$ is $\alpha$-compact (resp. $\alpha^*$-compact) crisp subset of $X$.

Proof: Let $f: X \to Y$ be a $\alpha$-Perfect map and $K$ be a $\alpha$-compact crisp subset of $Y$. Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ be any open $\alpha$-shading of $f^{-1}(K)$. Then for each $y \in K$, $f^{-1}(y)$ is $\alpha$-compact and $\mathcal{U}$ is a $\alpha$-shading of $f^{-1}(y)$. Therefore there exists a finite $\alpha$-subshading $\{ U_\lambda : \lambda \in M(y) \}$, where $M(y)$ is a finite subset of $\Lambda$. Thus $f^{-1}(y) \subseteq \bigvee \{ U_\lambda : \lambda \in M(y) \}$, which is open. Then by Theorem 3.2 [34] (Theorem 5.15 Ch.1) there exists an open fuzzy set $V_y$ in $Y$ such that $V_y(y) = 1$ and
f^{-1}(V_y) \leq \mathcal{V} \{ U_\lambda : \lambda \in M(y) \}. Now \{ V_y : y \in K \} is an open \( \alpha \)-shading of K and K is \( \alpha \)-compact. Therefore it has a finite \( \alpha \)-subshading \{ V_y : y \in \Lambda \}. Then M = \bigcup \{ M(y) : y \in \Lambda \} is a finite subset of \( \Lambda \). Further it can be verified that \{ U_\lambda : \lambda \in M \} is a finite \( \alpha \)-subshading of \{ U_\lambda : \lambda \in \Lambda \} and hence \( f^{-1}(K) \) is \( \alpha \)-compact. The proof for \( \alpha^* \)-case is similar.

The following result is an extension of the corresponding result in topology.

3.4 Theorem: The composition of two \( \alpha \)-Perfect (resp. \( \alpha^* \)-Perfect) maps is \( \alpha \)-Perfect (resp. \( \alpha^* \)-Perfect).

Proof: Let \( f : X \to Y \) and \( g : Y \to Z \) be \( \alpha \)-Perfect maps, where \( X, Y \) and \( Z \) are fts's. Since \( f \) and \( g \) are \( F \)-closed and \( F \)-continuous it can be verified that \( gof \) is \( F \)-closed and \( F \)-continuous. Now let \( z \in Z \). Since \( g \) is \( \alpha \)-Perfect \( g^{-1}(z) \) is \( \alpha \)-compact in \( Y \). Also since \( f \) is \( \alpha \)-Perfect \( f^{-1}(g^{-1}(z)) \) is \( \alpha \)-compact. Thus for each \( z \in Z, (gof)^{-1}(z) \) is \( \alpha \)-compact in \( X \). Hence \( gof : X \to Z \) is \( \alpha \)-Perfect map. The proof of \( \alpha^* \)-case is similar.

The concept of \( \alpha \)-local finiteness is invariant under \( \alpha \)-perfect maps.

3.5 Theorem: If \( f : X \to Y \) is a \( \alpha \)-Perfect map and \( \{ A_\lambda : \lambda \in \Lambda \} \) is a \( \alpha \)-locally finite family of fuzzy subsets of \( X \), then \( \{ f(A_\lambda) : \lambda \in \Lambda \} \) is a \( \alpha \)-locally finite family of fuzzy subsets of \( Y \).

Proof: Let \( y \in Y \). Then \( f^{-1}(y) \) is \( \alpha \)-compact in \( X \). Since \( \{ A_\lambda : \lambda \in \Lambda \} \) is \( \alpha \)-locally finite family, for each \( x \in f^{-1}(y) \) there is an open fuzzy set \( U_x \) with \( U_x \subseteq f^{-1}(y) \).
$U_x(x) = 1$ and $U_x \land A_\lambda$ is nonempty of order $\alpha$ for at most finitely many $\lambda \in \Lambda$. That is $\Lambda_x = \{ \lambda \in \Lambda : U_x \land A_\lambda$ is nonempty of order $\alpha \}$ is finite. Therefore $\{ U_x : x \in f^{-1}(y) \}$ is an open $\alpha$-shading of $f^{-1}(y)$ and hence has a finite $\alpha$-subshading, say $\{ U_x : x \in B \}$ where $B$ is a finite subset of $f^{-1}(y)$. Thus $f^{-1}(y) \leq \bigvee \{ U_x : x \in B \}$ which is an open fuzzy set in $X$. Therefore by Theorem 3.2 [34] (Theorem 5.15, Ch. I) there exists an open fuzzy set $V_y$ in $Y$ such that $V_y(y) = 1$ and $f^{-1}(y) \leq \bigvee \{ U_x : x \in B \}$. Now $\{ \lambda \in \Lambda : V_y \land f(A_\lambda) \}$ is nonempty of order $\alpha$ is finite; if $V_y \land f(A_\lambda)$ is nonempty of order $\alpha$ then $f^{-1}(\emptyset) \land A_\lambda$ is nonempty of order $\alpha$ which implies $U_x \land A_\lambda$ is nonempty of order $\alpha$ for some $x_0 \in B$. Therefore $\lambda \in \Lambda_{x_0}$. Therefore $\lambda \in M$ where $M = \bigcup \{ \Lambda_x : x \in B \}$ is a finite subset of $\Lambda$. Therefore $\{ \lambda \in \Lambda : V_y \land f(A_\lambda) \}$ is nonempty of order $\alpha$ is finite.

Thus $\{ f(A_\lambda) : \lambda \in \Lambda \}$ is $\alpha$-locally finite in $Y$. Hence the proof.

The concept of regularity in fuzzy topology has been introduced and studied by many researchers. The following definition is a modification of definition 1.9 [33].

**3.6 Definition:** Let $0 \leq \alpha < 1$ (resp. $0 < \alpha \leq 1$). A fts $(X, T)$ is said to be $\alpha$-Regular (resp. $\alpha^*$-Regular) if for each $x \in X$ and a closed fuzzy set $A$ with $A(x) = 0$, there exist open fuzzy sets $G$, $H$ such that $G(x) > \alpha$ (resp. $G(x) \geq \alpha$), $A \leq H$ and $G \leq 1 - H$.

The concept of $\alpha$-regularity has the following characterization.
3.7 Theorem: Let \((X, T)\) be a fts. Then the following statements are equivalent.

1) \((X, T)\) is \(\alpha\)-regular.

2) For each \(x \in X\) and \(G \in T\) with \(G(x) = 1\) there exists \(H \in T\) with \(H(x) > \alpha\) such that \(H \leq H \leq G\).

3) For each \(x \in X\) and a closed fuzzy set \(A\) with \(A(x) = 0\) there exists a \(H \in T\) with \(H(x) > \alpha\) such that \(A < H \lor H < A\).

Proof: \((1) \Rightarrow (2)\). Let \(x \in X\) and \(G \in T\) with \(G(x) = 1\). Then \(1 - G\) is a closed fuzzy set and \((1 - G)(x) = 1 - G(x) = 1 - 1 = 0\). Therefore by \((1)\) there exist open fuzzy sets \(H\) and \(K\) such that \(H(x) > \alpha\), \(1 - G \leq K\) and \(H \leq 1 - K\). Now \(H \leq 1 - K\) implies that \(H \leq 1 - K = 1 - K\), as \(K\) is an open fuzzy set. Therefore \(H \leq 1 - K\). But \(1 - G < K\) implies that \(1 - K < G\). Therefore \(H \leq G\). But \(H \leq H\). Therefore \(H \leq H \leq G\) and hence \((2)\) holds.

\((2) \Rightarrow (3)\). Let \(x \in X\) and \(A\) be a closed fuzzy set with \(A(x) = 0\). Then \(1 - A\) is an open fuzzy set and \((1 - A)(x) = 1 - A(x) = 1 - 0 = 1\). Therefore from \((2)\) there exists \(H \in T\) with \(H(x) > \alpha\) such that \(H \leq H \leq 1 - A\). Therefore \((3)\) holds.

\((3) \Rightarrow (1)\). Let \(x \in X\) and \(A\) be a closed fuzzy set with \(A(x) = 0\). From \((3)\) there exists open fuzzy set \(H\) with \(H(x) > \alpha\) such that \(A \leq 1 - H\) or \(H \leq 1 - A\). Let \(G = 1 - H\). Then clearly \(G\) is an open fuzzy set, and \(A \leq 1 - H = G\) so that \(A \leq G\). Also \(H \leq H = 1 - G\) so that \(H \leq 1 - G\). Thus for each \(x \in X\) and
closed fuzzy set $A$ with $A(x) = 0$, there exist open fuzzy sets $H, G$ with $H(x) > \alpha$, $A \leq G$ and $H \leq 1 - G$. Hence $X$ is $\alpha$-regular fts. Therefore (1) holds.

The concept of $\alpha$-regularity is invariant under $F$-continuous, $F$-open, $F$-closed surjections, which is proved in the following.

**3.8 Theorem**: Let $f : X \rightarrow Y$ be an $F$-continuous, $F$-open, $F$-closed surjection. If $X$ is a $\alpha$-regular (resp. $\alpha^*$-regular) fts then $Y$ is $\alpha$-regular (resp. $\alpha^*$-regular).

**Proof**: Let $x \in X, y \in Y$ such that $f(x) = y$. Let $A$ be a closed fuzzy set in $Y$ with $A(y) = 0$. Then $f^{-1}(A)(x) = A(f(x)) = A(y) = 0$ and $f^{-1}(A)$ is closed fuzzy set in $X$, as $f$ is $F$-continuous. Since $X$ is $\alpha$-regular fts, by Theorem 3.7 (3) there exists an open fuzzy set $G$ with $G(x) > \alpha$ such that $\bar{G} \leq 1 - f^{-1}(A) = f^{-1}(1 - A)$. Since $f$ is $F$-open, it follows that $f(G)$ is an open fuzzy set in $Y$ such that $[f(G)](y) > \alpha$ and $f(G) \leq f(\bar{G}) \leq f(f^{-1}(1 - A)) = 1 - A$, as $f$ is onto. Therefore $f(\bar{G}) \leq 1 - A$. Since $f$ is $F$-closed, $\bar{f}(G) \leq f(\bar{G})$. Therefore $\bar{f}(G) \leq 1 - A$. Thus for $y \in Y$ and a closed fuzzy set $A$ with $A(y) = 0$ there exists an open fuzzy set $f(G)$ with $[f(G)](y) > \alpha$ such that $\bar{f}(G) \leq 1 - A$. Hence by theorem 3.7 (3) it follows that, $Y$ is $\alpha$-regular. The proof of $\alpha^*$-case is similar.
4. NEARLY $\alpha$ - COMPACT FTS


In this section, a new concept of near $\alpha$ - compactness has been introduced and studied. Among other results, a characterization, a hereditary property and an invariance under maps of nearly $\alpha$ - compact fts, have been obtained.

The following definition is due to K. K. Azad [4].

4.1 Definition [4]: A fuzzy set $A$ of a fts $(X, T)$ is said to be a fuzzy regular open if it is the fuzzy interior of closed fuzzy set. That is $A$ is fuzzy regular open if

$$\text{Int}(\text{cl}A) = \overline{A} = A.$$  

A fuzzy set $A$ of a fts $(X, T)$ is said to be a fuzzy regular closed if it is the closure of interior of the fuzzy set. That is $A$ is fuzzy regular closed if

$$\text{cl}(\text{Int}A) = \overline{A} = A.$$  

Note that the complement of every fuzzy regular open (closed) set is fuzzy regular closed (open) set.

Every fuzzy regular open (closed) set is a fuzzy open (closed) set.

4.2 Definition [12]: A collection $\{A_i : i \in I\}$ of fuzzy sets $A_i$ nonempty set $X$ is a cover of $X$ iff

$$\bigvee_{i \in I} A_i = 1_X.$$
Near compactness was introduced and studied by Haydar E. S. [20] which is contained in the following.

4.3 Definition [20]: A fts (X, T) is said to be nearly compact if every fuzzy regular open cover of X has a finite subcover.

The concept of nearly α-compact fts is introduced in the following.

4.4 Definition: A fts (X, T) is called a nearly α-compact (resp. nearly α*-compact) if every fuzzy regular open α-shading (resp. α*-shading) of X has a finite α-subshading (resp. α*-subshading).

Nearly α-compactness is a generalization of α-compactness, which is justified in the following.

4.5 Theorem: Every α-compact fts is nearly α-compact.

Proof: Let (X, T) be an α-compact fts. To prove that (X, T) is nearly α-compact fts. Let U be any fuzzy regular open α-shading of X. Then U is an open α-shading of X. Since (X, T) is α-compact, U has a finite α-subshading say V = {V_i; i = 1, 2, 3, ..., n}. Then V' = {V^α_i; i = 1, 2, 3, ..., n} is the required finite α-subshading of U. Thus every fuzzy regular open α-shading has a finite α-subshading. Hence (X, T) is nearly α-compact fts.

The concept of fuzzy semiregularization was introduced and studied by M. N. Mukherjee and B. Ghosh [37] which is contained in the following.
4.6 Definition [37]: Let \((X, T)\) be a fts. Consider the set of all fuzzy regular open sets in \(X\). Then it is easy to see that it forms a base for some fuzzy topology on \(X\). We call this fuzzy topology, the fuzzy semiregularization topology of \(T\) and it is to be denoted by \(T_s\). Clearly \(T_s \subseteq T\).

The fts \((X, T_s)\) is called the fuzzy semiregularization space or simply the fuzzy semiregularization of \((X, T)\).

The fuzzy semiregular spaces were also introduced and studied by K.K.Azad [4].

From this definition it follows that a fts \((X, T)\) is fuzzy semiregular iff \(T_s = T\).

The concept of near \(\alpha\) - compactness is characterized in terms of its semiregular fts.

4.7 Theorem: A fts \((X, T)\) is nearly \(\alpha\) - compact (resp. \(\alpha^*\) - compact) iff its semiregular fts is \(\alpha\) - compact (resp. \(\alpha^*\) - compact).

Proof: Let \(X\) be a nearly \(\alpha\) - compact fts. Let \(X_s\) be a semiregular fts of \(X\). To prove that \(X_s\) is \(\alpha\) - compact fts. Let \(\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}\) be any open \(\alpha\) - shading of \(X_s\). Then for each \(U_\lambda\) there exists an index set \(I_\lambda\) such that \(U_\lambda = \bigvee_{\gamma \in I_\lambda} W_\gamma\), where for each \(\gamma \in I_\lambda\), \(W_\gamma\) is fuzzy regular open set in \(X\). Now the family \(\{ W_\gamma : \gamma \in I_\lambda, \lambda \in \Lambda \}\) is an \(\alpha\) - shading of \(X\); For let \(x \in X\). Since \(\mathcal{U}\) is \(\alpha\) - shading of \(X\), there exists a \(U_\lambda\) in
such that $U_\gamma(x) > \alpha$, which implies $(\bigvee_{r \in I_\lambda} W_\gamma(x)) > \alpha$. That is, $\sup_{r \in I_\lambda} \{ W_\gamma(x) \} > \alpha$. That is, $W_\gamma(x) > \alpha$ for some $\gamma \in I_\lambda$. It follows that $\{ W_\gamma : \gamma \in I_\lambda, \lambda \in \Lambda \}$ is fuzzy regular open $\alpha$-shading of $X$. Since $X$ is nearly $\alpha$-compact, it follows that $\{ W_\gamma : \gamma \in I_\lambda, \lambda \in \Lambda \}$ has a finite $\alpha$-subshading say $\mathcal{V} = \{ W_{\gamma_i} : \gamma_i \in I_\lambda, \lambda \in \Lambda, i = 1,2,3,\ldots,n \}$.

Now for $\gamma_1 = 1,2,3,\ldots,n$, there exists $\lambda_i$ such that $W_{\gamma_i} \leq \bigvee_{r \in I_\lambda} W_\gamma = U_\lambda$. Then $\mathcal{U}' = \{ U_\lambda : i = 1,2,3,\ldots,n \}$ is the required $\alpha$-subshading of $\mathcal{U}$. Now $\mathcal{U}'$ is an $\alpha$-shading of $X$. Let $x \in X$. Since $\mathcal{V}$ is $\alpha$-shading of $X$ there exists $W_{\gamma_{i_0}}$ in $\mathcal{V}$ such that $W_{\gamma_{i_0}}(x) > \alpha$ and $U_{\lambda_{i_0}} = \bigvee_{r \in I_\lambda} W_{\gamma_r}$, which implies $U_{\lambda_{i_0}}(x) \geq W_{\gamma_{i_0}}(x) > \alpha$.

It follows that $\mathcal{U}'$ is $\alpha$-shading of $X$. Thus every open $\alpha$-shading of $X_S$ has a finite $\alpha$-subshading. Hence $X_S$ is $\alpha$-compact fts.

Conversely, Let $X_S$ be $\alpha$-compact fts. To prove that $X$ is nearly $\alpha$-compact fts. Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ be any fuzzy regular open $\alpha$-shading of $X$. Then it is also a open $\alpha$-shading of $X_S$. Since $X_S$ is $\alpha$-compact, $\mathcal{U}$ has a finite $\alpha$-subshading say $\mathcal{V} = \{ U_{\lambda_i} : i = 1,2,3,\ldots,n \}$. Then $\mathcal{V}' = \{ \hat{U}_{\lambda_i} : i = 1,2,3,\ldots,n \}$ is the required finite $\alpha$-subshading of $\mathcal{U}$. Now, $\mathcal{V}'$ is $\alpha$-shading of $X$. Let $x \in X$.

Since $\mathcal{V}$ is $\alpha$-shading, there exists $U_{\lambda_{i_0}}$ such that $U_{\lambda_{i_0}}(x) > \alpha$. Now $U_{\lambda_{i_0}} \leq \hat{U}_{\lambda_{i_0}}$.
which implies $U^\lambda_a = U^{\lambda_0}_a \leq U^{\lambda_0}_a$. Therefore $U^\lambda_a(x) \geq U_a^\lambda(x) > \alpha$. Thus $\bar{U}^\lambda_a(x) > \alpha$.

Therefore $\mathcal{V}$ is finite $\alpha$-subshading of $\mathcal{U}$. Hence $X$ is nearly $\alpha$-compact fts.

4.8 Corollary: Nearly $\alpha$-compact fts and $\alpha$-compact fts are equivalent in semiregular fts.

Nearly $\alpha$-compactness has the following hereditary property.

4.9 Theorem: Every closed crisp subspace of a nearly $\alpha$-compact fts is nearly $\alpha$-compact.

Proof: Let $(Y, T_Y)$ be a closed crisp subspace of a nearly $\alpha$-compact fuzzy topological space $(X, T)$. To prove that $(Y, T_Y)$ is nearly $\alpha$-compact fts. Let $\mathcal{U} = \{ U_\lambda; \lambda \in \Lambda \}$ be any fuzzy regular open $\alpha$-shading of $Y$. Since $U_\lambda$ is an open fuzzy set in $Y$, there exists open fuzzy set $V_\lambda$ in $X$ such that $U^\lambda_\lambda = Y \wedge V_\lambda$. Then $\{ V_\lambda; \lambda \in \Lambda \} \cup \{ 1 - Y \}$ is an open $\alpha$-shading of $X$; Since $Y$ is closed, $1 - Y$ is an open fuzzy set. Let $x \in X$ then $x \in Y$ or $x \not\in Y$. If $x \not\in Y$ then $x \in 1 - Y$. That is $(1 - Y)(x) = 1 > \alpha$, where $(1 - Y) \in \{ V_\lambda; \lambda \in \Lambda \} \cup \{ 1 - Y \}$. Suppose $x \in Y$. Since $\{ U_\lambda; \lambda \in \Lambda \}$ is an $\alpha$-shading of $Y$, there exists $U_\lambda$ such that $U^\lambda_\lambda(x) > \alpha$.

But $U_\lambda = Y \wedge V_\lambda > \alpha$. Therefore $U_\lambda^\lambda(x) = \min \{ Y(x), V_\lambda(x) \} > \alpha$. But $Y(x) = 1$. Therefore $V_\lambda(x) > \alpha$. Thus $\{ V_\lambda; \lambda \in \Lambda \} \cup \{ 1 - Y \}$ is an open $\alpha$-shading of $X$.

Let $\mathcal{Y} = \{ \varphi_\lambda; \lambda \in \Lambda \} \cup \{ \varphi_\lambda \wedge (1 - Y)^- \}$ be a family of fuzzy regular open sets of $X$. $\mathcal{Y}$ is an $\alpha$-shading of $X$; Let $x \in X$, then $x \in Y$ or $x \not\in Y$. If $x \not\in Y$ then
x \in (1 - Y). That is \((1 - Y)(x) = 1 > \alpha\). As Y is closed, 1 - Y is an open fuzzy set. Now \((1 - Y) \leq (1 - Y)^0\). Therefore \((1 - Y)^0 \leq [(1 - Y)^0]\). That is \((1 - Y) \leq [(1 - Y)^0]\), since 1 - Y is open. Therefore \([(1 - Y)^0]\) \geq (1 - Y)(x) > \alpha, Which implies \([(1 - Y)^0]\) \geq (1 - Y)(x) > \alpha, where \([(1 - Y)^0]\) \geq \forall. If x \in Y then there exists \(U_\lambda\) in \(\{ U_\lambda: \lambda \in \Lambda \}\) such that \(U_\lambda(x) > \alpha\), since \(\{ U_\lambda: \lambda \in \Lambda \}\) is an \(\alpha\) - shading of Y. But \(U_\lambda = Y \wedge V_\lambda\). Therefore \((Y \wedge V_\lambda)(x) > \alpha\). That is Min\{Y(x), V_\lambda(x)\} > \alpha which implies \(V_\lambda(x) > \alpha\) as \(Y(x) = 1\). Also \(V_\lambda\) is open, so \(V_\lambda = V_\lambda^0\) and \(V_\lambda \leq \bar{V}_\lambda\) which implies \(V_\lambda^0 \leq \bar{V}_\lambda^0\). Therfore \(\bar{V}_\lambda^0(x) \geq V_\lambda^0(x) = V_\lambda(x) > \alpha\). That is \(\bar{V}_\lambda^0(x) > \alpha\) where \(\bar{V}_\lambda^0 \in \forall\). Therefore \(\forall\) is a fuzzy regular open \(\alpha\) - shading of X.

Since X is nearly \(\alpha\) - compact, \(\forall\) has a finite \(\alpha\) - subshading, say \(\forall' = \{ \bar{V}_{1}^0, \bar{V}_{2}^0, ..., \bar{V}_{\lambda_k}^0 \}\) \(\cap [(1 - Y)^0]\) for X. Then the family \(\forall' = \{ \bar{U}_{\lambda_1}^0, \bar{U}_{\lambda_2}^0, ..., \bar{U}_{\lambda_k}^0 \}\) is a finite \(\alpha\) - subshading of \(\forall\) for Y. Let \(x \in Y\). Then x \in X and x \notin 1 - Y. Since \(\forall'\) is a finite \(\alpha\) - shading of X, there exists \(\bar{V}_\lambda\), i.e \(\{ 1,2,3, ..., k \}\) such that \(\bar{V}_\lambda(x) > \alpha\). Also \(x \in Y\) which implies \(Y(x) = 1 > \alpha\). Therefore \(U_\lambda(x) \geq (Y \wedge \bar{V}_\lambda)(x) > \alpha\). Therefore \(U_\lambda(x) > \alpha\). But \(U_\lambda\) is open. Therefore \(U_\lambda^0 = U_\lambda\) and \(U_\lambda \leq U^0_\lambda\) which implies \(U_\lambda^0 \leq U^0_\lambda\). That is

42
Therefore \( U^\alpha \leq U^\alpha \). Therefore \( U^\alpha \geq U^\alpha > \alpha \). Thus \( U^\alpha (x) > \alpha \). Thus for each \( x \in Y \),

there exists \( U^\alpha_i \), \( i \in \{1, 2, 3, \ldots, k\} \) such that \( U^\alpha_i (x) > \alpha \). It follows that \( \mathcal{U} \) is an \( \alpha \)-shading of \( Y \). Thus every fuzzy regular open \( \alpha \)-shading \( \mathcal{U} \) of \( Y \) has a finite \( \alpha \)-subshading \( \mathcal{U}' \). Hence \( Y \) is nearly \( \alpha \)-compact fts.

Similar proof can be proved for \( \alpha^* \)-case.

The concept of fuzzy almost continuous maps and fuzzy almost open maps are given below which are needed in the sequel.

4.10 Definition [4]: A mapping \( f : (X, T) \rightarrow (Y, S) \) from a fts \( (X, T) \) to another fts \( (Y, S) \) is called a fuzzy almost continuous mapping if \( f^{-1}(G) \in T \) for every fuzzy regular open set \( G \in S \).

4.11 Definition [55]: A mapping \( f : (X, T) \rightarrow (Y, S) \) from a fts \( (X, T) \) to another fts \( (Y, S) \) is called a fuzzy almost open if the image of every fuzzy regular open set in \( X \) is fuzzy open set in \( Y \).

The invariance of nearly \( \alpha \)-compact fts under maps is proved in the following.

4.12 Theorem: The image of a nearly \( \alpha \)-compact (resp. nearly \( \alpha^* \)-compact) fts under fuzzy almost continuous and fuzzy open onto function is nearly \( \alpha \)-compact (resp. nearly \( \alpha^* \)-compact).
Proof: Let \( f : X \to Y \) be a fuzzy almost continuous, fuzzy almost open onto function from a nearly \( \alpha \)-compact fts \( X \) onto a fts \( Y \). To prove that \( Y \) is nearly \( \alpha \)-compact fts. Let \( \mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \} \) be any fuzzy regular open \( \alpha \)-shading of \( Y \).

Since \( f \) is almost continuous it follows that \( \{ f^{-1}(\overline{U_\lambda^0}) : \lambda \in \Lambda \} \) is a family of open fuzzy set in \( X \). Then \( \mathcal{V} = \{ ( [ f^{-1}(U_\lambda^0) ]^{-})^0 : \lambda \in \Lambda \} \) is a family of fuzzy regular open sets in \( X \).

\( \mathcal{V} \) is an \( \alpha \)-shading of \( X \); Let \( x \in X \). Then \( f(x) \in Y \). Since \( \mathcal{U} \) is an \( \alpha \)-shading of \( Y \), there exists \( U_{\lambda}^0 \) in \( \mathcal{U} \) such that \(( \overline{U_{\lambda}^0} ) f(x) > \alpha \). Then \( f^{-1}(\overline{U_{\lambda}^0}(x)) > \alpha \). That is \( f^{-1}(\overline{U_{\lambda}^0})](x) > \alpha \). But \( f^{-1}(\overline{U_{\lambda}^0}) \) is open fuzzy set.

Therefore \( [ f^{-1}(\overline{U_{\lambda}^0}) ]^0 = f^{-1}(\overline{U_{\lambda}^0}) \). Also \( f^{-1}(\overline{U_{\lambda}^0}) \leq [ f^{-1}(\overline{U_{\lambda}^0}) ]^{-}. \) Therefore \( [ f^{-1}(\overline{U_{\lambda}^0}) ]^0 \leq ([ f^{-1}(\overline{U_{\lambda}^0}) ]^{-})^0. \) That is \( f^{-1}(\overline{U_{\lambda}^0}) \leq ([ f^{-1}(\overline{U_{\lambda}^0}) ]^{-})^0 \) which implies \( ([ f^{-1}(\overline{U_{\lambda}^0}) ]^{-})^0(x) > \alpha \). Thus for each \( x \in X \), there exists \(( [ f^{-1}(\overline{U_{\lambda}^0}) ]^{-})^0 \in \mathcal{V} \) such that \(( [ f^{-1}(\overline{U_{\lambda}^0}) ]^{-})^0(x) > \alpha \). Therefore \( \mathcal{V} \) is an \( \alpha \)-shading of \( X \). Since \( X \) is nearly \( \alpha \)-compact, \( \mathcal{V} \) has a finite \( \alpha \)-subshading say \( \mathcal{V}' = \{ ( [ f^{-1}(\overline{U_i^0}) ]^{-})^0 : i = 1, 2, 3, \ldots, k \} \) for \( X \). Then the family \( \mathcal{U}' = \{ \overline{U_i^0} : i = 1, 2, 3, \ldots, k \} \) is a finite \( \alpha \)-subshading of \( \mathcal{U} \) for \( Y \); Let \( y \in Y \), then there exists some \( x \in X \) such that \( f(x) = y \). Since \( \mathcal{V}' \) is an \( \alpha \)-shading of \( X \), there
exists \( \{ f^{-1}(\overline{U}_{A_i}) \} \in \psi' \) for some \( i_0 \in \{ 1,2,3, \ldots, k \} \) such that \[ \left( \left( f^{-1}(\overline{U}_{A_i}) \right)^0 \right)(x) > \alpha \quad \text{......(I).} \]

Now \( \overline{U}_{A_i} \subseteq \overline{U}_{A_{i_0}} \) for each \( i_0 \in \{ 1,2,3, \ldots, k \} \). Therefore \( f^{-1}[\overline{U}_{A_{i_0}}] \leq f^{-1}[\overline{U}_{A_{i_0}}] \) \( \text{.........(II).} \). But \( \overline{U}_{A_{i_0}} \) is open fuzzy set, so that \( \overline{U}_{A_{i_0}} = (\overline{U}_{A_{i_0}})^0 \) which implies \( \overline{U}_{A_{i_0}} = (\overline{U}_{A_{i_0}})^0 \). Therefore \( f^{-1}[\overline{U}_{A_{i_0}}] = f^{-1}(\overline{U}_{A_{i_0}})^0 \). Also \((\overline{U}_{A_{i_0}})^0)^0\) is fuzzy regular closed and \( f \) is almost continuous. Therefore \( f^{-1}(\overline{U}_{A_{i_0}})^0 \) must be a fuzzy closed set. Hence \( f^{-1}[\overline{U}_{A_{i_0}}] \) is a fuzzy closed set in \( X \).

From (II) it follows that \( (f^{-1}[\overline{U}_{A_{i_0}}])^0 \leq (f^{-1}[\overline{U}_{A_{i_0}}])^0 = f^{-1}[\overline{U}_{A_{i_0}}] \). That is \[ \left( f^{-1}[\overline{U}_{A_{i_0}}] \right)^0 \leq \left( f^{-1}[\overline{U}_{A_{i_0}}] \right)^0 \leq f^{-1}[\overline{U}_{A_{i_0}}] \] which implies that \( f(\overline{U}_{A_{i_0}}) \) is fuzzy regular open set, and \( f \) is fuzzy almost open function, it follows that \( f([f^{-1}[\overline{U}_{A_{i_0}}])^0]) \) must be an open fuzzy set in \( Y \).

Therefore \( f([f^{-1}[\overline{U}_{A_{i_0}}])^0]) \leq \overline{U}_{A_{i_0}} \).

That is \( f([f^{-1}[\overline{U}_{A_{i_0}}])^0] \leq \overline{U}_{A_{i_0}} \).

Which implies that \( f^{-1}(f([f^{-1}[\overline{U}_{A_{i_0}}])^0])) \leq f^{-1}(\overline{U}_{A_{i_0}}) \).
That is \[ (f^{-1}(\overline{U}_{\kappa_0}^0))^0 \leq \overline{f(U_{\kappa_0})}, \]

Which implies \([ f^{-1}(\overline{U}_{\kappa_0}^0) ](x) \geq \left( [(f^{-1}(\overline{U}_{\kappa_0}^0))^0](x) > \alpha, \text{ from (1).} \]

Therefore \( f^{-1}(\overline{U}_{\kappa_0}^0)(x) > \alpha. \) That is \( (\overline{U}_{\kappa_0}^0)(f(x)) > \alpha \) which implies \( \overline{U}_{\kappa_0}^0(y) > \alpha. \) Thus for each \( y \in Y, \) there exists \( \overline{U}_{\kappa_0}^0 \) in \( \mathcal{U} \) such that \( \overline{U}_{\kappa_0}^0(y) > \alpha. \) Therefore \( \mathcal{U}' \) is a finite \( \alpha \)-subshading of \( \mathcal{U} \) for \( Y. \) Therefore every fuzzy regular open \( \alpha \)-shading \( \mathcal{U} \) of \( Y \) has a finite \( \alpha \)-subshading \( \mathcal{U}' = \{ \overline{U}_{\kappa_0}^0 : i = 1, 2, 3, \ldots, k \}. \) Hence \( Y \) is a nearly \( \alpha \)-compact fts.

5. NEARLY \( \alpha \)-LINDELÖF FTS


In this section, nearly \( \alpha \)-Lindelöf property for fts has been introduced and studied. Its relation with \( \alpha \)-Lindelöf property, \( \alpha \)-compactness and nearly \( \alpha \)-compactness is given. It has been characterized in terms of its semiregular fts. A subspace property and a invariance under maps have also been obtained.
The following concept is due to S. R. Malghan and S. S. Benchalli [33].

5.1 Definition [33]: Let \( \alpha \in [0,1) \) (resp. \( \alpha \in (0,1] \)). A fts \((X,T)\) is said to be \( \alpha \)-Lindelof (resp. \( \alpha^* \)-Lindelof) if every open \( \alpha \)-shading (resp. \( \alpha^* \)-shading) of \( X \) has a countable \( \alpha \)-subshading (resp. countable \( \alpha^* \)-subshading).

Nearly \( \alpha \)-Lindelof property for fts is introduced in the following.

5.2 Definition: Let \( \alpha \in [0,1) \) (resp. \( \alpha \in (0,1] \)). A fts \((X,T)\) is said to be nearly \( \alpha \)-Lindelof (resp. nearly \( \alpha^* \)-Lindelof) if every fuzzy regular open \( \alpha \)-shading (resp. \( \alpha^* \)-shading) of \( X \) has a countable \( \alpha \)-subshading (resp. countable \( \alpha^* \)-subshading).

5.3 Theorem: Every \( \alpha \)-Lindelof (resp. \( \alpha^* \)-Lindelof) fts is nearly \( \alpha \)-Lindelof (resp. nearly \( \alpha^* \)-Lindelof).

Proof: Let \((X,T)\) be a \( \alpha \)-Lindelof fts. To prove that \((X,T)\) is Nearly \( \alpha \)-Lindelof fts. Let \( \mathcal{U} \) be any fuzzy regular open \( \alpha \)-shading of \( X \), then \( \mathcal{U} \) is fuzzy open \( \alpha \)-shading of \( X \). Since \( X \) is \( \alpha \)-Lindelof, it follows that \( \mathcal{U} \) has a countable \( \alpha \)-subshading which is the required countable \( \alpha \)-subshading of \( \mathcal{U} \). Thus every fuzzy regular open \( \alpha \)-shading \( \mathcal{U} \) of \( X \) has a countable \( \alpha \)-subshading. Hence \( X \) is nearly \( \alpha \)-Lindelof fts. The proof of \( \alpha^* \)-case is analogous.
5.4 **Theorem**: If a fts is $\alpha$-compact then it is Nearly $\alpha$-Lindelof.

5.5 **Theorem**: If a fts is Nearly $\alpha$-compact then it isNearly $\alpha$-Lindelof.

Nearly $\alpha$-Lindelof property is characterized in terms of its semiregular fts, in the following.

5.6 **Theorem**: A fts $(X, T)$ is Nearly $\alpha$-Lindelof (resp. Nearly $\alpha^*$-Lindelof) iff its semiregular fts is $\alpha$-Lindelof (resp. $\alpha^*$-Lindelof).

**Proof**: Let $X$ be a Nearly $\alpha$-Lindelof fts. Let $X_s$ be a semiregular fts of $X$. To prove that $X_s$ is $\alpha$-Lindelof fts. Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ be any open $\alpha$-shading of $X_s$. Then for each $U_\lambda$ there exists an index set $I_\lambda \subset \Lambda$, such that $U_\lambda = \bigvee_{\gamma \in I_\lambda} W_\gamma$ where each $W_\gamma$ is fuzzy regular open set in $X$. It can be shown that $\{ W_\gamma : \gamma \in I_\lambda, \lambda \in \Lambda \}$ is an $\alpha$-shading of $X$ and it is a family of fuzzy regular open sets of $X$. Since $X$ is nearly $\alpha$-Lindelof, $\{ W_\gamma : \gamma \in I_\lambda, \lambda \in \Lambda \}$ has a countable $\alpha$-subshading say $\mathcal{U}' = \{ W_{r_n} : n \in N \}$.

Now for each $\gamma_n$, there exists a $\lambda$ such that $W_{r_n} \leq \bigvee_{\gamma \in I_{\lambda_n}} W_\gamma = U_{\lambda_n}$. Then $\mathcal{U}' = \{ U_{\lambda} : n \in N \}$ is the required $\alpha$-subshading of $\mathcal{U}$. Thus every fuzzy regular open $\alpha$-shading $\mathcal{U}$ has a countable $\alpha$-subshading $\mathcal{U}'$. Hence $X_s$ is $\alpha$-Lindelof fts.
Conversely, let $X$ be a $\alpha$-Lindelof fls. To prove, $X$ is nearly $\alpha$-Lindelof fls. Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ be any fuzzy regular open $\alpha$-shading of $X$. Then $\mathcal{U}$ is a fuzzy open $\alpha$-shading of $X$. Since $X$ is $\alpha$-Lindelof, it follows that $\mathcal{U}$ has a countable $\alpha$-subshading say $\mathcal{V} = \{ U_\lambda : n \in \mathbb{N} \}$. Then $\mathcal{V}' = \{ \overline{U_\lambda} : n \in \mathbb{N} \}$ is the required countable $\alpha$-subshading of $\mathcal{U}$ for $X$. Therefore $X$ is nearly $\alpha$-Lindelof fls. The proof of $\alpha^*$-case is similar.

5.7 Corollary: Nearly $\alpha$-Lindelof and $\alpha$-Lindelof fls are equivalent in semiregular fls.

A hereditary property for nearly $\alpha$-Lindelof fls has been obtained in the following.

5.8 Theorem: Every closed crisp subspace of a nearly $\alpha$-Lindelof (resp. nearly $\alpha^*$-Lindelof) fls is Nearly $\alpha$-Lindelof (resp. Nearly $\alpha^*$-Lindelof).

Proof: Let $Y$ be a closed crisp subspace of a nearly $\alpha$-Lindelof fls $X$. To prove that $Y$ is nearly $\alpha$-Lindelof fls. Let $\mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \}$ be any fuzzy regular open $\alpha$-shading of $Y$. For each $\lambda \in \Lambda$, $U_\lambda$ is open in $Y$. Therefore there is an open fuzzy set $V_\lambda$ in $X$ such that $U_\lambda = Y \wedge V_\lambda$. Then the family $\{ V_\lambda : \lambda \in \Lambda \} \cup \{ 1 - Y \}$ is an open $\alpha$-shading of $X$. 

49
Let \( \mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \} \cup \{ [(1 - Y)^-]^0 \} \) be a family of fuzzy regular open sets in \( X \). Then it can be shown that \( \mathcal{V} \) is \( \alpha \) - shading of \( X \). Since \( X \) is nearly \( \alpha \) - Lindelof, \( \mathcal{V} \) has a countable \( \alpha \) - subshading say \( \mathcal{V}' = \{ V_\lambda : n \in N \} \cup \{ [(1 - Y)^-]^n \} \). Then the family \( \mathcal{U} = \{ U_\lambda : n \in N \} \) is the required countable \( \alpha \) - subshading of \( \mathcal{U} \) for \( Y \). Therefore \( Y \) is nearly \( \alpha \) - Lindelof fts. For \( \alpha^* \) - case the proof is similar.

The invariance of nearly \( \alpha \) - Lindelof property, under maps is given in the following.

5.9 Theorem: The image of a Nearly \( \alpha \) - Lindelof (resp. Nearly \( \alpha^* \) - Lindelof) fts under a fuzzy almost continuous, fuzzy almost open onto function is Nearly \( \alpha \) - Lindelof (resp. Nearly \( \alpha^* \) - Lindelof) fts.

Proof: The proof is similar to that of Theorem 4.12 and therefore is omitted.